1. (a) $P^{-1} = \dfrac{\operatorname{adj} P}{|P|} = \dfrac{1}{6}\begin{pmatrix}1 & -1\\ 4 & 2\end{pmatrix}^{t} = \dfrac{1}{6}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$ 1A

$$P^{-1}AP = \frac{1}{6}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}3 & 8\\ 1 & 5\end{pmatrix}\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}$$ 1A

(b) Let $D = \begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}$, then $A = PDP^{-1}$. 1M

$$A^n = (PDP^{-1})^n$$

$$= PD^nP^{-1}$$ 1M

$$= \begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}\begin{pmatrix}7 & 0\\ 0 & 1\end{pmatrix}^n \frac{1}{6}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}2 & -4\\ 1 & 1\end{pmatrix}\begin{pmatrix}7^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$ 1A

$$= \frac{1}{6}\begin{pmatrix}2\cdot 7^n & -4\\ 7^n & 1\end{pmatrix}\begin{pmatrix}1 & 4\\ -1 & 2\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}2\cdot 7^n+4 & 8\cdot 7^n-8\\ 7^n-1 & 4\cdot 7^n+2\end{pmatrix}$$ 1A

(6)

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$= (4-\lambda)[24 + 10\lambda + \lambda^2 - 45]$

$+ 3(12 + 3\lambda + 7) + (25+\lambda)$

$= (4-\lambda)(\lambda^2 + 10\lambda - 25) + 3(3\lambda + 25) + (\lambda + 25)$

Solution	Marks
2. (*) is equivalent to $\begin{pmatrix} 4-\lambda & 3 & 1 \\ 3 & -(4+\lambda) & 7 \\ 1 & 7 & -(6+\lambda) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{0}$.	1M
Consider $\Delta = \begin{vmatrix} 4-\lambda & 3 & 1 \\ 3 & -(4+\lambda) & 7 \\ 1 & 7 & -(6+\lambda) \end{vmatrix}$	
$= \lambda(\lambda^2+6\lambda-75)$ $\quad -\lambda^3-6\lambda^2+75\lambda$.	1A
For (*) to have nontrivial solutions, $\Delta = 0$	1M
$\Rightarrow \quad \lambda = 0$ or $-3\pm2\sqrt{21}$	
$\because \quad \lambda$ is an integer,	
$\therefore \quad \lambda = 0$.	1A
Thus the augmented matrix is $\left(\begin{array}{ccc\|c} 4 & 3 & 1 & 0 \\ 3 & -4 & 7 & 0 \\ 1 & 7 & -6 & 0 \end{array}\right)$	
When reduced to Echelon form, it becomes $\left(\begin{array}{ccc\|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$	1A
$\therefore \quad \begin{cases} x+z=0 \\ y-z=0 \end{cases}$	
Let $z=t$, then $x=-t$ and $y=t$.	
S.S. $= \{(-t,t,t) : t\in\mathbb{R}\}$.	1A
	(6)

$$\Delta = (4-\lambda)\begin{vmatrix} -(4+\lambda) & 7 \\ 7 & -(6+\lambda) \end{vmatrix}$$

$$-3\begin{vmatrix} 3 & 7 \\ 1 & -(6+\lambda) \end{vmatrix} + \begin{vmatrix} 3 & -(4+\lambda) \\ 1 & 7 \end{vmatrix}$$

$$= (4-\lambda)\left[(4+\lambda)(6+\lambda)-49\right]$$

$$-3\left[-3(6+\lambda)-7\right] + \left[21+(4+\lambda)\right]$$

$$= (4-\lambda)\left[24+10\lambda+\lambda^2-49\right]$$

$$+3(18+3\lambda+7) + (25+\lambda)$$

$$= (4-\lambda)(\lambda^2+10\lambda-25)+3(3\lambda+25)+(\lambda+25)$$

$$= 4\lambda^2+40\lambda-100-\lambda^3-10\lambda^2+25\lambda+9\lambda+75+\lambda$$

