SECTION A (40 marks)
Answer ALL questions in this section.
Write your answers in the AL(C1) answer book.

1. (a) Let $A = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}$ where $a, b \in \mathbf{R}$ and $a \neq b$.

   Prove that $A^n = \begin{pmatrix} a^n & \dfrac{a^n - b^n}{a - b} \\ 0 & b^n \end{pmatrix}$ for all positive integers n.

   (b) Hence, or otherwise, evaluate $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{95}$.

(6 marks)

2. Let n be an integer and $n > 1$. By considering the binomial expansion of $(1+x)^n$, or otherwise,

   (a) show that $C_1^n + 2C_2^n + 3C_3^n + \ldots + nC_n^n = 2^{n-1}n$;

   (b) evaluate $\dfrac{1}{(n-1)!} + \dfrac{-2}{2!(n-2)!} + \dfrac{3}{3!(n-3)!} + \ldots + \dfrac{(-1)^{n-1}n}{n!}$.

(5 marks)

3. (a) If a_1, a_2, a_3, a_4, p, q, α, β are real numbers such that $x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = (x^2 + px + q)^2 - (\alpha x + \beta)^2$ for all x,

   show that $\begin{cases} \alpha^2 = \dfrac{a_1^2}{4} + 2q - a_2 \\ \alpha\beta = \dfrac{1}{2}(a_1q - a_3) \\ \beta^2 = q^2 - a_4 \end{cases}$.

   (b) Find the possible real values of p, q, α, β such that $x^4 + 4x^3 - 12x^2 + 24x - 9 = (x^2 + px + q)^2 - (\alpha x + \beta)^2$ for all x.

   (c) Solve $x^4 + 4x^3 - 12x^2 + 24x - 9 = 0$.

(7 marks)

4. Let $\mathrm{f} : [-1, 1] \to [0, \pi]$, $\mathrm{f}(x) = \arccos x$ and $\mathrm{g} : \mathbf{R} \to \mathbf{R}$, $\mathrm{g}(x) = \mathrm{f}(\cos x)$.

   (a) Show that $\mathrm{g}(x)$ is even and periodic.

   (b) Find $\mathrm{g}(x)$ for $x \in [0, \pi]$.
   Hence sketch the graph of $\mathrm{g}(x)$ for $x \in [-2\pi, 2\pi]$.

(5 marks)

5. (a) For $x > 0$, prove that $\ln x \le x - 1$ where the equality holds if and only if $x = 1$.

   (b) Prove that $\ln \dfrac{r}{r-1} < \dfrac{1}{r-1}$ for $r > 1$.

   Hence deduce that $\ln n < \displaystyle\sum_{k=1}^{n-1} \frac{1}{k}$ for $n = 2, 3, 4, \ldots$.

(7 marks)

6. Let $\{a_n\}$ be a sequence of non-negative integers such that

$$n \le \sum_{k=1}^{n} a_k^2 \le n+1+(-1)^n \quad \text{for} \quad n = 1,2,3,\ldots .$$

Prove that $a_n = 1$ for $n \ge 1$.

(4 marks)

7. Let $a \in \mathbf{C}$ and $a \ne 0$.

(a) Show that if $|z| = |z-a|$, then $\operatorname{Re}\left(\frac{z}{a}\right) = \frac{1}{2}$.

(b) If $|z| = |z-a| = |a|$, express z in terms of a .

(6 marks)

SECTION B (60 marks)

Answer any **FOUR** questions from this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

8. Let M_{mn} be the set of all $m \times n$ matrices.

(a) Let $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in M_{22}$.

(i) Show that if $A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \begin{pmatrix} s_1 & s_2 \end{pmatrix}$, where $u_1, u_2, s_1, s_2 \in \mathbf{R}$, then $\det A = 0$.

(ii) Conversely, show that if $\det A = 0$, then $A = BC$ for some $B \in M_{21}$ and $C \in M_{12}$.