Solution	Marks
3. (a) $\because \quad x^3+px+q=0$	
$\therefore \quad x(x^2+p)=-q$	
$\quad x^2(x^2+p)^2=q^2$	1A
Let $y=x^2$, then α^2, β^2 and γ^2 are the roots of the equation	1M
$\quad y(y+p)^2=q^2$	
i.e. $y^3+2py^2+p^2y-q^2=0$ or $x^3+2px^2+p^2x-q^2=0$.	1A
Alternatively	
$\because \quad (x-\alpha^2)(x-\beta^2)(x-\gamma^2)$	
$= x^3-(\alpha^2+\beta^2+\gamma^2)x^2+(\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2)x-\alpha^2\beta^2\gamma^2$	1M
and $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta) = -2p$	
$\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2 = (\beta\gamma+\gamma\alpha+\alpha\beta)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma) = p^2$	
$\alpha^2\beta^2\gamma^2 = q^2$	1A
$\therefore$ the equation is $x^3+2px^2+p^2x-q^2=0$.	1A
(b) $\because \quad \begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = (x+5)(x-2)(x-3)$	
$\therefore$ The roots of $\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$ are -5, 2, 3.	1A
Note that $\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = x^3-19x+30$,	
putting $p=-19$ and $q=30$ in (a),	1M
$\because \quad 2p=-38$, $p^2=361$ and $q^2=900$	
$\therefore$ The roots of $x^3-38x^2+361x-900=0$ are 4, 9, 25.	1A
	(6)

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Solution	Marks
4. (a) $\sum_{k=1}^{n} a_k b_k = s_1 b_1 + (s_2 - s_1) b_2 + \ldots + (s_n - s_{n-1}) b_n$	1M
$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$	1

Alternatively

The statement clearly holds for n=1.

Assume it is true for n=m, then

$$\sum_{k=1}^{m+1} a_k b_k$$

$$= \sum_{k=1}^{m} a_k b_k + a_{m+1} b_{m+1}$$

$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{m-1}(b_{m-1} - b_m) + s_m b_m + (s_{m+1} - s_m) b_{m+1} \quad \text{1M}$$

$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_m(b_m - b_{m+1}) + s_{m+1} b_{m+1} \quad \text{1}$$

The statement is proved by using the principle of mathematical induction.

(b) For $k = 1, 2, \ldots, n-1$,

$\because$ $b_k \geq b_{k+1}$ and $m \leq s_k \leq M$,

$\therefore$

$$m(b_1 - b_2) \leq s_1(b_1 - b_2) \leq M(b_1 - b_2)$$
$$m(b_2 - b_3) \leq s_2(b_2 - b_3) \leq M(b_2 - b_3)$$
$$\vdots$$
$$m(b_{n-1} - b_n) \leq s_{n-1}(b_{n-1} - b_n) \leq M(b_{n-1} - b_n) \quad \text{1}$$

For $k = n$,

$\because$ $b_n \geq 0$ and $m \leq s_n \leq M$

$\therefore$ $mb_n \leq s_n b_n \leq Mb_n$ 1

Summing up the inequalities, we have

$$mb_1 \leq \sum_{k=1}^{n} a_k b_k \leq Mb_1 . \quad \text{1}$$

(5)

Solution	Marks
5. By the given property, we have	
$a_1=\left(\frac{1+a_1}{2}\right)^2$	
$a_1^2-2a_1+1=0$	
$\therefore\ a_1=1.$	1A
$a_n=2n-1$ holds for $n=1$.	
Assume $a_k=2k-1$ for some $k\in\mathbb{N}$.	
$\because\ a_1+a_2+\dots+a_{k+1}=\left(\frac{1+a_{k+1}}{2}\right)^2$	
$\therefore\ \left(\frac{1+a_k}{2}\right)^2+a_{k+1}=\left(\frac{1+a_{k+1}}{2}\right)^2$	1M
$\Rightarrow\ \left(\frac{1+2k-1}{2}\right)^2+a_{k+1}=\left(\frac{1+a_{k+1}}{2}\right)^2$	1M
$\Rightarrow\ 4k^2+4a_{k+1}=1+2a_{k+1}+a_{k+1}^2$	
$\Rightarrow\ a_{k+1}^2-2a_{k+1}+(1-4k^2)=0$	
$\Rightarrow\ [a_{k+1}-(1-2k)][a_{k+1}-(1+2k)]=0$	
$\Rightarrow\ a_{k+1}=1+2k \qquad (\because a_{k+1}>0)$	1+1A
$\Rightarrow\ a_{k+1}=2(k+1)-1$.	
Hence $P(k+1)$ is also true.	
By the principle of mathematical induction, $a_n=2n-1$ for $n\in\mathbb{N}$.	
	(5)

Solution	Marks
6. (a) $z+\bar{z}=0 \Rightarrow 2Re(z)=0$	
$\Rightarrow z$ is purely imaginary $(\because z\neq 0)$	1
$\Rightarrow \text{Arg } z=\pm\frac{\pi}{2}$	1
Alternatively	
Let $z=x+iy$, then $(x+iy)+(x-iy)=0$	
$\Rightarrow x=0$	
$\Rightarrow z=iy$ where $y\neq 0 \quad (\because z\neq 0)$	1
$\Rightarrow \text{Arg } z=\pm\frac{\pi}{2}$	1
(b) $\lvert z_1+z_2\rvert^2=\lvert z_1-z_2\rvert^2$	
$(z_1+z_2)\overline{(z_1+z_2)}=(z_1-z_2)\overline{(z_1-z_2)}$	1M
$(z_1+z_2)(\overline{z_1}+\overline{z_2})=(z_1-z_2)(\overline{z_1}-\overline{z_2})$	
$\lvert z_1\rvert^2+\lvert z_2\rvert^2+z_1\overline{z_2}+\overline{z_1}z_2=\lvert z_1\rvert^2+\lvert z_2\rvert^2-z_1\overline{z_2}-\overline{z_1}z_2$	
$2(z_1\overline{z_2}+\overline{z_1}z_2)=0$	
$\because z_2\neq 0$	
$\therefore \frac{z_1}{z_2}+\overline{\frac{z_1}{z_2}}=0$	1
Since $\frac{z_1}{z_2}\neq 0$, by (a),	
$\text{Arg }\frac{z_1}{z_2}=\pm\frac{\pi}{2}$	1
	(5)

Solution	Marks
7. (a) $\because (1+x)^n+(1+x)^{n+1}+\ldots+(1+x)^{n+m} = \dfrac{(1+x)^{n+m+1}-(1+x)^n}{x}$ for $x\neq 0$	
Comparing the coefficients of x^n on both sides, we have	1M
$C_n^n+C_n^{n+1}+\ldots+C_n^{n+m}$ = coefficient of x^{n+1} in $(1+x)^{n+m+1}$	
$= C_{n+1}^{n+m+1}$	1
(b) $\displaystyle\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3)$	
$= (4!)\displaystyle\sum_{r=5}^{m+4} C_4^r$	1M
$= 24\left[(C_4^4+C_4^5+\ldots+C_4^{m+4})-C_4^4\right]$	
$= 24\left(C_5^{m+5}-1\right)$	1
Hence for $k\geq 4$,	
$\displaystyle\sum_{r=0}^{k} r(r-1)(r-2)(r-3)$	
$= \begin{cases} 4! & \text{if } k=4 \\ 4!+\displaystyle\sum_{r=5}^{k} r(r-1)(r-2)(r-3) & \text{if } k\geq 5 \end{cases}$	1A + 1A
$= \begin{cases} 24 & \text{if } k=4 \\ 24+24\left(C_5^{k+1}-1\right) & \text{if } k\geq 5 \end{cases}$	
$= 24\,C_5^{k+1}$	1A
	(7)