(5 marks)

(b) Let $D = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in M_{33}$.

(i) Show that if $D = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} \begin{pmatrix} s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{pmatrix}$, where $u_i, v_i, s_i, t_i \in \mathbf{R}$ $(i = 1,2,3)$, then $\det D = 0$.

(ii) Suppose there are α , $\beta \in \mathbf{R}$ such that $c_i = \alpha a_i + \beta b_i$ for $i = 1, 2, 3$. Find $S \in M_{32}$ and $T \in M_{23}$ such that $D = ST$.

(iii) Show that if $\det D = 0$, then $D = PQ$ for some $P \in M_{32}$ and $Q \in M_{23}$.

(10 marks)

9. Consider the following systems of linear equations

$$(S): \begin{cases} 2x + 2y - z = k \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \end{cases}$$

and

$$(T): \begin{cases} 6x + 6y - 3z = 2 \\ hx - 3y - z = 0 \\ -3x + hy + z = 0 \\ -5x - 2y + 6z = h \end{cases}$$

(a) Show that (S) has a unique solution if and only if $h^2 \neq 9$. Solve (S) in this case.

(3 marks)

(b) For each of the following cases, find the value(s) of k for which (S) is consistent, and solve (S) :

(i) $h = 3$,

(ii) $h = -3$.

(7 marks)

(c) Find the values of h for which (T) is consistent. Solve (T) for each of these values of h .

(5 marks)

10. Let α , β and γ be real and distinct and

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 + px^2 + qx + r \ .$$

(a) Show that

(i) $\dfrac{1}{x-\alpha} + \dfrac{1}{x-\beta} + \dfrac{1}{x-\gamma} = \dfrac{3x^2+2px+q}{x^3+px^2+qx+r}$;

(ii) $3\alpha^2 + 2p\alpha + q = (\alpha-\beta)(\alpha-\gamma)$.

(4 marks)

(b) Let $f(x)$ be a real polynomial. Suppose $Ax^2 + Bx + C$ is the remainder when $(3x^2+2px+q)f(x)$ is divided by $x^3 + px^2 + qx + r$.

(i) Prove that $\dfrac{f(\alpha)}{x-\alpha} + \dfrac{f(\beta)}{x-\beta} + \dfrac{f(\gamma)}{x-\gamma} = \dfrac{Ax^2+Bx+C}{x^3+px^2+qx+r}$.

(ii) Express A , B and C in terms of α , β , γ , $f(\alpha)$, $f(\beta)$ and $f(\gamma)$.

(11 marks)

11. Let P, Q be two points on a circle with centre C such that P, Q, C are non-collinear and taken anti-clockwise. $\angle PCQ = \alpha$ and M is the mid-point of PQ. Let z_P, z_Q, z_C and z_M be the complex numbers represented by P, Q, C and M respectively.

(a) Show that $z_C - z_M = i(z_M - z_P)\cot\frac{\alpha}{2}$.

(5 marks)

(b) Express z_C and the radius r of the circle in terms of z_P, z_Q and α.

(4 marks)

(c) (i) Show that any circle in the complex plane can be represented by an equation of the form

$$z\bar{z} + az + b\bar{z} + c = 0$$

where $a, b \in \mathbf{C}$ and $c \in \mathbf{R}$.

(ii) Let $\mathscr{C}$: $z\bar{z} + az + b\bar{z} + c = 0$ be a circle passing through the points representing $1+i$ and $-i$. If the chord joining these two points subtends an angle $\frac{\pi}{3}$ at the centre, find the values of a, b and c.

(6 marks)

12. Let $p > 0$ and $p \neq 1$. $\{a_n\}$ is a sequence of positive numbers defined by $\begin{cases} a_0 = 2 \\ a_n = \frac{1}{\sqrt[p]{n}} + \frac{1}{p}a_{n-1}, & n = 1, 2, 3, \ldots \end{cases}$.