Solution	Marks
8. (a) $\det(M) = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$	
$= \begin{vmatrix} a+b+c & b & c \\ c+a+b & a & b \\ b+c+a & c & a \end{vmatrix}$	
$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$	1A
$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$	1A
$= \frac{1}{2}(a+b+c)[(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+c^2)]$	
$= \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$	1
$\because$ a, b and c are non-negative real numbers	
$\therefore$ $\det(M) \geq 0$	
$\because$ $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$	
$\geq a^2+b^2+c^2-ab-bc-ca$	
$\therefore$ $\det(M) \leq (a+b+c)^3$	1
	(4)
(b) The statement clearly holds for $n=1$.	
Assume the statement is true for $n=k$.	
$\because$ $M^{k+1} = M^k M$	
By expanding $M^k M$, we can obtain that M^{k+1} is of the required form and	
$\begin{cases} a_{k+1} = aa_k+cb_k+bc_k \\ b_{k+1} = ba_k+ab_k+cc_k \\ c_{k+1} = ca_k+bb_k+ac_k \end{cases}$.	1M + 1A
Obviously, $a_{k+1}, b_{k+1}, c_{k+1} \geq 0$.	1
Further, $a_{k+1}+b_{k+1}+c_{k+1} = (a+b+c)(a_k+b_k+c_k)$	
$= (a+b+c)(a+b+c)^k$	
$= (a+b+c)^{k+1}$	1
By the principle of mathematical induction, the statement is true for all positive integers n.	
	(4)

Solution	Marks
8. (c) (i) If $a+b+c=1$ and at least two of a, b, c are non-zero, by (a),	
$0 \le \det(M) = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$	
$= (a+b+c)^2-3(ab+bc+ca)$	1A
< 1 is a [illegible]	1A
$\therefore \lim\limits_{n\to\infty} \det(M^n) = \lim\limits_{n\to\infty} [\det(M)]^n = 0$	1
(ii) $\because \det(M^n) = \frac{1}{2}[(a_n-b_n)^2+(b_n-c_n)^2+(c_n-a_n)^2]$	
$\ge \frac{1}{2}(a_n-b_n)^2$	
≥ 0	1
and $\lim\limits_{n\to\infty} \det(M^n) = 0$	
$\therefore \lim\limits_{n\to\infty} \frac{1}{2}(a_n-b_n) = 0 \Rightarrow \lim\limits_{n\to\infty} (a_n-b_n) = 0$.	
Similarly, $\lim\limits_{n\to\infty} (a_n-c_n) = 0$.	1
(iii) $\because \lim\limits_{n\to\infty} (3a_n-1) = \lim\limits_{n\to\infty} [3a_n-(a_n+b_n+c_n)]$	1M
$= \lim\limits_{n\to\infty} [(a_n-b_n)+(a_n-c_n)]$	
$= 0$	
$\therefore \lim\limits_{n\to\infty} a_n = \frac{1}{3}$.	1
	(7)

Solution	Marks

9\. (a) (i) Suppose on the contrary that (II) has a solution (x_0, y_0), then $(x_0, y_0, 1)$ would be a soln. of (I) other than the trivial solution. — 1

(ii) (u, v) is a solution of (II)

$\Rightarrow$ $(u, v, 1)$ is a particular solution of (I) — 1

$\Rightarrow$ (ut, vt, t) are solutions of (I) $\forall\ t \in \mathbb{R}$ — 1

(iii) If (x_0, y_0, z_0) is a solution of (I) and $z_0 \neq 0$,

then $\left(\frac{x_0}{z_0}, \frac{y_0}{z_0}, 1\right)$ would be a solution of (II)

which contradicts the condition that (II) has no solution. — 1

Hence solutions of (I) must be in the form $(x_0, y_0, 0)$. — 1A

(5)

(b) (i) $$\Delta = \begin{vmatrix} -(3+k) & 1 & -1 \\ -7 & 5-k & -1 \\ -6 & 6 & k-2 \end{vmatrix} = (k+2)(k-2)(k-4)$$

(III) has non-trivial solutions when $\Delta = 0$ — 1M

i.e. when $k = -2, 2, 4$ — 1A

(ii) By (a)(i)/(ii), the possible values of k are $-2, 2, 4$. — 1M

(1) If $k = -2$, the augmented matrix of (IV) is $\left(\begin{array}{cc|c} -1 & 1 & 1 \\ -7 & 7 & 1 \\ -6 & 6 & 4 \end{array}\right)$.

which is inconsistent. — 1

(2) If $k = 2$, the augmented matrix of (IV) is $\left(\begin{array}{cc|c} -5 & 1 & 1 \\ -7 & 3 & 1 \\ -6 & 6 & 0 \end{array}\right)$.

It becomes $\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{array}\right)$ when reduced to echelon form.

$\therefore$ (IV) is consistent with solution $(-\frac{1}{4}, -\frac{1}{4})$. — 1A

Solution	Marks
(3) If $k=4$, the augmented matrix of (IV) is $\left(\begin{array}{cc\|c} -7 & 1 & 1 \\ -7 & 1 & 1 \\ -6 & 6 & -2 \end{array}\right)$.	
It becomes $\left(\begin{array}{cc\|c} 1 & 0 & -\frac{2}{9} \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 0 \end{array}\right)$ when reduced to echelon form.	
$\therefore$ (IV) is consistent with solution $(-\frac{2}{9}, -\frac{5}{9})$.	1A
(iii) (1) If $k=-2$, the augmented matrix of (III) is $\left(\begin{array}{ccc\|c} -1 & 1 & -1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & -4 & 0 \end{array}\right)$.	
It becomes $\left(\begin{array}{ccc\|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ when reduced to echelon form.	1
$\therefore$ S.S. $=\{(t,-t,0) : t\in\mathbb{R}\}$	1A
(2) If $k=2$, by the results of (a)(ii) and (b)(ii), S.S. $=\{(t,t,-4t) : t\in\mathbb{R}\}$	1A
(3) If $k=4$, by the results of (a)(ii) and (b)(ii), S.S. $=\{(2t,5t,-9t) : t\in\mathbb{R}\}$	1
	(10)

Solution	Marks
10. (a) $\underline{a}, \underline{b}$ and $\underline{c}$ are linearly dependent	
$\Leftrightarrow$ $\exists\, x, y, z \in \mathbb{R}$, not all zero, such that $x\underline{a} + y\underline{b} + z\underline{c} = \underline{0}$	2
$\Leftrightarrow$ The system of homogeneous equations $\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$ has non-trivial solution	2
$\Leftrightarrow$ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$	1
	(5)
(b) For any $\underline{x} = (k_1, k_2, k_3)$ in $\mathbb{R}^3$, consider the following system of linear equations: $\begin{cases} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{cases}$	1
$\because$ $\underline{a}, \underline{b}$ and $\underline{c}$ are linearly independent, by (a), $\Delta \neq 0$	1
$\Rightarrow$ the system of equations has a unique solution (x_1, x_2, x_3)	1
$\Rightarrow$ there are unique $x_1, x_2, x_3 \in \mathbb{R}$ such that $\underline{x} = x_1\underline{a} + x_2\underline{b} + x_3\underline{c}$.	1
	(4)
(c) (i) S represents a point when $\underline{a} = \underline{b} = \underline{c} = \underline{0}$.	1A
(ii) S represents a line if one of the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ is non-zero,	1A
and the other two are scalar multiples of it.	1A
(iii) S represents a plane when two of the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are linearly independent	1A
and the third vector is a linear combination of them.	1A
(iv) S represents the space when the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are linearly independent.	1A
	(6)