(a) Prove that $\lim_{n\to\infty} a_n = 0$ if the limit exists. (2 marks)

(b) (i) If $2 = a_0 < a_1 < a_2 < \ldots$, show that $\lim_{n\to\infty} a_n$ does not exist.

(ii) If $a_{k-1} \geq a_k$ for some $k \geq 1$, show that $a_{n-1} \geq a_n$ for $n \geq k$ and deduce that $\lim_{n\to\infty} a_n = 0$.

(4 marks)

(c) (i) If $0 < p < 1$, show that $\lim_{n\to\infty} a_n$ does not exist.

(ii) If $p \geq 2$, show that $\lim_{n\to\infty} a_n = 0$.

(4 marks)

(d) Suppose $1 < p < 2$.

(i) Prove by mathematical induction that $a_n < \frac{2}{p-1}$ for $n \geq 0$.

(ii) Prove that $\lim_{n\to\infty} a_n = 0$.

(5 marks)

13. Let a and b be positive numbers.

(a) Prove that

$$a^a b^b \geq a^b b^a$$

where, if the equality holds, then $a = b$.

(4 marks)

(b) Using (a), or otherwise, prove that

$$\left(\frac{a+b}{2}\right)^{a+b} \geq a^b b^a$$

where, if the equality holds, then $a = b$.

(3 marks)

(c) Show that $x^x(1-x)^{1-x} \geq \frac{1}{2}$ for $0 < x < 1$

where, if the equality holds, then $x = \frac{1}{2}$.

Deduce that $a^a b^b \geq \left(\frac{a+b}{2}\right)^{a+b}$

where, if the equality holds, then $a = b$.

(8 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1995

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm-5.00 pm (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)
Answer **ALL** questions in this section.
Write your answers in the AL(C1) answer book.

1. (a) Evaluate $\lim_{x \to 0} \left(\frac{a^x + b^x + 1}{3} \right)^{\frac{1}{x}}$ where $a, b > 0$.

   (b) By considering a suitable definite integral, evaluate

   $$\lim_{n \to \infty} \left(\frac{1^2}{n^3 + 1^3} + \frac{2^2}{n^3 + 2^3} + \dots + \frac{n^2}{n^3 + n^3} \right) .$$

(6 marks)

2. (a) Using the substitution $x = \sin^2\theta \quad (0 < \theta < \frac{\pi}{2})$, prove that

   $$\int \frac{f(x)}{\sqrt{x(1-x)}}\, dx = 2 \int f(\sin^2\theta)\, d\theta .$$

   (b) Hence, or otherwise, evaluate

   $$\int \frac{dx}{\sqrt{x(1-x)}} \quad \text{and} \quad \int \sqrt{\frac{x}{1-x}}\, dx .$$

(5 marks)

3. Consider the parabola $y^2 = 4ax$.

   (a) Prove that the equation of the normal at $P(at^2, 2at)$ is

   $$y + tx = 2at + at^3 . \quad \dots\dots\dots\dots\dots\dots (*)$$

   (b) $P_i(at_i^2, 2at_i)$, $i = 1, 2, 3$, are three distinct points on the parabola. Suppose the normals at these points are concurrent. By considering (*) as a cubic equation in t , or otherwise, show that $t_1 + t_2 + t_3 = 0$.

(5 marks)

4. For $x \ge 0$, define $F(x) = \int_0^x \frac{\sin t}{t+1}\, dt$.

   (a) Find the value of x_0 for which $F(x) \le F(x_0)$ for all $x \in [0, 2\pi]$.

   (b) By considering $F(0)$ and $F(2\pi)$, show that $F(x) > 0$ for all $x \in (0, 2\pi)$.

(7 marks)

5. Figure 1 shows the graphs of the circle $\Gamma_1 : r = -2\cos\theta$ and the cardioid $\Gamma_2 : r = 2 + 2\cos\theta$.