Solution	Marks
11. (a) (i) For any $z_1, z_2 \in \mathbb{C}$ with $f(z_1) = f(z_2)$, we have	
$f(z_1 - z_2) = f(z_1) - f(z_2) = 0$	
$\therefore$ $z_1 - z_2 = 0$ or $z_1 = z_2$	1
Hence f is injective.	
(ii) For any $z \in \mathbb{C}$, $z = x + yi$ for some $x, y \in \mathbb{R}$.	
$f(z) = f(1)x + f(i)y$	
$= f(1)x + if(1)y$	
$= (x+yi)f(1)$	
$= zf(1)$	1
$\because$ $f(i) = if(1) \neq 0$, $\therefore$ $f(1) \neq 0$.	
Hence $f(z) = 0 \Rightarrow z = 0$.	
By the result of (a)(i), f is injective.	1
For any $z \in \mathbb{C}$, $\dfrac{z}{f(1)} \in \mathbb{C}$ and	
$f\left(\dfrac{z}{f(1)}\right) = f(1)\dfrac{z}{f(1)} = z$.	1
$\therefore$ f is surjective.	
	(4)
(b) (i) For any $z_1, z_2 \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}$,	
$g(\alpha z_1 + \beta z_2) = \lambda(\alpha z_1 + \beta z_2) + \mu(\overline{\alpha z_1 + \beta z_2})$	1
$= \alpha(\lambda z_1 + \mu \overline{z_1}) + \beta(\lambda z_2 + \mu \overline{z_2})$	
$= \alpha g(z_1) + \beta g(z_2)$	1
$\therefore$ g is real linear.	
(ii) $[\Rightarrow]$:	
If g is not injective, $\exists$ $z \in \mathbb{C}$ such that	
$z \neq 0$ and $g(z) = 0$.	1
$\Rightarrow$ $\lambda z + \mu \overline{z} = 0$	
$\Rightarrow$ $\lvert\lambda\rvert \lvert z\rvert = \lvert\mu\rvert \lvert\overline{z}\rvert$	
$\Rightarrow$ $\lvert\lambda\rvert = \lvert\mu\rvert$.	1

Solution	Marks

(⇐):

If $|\lambda| = |\mu| = 0$, then $g(z) = 0$ which is not injective. 1

If $|\lambda| = |\mu| \neq 0$, then $\lambda = \gamma\mu$ for some $\gamma \in C$ with $|\gamma| = 1$.

Consider the equation $g(z) = 0$,

$$\lambda z + \mu\bar{z} = 0$$

$$\mu(\gamma z + \bar{z}) = 0$$

$$\gamma z + \bar{z} = 0 \qquad 1$$

Let $z = x+yi$ and $\gamma = a+bi$ where, $x, y, a, b \in \mathbb{R}$, then

$$(a+bi)(x+yi) + (x-yi) = 0 \qquad 1$$

$$[(a+1)x - by] + [bx + (a-1)y]i = 0$$

$$\Rightarrow (*) \begin{cases} (a+1)x - by = 0 \\ bx + (a-1)y = 0 \end{cases}$$

$$\because \begin{vmatrix} a+1 & -b \\ b & a-1 \end{vmatrix} = a^2 - 1 + b^2$$

$$= 0. \qquad (\because a^2 + b^2 = 1)$$

$\therefore$ The system (*) has nontrivial solution 1

$\Rightarrow$ g is not injective.

(8)

(c) Let $z = x+yi$ where $x, y \in \mathbb{R}$, then

$$f(z) = f(x+yi)$$

$$= xf(1) + yf(i) \qquad 1A$$

$$= \left(\frac{z+\bar{z}}{2}\right)f(1) - \left(\frac{z-\bar{z}}{2}i\right)f(i) \qquad 1A$$

$$= \frac{1}{2}[f(1) - if(i)]z + \frac{1}{2}[f(1) + if(i)]\bar{z}$$

i.e. $a = \frac{1}{2}[f(1) - if(i)]$, $b = \frac{1}{2}[f(1) + if(i)]$. 1A

(3)