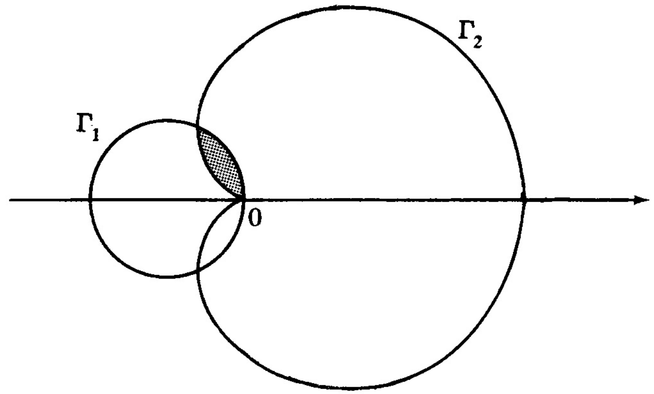

Figure 1

   (a) Find the polar coordinates of all the intersecting points of Γ_1 and Γ_2 other than the pole.

   (b) Find the area of the shaded region.

(6 marks)

6. Let r be a real number. Define $y = \left(\frac{x+1}{x-1}\right)^r$ for $x > 1$.

(a) Show that $\frac{dy}{dx} = \frac{-2ry}{x^2-1}$.

(b) For $n = 1, 2, 3, \ldots$, show that

$$(x^2-1)y^{(n+1)} + 2(nx+r)y^{(n)} + (n^2-n)y^{(n-1)} = 0 ,$$

where $y^{(0)} = y$ and $y^{(k)} = \frac{d^k y}{dx^k}$ for $k \geq 1$.

(5 marks)

7. Let $f : \mathbf{R} \to (-1, \infty)$ be a differentiable function.

(a) Differentiate $\ln[1 + f(x)]$.

(b) If $f(x) = x^3 + \int_0^x 3t^2 f(t)\,dt$ for all $x \in \mathbf{R}$, by considering $f'(x)$, find $f(x)$.

(6 marks)

SECTION B (60 marks)
Answer any **FOUR** questions from this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

8. Let $I_k = \int_0^1 \frac{(-1)^k(1-x)x^{3k}}{1+x^3}\,dx$, $k = 0, 1, 2, \ldots$.

(a) Evaluate I_0 . (4 marks)

(b) Prove that $-\frac{1}{3k+1} \leq I_k \leq \frac{1}{3k+1}$. (3 marks)

(c) Express $I_{k+1} - I_k$ in terms of k . (3 marks)

(d) For $n = 0, 1, 2, \ldots$, let $b_n = \sum_{k=0}^{n} \frac{(-1)^k}{(3k+1)(3k+2)}$.

Using (a) and (c), express I_{n+1} in terms of b_n .

Hence use (b) to evaluate $\lim_{n\to\infty} b_n$. (5 marks)

9. Let $f(x) = \dfrac{|x|}{(x+1)^2}$, where $x \neq -1$.

(a) (i) Find $f'(x)$ and $f''(x)$ for $x > 0$.

(ii) Find $f'(x)$ and $f''(x)$ for $x < 0$.

(iii) Show that $f'(0)$ does not exist.

(4 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) < 0$, (ii) $f'(x) > 0$,

(iii) $f''(x) < 0$, (iv) $f''(x) > 0$.

(4 marks)

(c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

(3 marks)

(d) Find the asymptote(s) and sketch the graph of $f(x)$.

(4 marks)

10. For any $\beta > 0$, define a sequence of real numbers as follows:

$$a_1 = \beta + 1 , \quad a_n = a_{n-1} + \frac{\beta}{a_{n-1}} \quad \text{for } n > 1 .$$

(a) Prove that

(i) $a_n^2 \geq a_{n-1}^2 + 2\beta$ for $n \geq 2$;

(ii) $a_n^2 \geq \beta^2 + 2n\beta + 1$ for $n \geq 1$.

(2 marks)

(b) Using (a), show that for $n \geq 2$,

$$a_n^2 \leq \beta^2 + 2n\beta + 1 + \sum_{k=1}^{n-1} \frac{\beta^2}{\beta^2 + 2k\beta + 1} .$$

(3 marks)

(c) Prove that for $k \geq 1$,

$$\frac{1}{\beta^2 + 2k\beta + 1} \leq \int_{k-1}^{k} \frac{1}{\beta^2 + 2\beta x + 1}\, dx .$$

(2 marks)

(d) Using the above results, show that $\lim\limits_{n\to\infty} \dfrac{a_n^2}{n}$ exists and find the limit.

State with reasons whether $\lim\limits_{n\to\infty} \dfrac{a_n^2}{\sqrt{n}}$ exists.

(8 marks)

11. (a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2\sin\theta + \cos\theta + 2}$.

(5 marks)

(b) Let $f(\theta) = a\sin\theta + b\cos\theta + c$ and

$g(\theta) = A\sin\theta + B\cos\theta + C$

where A , B are not both zero.

Show that there exist real numbers p , q and r such that

$$f(\theta) = p\,g(\theta) + q\,g'(\theta) + r$$

for all real numbers θ .

(5 marks)

(c) Hence, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \frac{7\sin\theta - 4\cos\theta + 3}{2\sin\theta + \cos\theta + 2}\,d\theta .$$

(5 marks)

12. Consider the lines

$$L_1 : \frac{x-5}{2} = \frac{y-1}{2} = \frac{z}{-1}$$

and $$L_2 : \frac{x-4}{2} = \frac{y+8}{5} = \frac{z-1}{2} .$$

(a) Show that L_1 and L_2 do not intersect. (2 marks)

(b) Let L be the line perpendicular to L_1 and L_2 intersecting L_1 at A and L_2 at B .

(i) Find the coordinates of A and B .

(ii) Find the equations of L .

(7 marks)

(c) Let π be the plane containing the point A and perpendicular to L_1 .

(i) Find the equation of π .

(ii) Show that B lies on π .

(iii) Find the equations of the projection of L_2 on π .

(6 marks)

13. (a) Suppose $f(x)$, $g(x)$ are continuously differentiable functions such that $f'(x) \geq 0$ for $a \leq x \leq b$.

(i) Let $w(x) = \int_a^x g(t)\,dt$. Show that

$$\int_a^b f(x)g(x)\,dx = f(b)\int_a^b g(x)\,dx - \int_a^b f'(x)w(x)\,dx .$$

(ii) Using the Theorem (*) below, show that

$$\int_a^b f(x)g(x)\,dx = f(b)\int_c^b g(x)\,dx + f(a)\int_a^c g(x)\,dx$$

for some $c \in [a, b]$.

[**Theorem (*)** : If $w(x)$, $u(x)$ are continuous functions and $u(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b w(x)u(x)\,dx = w(c)\int_a^b u(x)\,dx \text{ for some } c \in [a, b] .$$]

(5 marks)

(b) Let $F(x)$ be a function with a continuous second derivative such that $F''(x) \geq 0$ and $F'(x) \geq m > 0$ for $a \leq x \leq b$. Using (a) with $f(x) = -\frac{1}{F'(x)}$ and $g(x) = -F'(x)\cos F(x)$, show that

$$\left| \int_a^b \cos F(x)\,dx \right| \leq \frac{4}{m} .$$

(5 marks)

(c) (i) Show that $\int_0^1 \cos(x^n)\,dx \leq \int_0^1 \cos(x^{n+1})\,dx$.

Hence show that $\lim_{n\to\infty} \int_0^1 \cos(x^n)\,dx$ exists.

(ii) Using (b), or otherwise, show that $\lim_{n\to\infty} \int_0^{2\pi} \cos(x^n)\,dx$ exists.

(5 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 1996

PURE MATHEMATICS A-LEVEL PAPER I

9.00 am–12.00 noon (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.