Solution	Marks

12. (a) (i)

$$p(x) = (x-z_1)(x-\overline{z_1})(x-z_2)(x-\overline{z_2})$$ 1M

$$= [x^2-(z_1+\overline{z_1})x+z_1\overline{z_1}][x^2-(z_2+\overline{z_2})x+z_2\overline{z_2}]$$ 1A

$$= (x^2-2x\cos\theta_1+1)(x^2-2x\cos\theta_2+1).$$ 1

(ii) Taking logarithm and differentiate w.r.t. x on both sides, we have

$$\frac{p'(x)}{p(x)} = \frac{2(x-\cos\theta_1)}{x^2-2x\cos\theta_1+1} + \frac{2(x-\cos\theta_2)}{x^2-2x\cos\theta_2+1}$$ 1M

$$\therefore\ p'(x) = 2p(x)\left(\frac{x-\cos\theta_1}{x^2-2x\cos\theta_1+1} + \frac{x-\cos\theta_2}{x^2-2x\cos\theta_2+1}\right)$$ 1

(5)

(b)

$$\frac{p(x)}{x-w} = \frac{p(x)-p(w)}{x-w} \quad (\because p(w)=0)$$ 1M

$$= \frac{(x^4-w^4)+a_1(x^3-w^3)+a_2(x^2-w^2)+a_3(x-w)}{x-w}$$

$$= (x^3+x^2w+xw^2+w^3)+a_1(x^2+xw+w^2)+a_2(x+w)+a_3$$ 1A

$$= x^3+(w+a_1)x^2+(w^2+a_1w+a_2)x+(w^3+a_1w^2+a_2w+a_3)$$ 1

(3)

(c) For $r=1,2$,

$$2p(x)\frac{x-\cos\theta_r}{x^2-2x\cos\theta_r+1}$$

$$= \frac{p(x)}{x-z_r}+\frac{p(x)}{x-\overline{z_r}}$$ 1M

$$= 2x^3+[(z_r+\overline{z_r})+2a_1]x^2+[(z_r^2+\overline{z_r}^2)+a_1(z_r+\overline{z_r})+2a_2]x$$
$$+[(z_r^3+\overline{z_r}^3)+a_1(z_r^2+\overline{z_r}^2)+a_2(z_r+\overline{z_r})+2a_3] \quad \text{by (b)}$$ 1A

Hence by (a)(ii), we have

$$p'(x) = 4x^3+(s_1+4a_1)x^2+(s_2+a_1s_1+4a_2)x+(s_3+s_2a_1+s_1a_2+4a_3).$$ 1

On the other hand,

$$p'(x) = 4x^3+3a_1x^2+2a_2x+a_3.$$

By comparing coefficients,

$$\begin{cases} 3a_1=s_1+4a_1 \\ 2a_2=s_2+a_1s_1+4a_2 \\ a_3=s_3+s_2a_1+s_1a_2+4a_3 \end{cases}$$ 1M

$$\therefore \begin{cases} s_1+a_1=0 \\ s_2+a_1s_1+2a_2=0 \\ s_3+s_2a_1+s_1a_2+3a_3=0 \end{cases}$$ 1

When $n=4$,

$$s_4+a_1s_3+a_2s_2+a_3s_1+4a_4 = p(z_1)+p(\overline{z_1})+p(z_2)+p(\overline{z_2})$$ 1M

$$= 0.$$ 1

(7)

Solution	Marks
13. (a) $A(m+1,n+1)-A(m,n+1)$	
$=(1-x^{m+1})(1-x^{m+2})\ldots(1-x^{m+n+1})-(1-x^{m})(1-x^{m+1})\ldots(1-x^{m+n})$	
$=(1-x^{m+1})(1-x^{m+2})\ldots(1-x^{m+n})[(1-x^{m+n+1})-(1-x^{m})]$	1A
$=A(m+1,n)\,x^{m}(1-x^{n+1})$	
which is divisible by $(1-x^{n+1})A(m+1,n)$.	1
	(2)
(b) (i) $\because A(1,n)=(1-x)(1-x^2)\ldots(1-x^n)=B(n)$	
$\therefore A(1,n)$ is divisible by $B(n)$.	1
$\because A(m,1)=(1-x^m)=(1-x)(1+x+\ldots+x^{m-1})$ and $B(1)=(1-x)$	
$\therefore A(m,1)$ is divisible by $B(1)$.	1
Hence $P(1,n)$ and $P(m,1)$ are true.	
(ii) By (a), $A(m+1,n+1)-A(m,n+1)$ is divisible by $(1-x^{n+1})A(m+1,n)$	
$\therefore A(m+1,n+1)-A(m,n+1)=q(x)(1-x^{n+1})A(m+1,n)$ for some $q(x)\in P(x)$	1
[More precisely, $q(x)=x^m$.]	
$\because P(m,n+1)$ is true	
$\therefore A(m,n+1)$ is divisible by $B(n+1)$	1
$\because P(m+1,n)$ is true	
$\therefore A(m+1,n)$ is divisible by $B(n)$	1
– RHS is divisible by $B(n+1)$ ($\because B(n+1)=B(n)(1-x^{n+1})$.)	1
$\therefore A(m+1,n+1)$ is divisible by $B(n+1)$.	
Hence $P(m+1,n+1)$ is true.	1
(iii) By (b)(i), $P(1,k+1)$ is true.	1
If $P(j,k+1)$ is true for some +ve integer j,	
$\because P(j+1,k)$ is given to be true,	1
$\therefore$ by b(ii), $P(j+1,k+1)$ is also true.	1
By the principle of mathematical induction,	
$P(m,k+1)$ is true for all +ve integers m.	1
	(10)
(c) By (b)(i), $P(m,1)$ is true for all +ve integers m.	1
Assume $P(m,k)$ is true for all +ve integers m,	
$P(m,k+1)$ is also true for all +ve integers m by b(iii).	1
By the principle of mathematical induction,	
$P(m,n)$ is true for all +ve integers m and n.	1
	(3)

1. (a) $\lim_{x\to 1}\frac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{5}}} = \lim_{x\to 1}\frac{-\frac{1}{2}x^{-\frac{1}{2}}}{-\frac{1}{5}x^{-\frac{4}{5}}}$ 1A

$= \lim_{x\to 1}\frac{5}{2}x^{-\frac{3}{10}}$

$= \frac{5}{2}.$ 1A

no marks for the part if working is not shown

Alternatively

Let $t=x^{\frac{1}{10}}$, then $x=t^{10}$ and $t\to 1$ as $x\to 1$.

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{1-\sqrt[5]{x}} = \lim_{t\to 1}\frac{1-t^5}{1-t^2}$$

$$= \lim_{t\to 1}\frac{(1-t)(1+t+t^2+t^3+t^4)}{(1-t)(1+t)}$$ 1A

$$= \lim_{t\to 1}\frac{1+t+t^2+t^3+t^4}{1+t}$$

$$= \frac{5}{2}.$$ 1A

(b) $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{\cos x}-\frac{\sin x}{\cos x}\right)$

$= \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x}$ $\left(\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{1+\sin x}\right)$ 1A

$= \lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-\sin x}$

$= 0.$ 1A

no marks for this part if working is not shown

without $x\to\frac{\pi}{2}$ 2nd step is wrong, give 1A and 2

Alternatively

Let $t=\frac{\pi}{2}-x$, then $x=\frac{\pi}{2}-t$ and $t\to 0$ as $x\to\frac{\pi}{2}$.

$$\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim_{t\to 0}\left[\sec\left(\frac{\pi}{2}-t\right)-\tan\left(\frac{\pi}{2}-t\right)\right]$$

$$= \lim_{t\to 0}\left(\frac{1}{\sin t}-\cot t\right)$$

$$= \lim_{t\to 0}\frac{1-\cos t}{\sin t}$$ 1A

$$= \lim_{t\to 0}\frac{2\sin^2\frac{t}{2}}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= \lim_{t\to 0}\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}$$

$$= 0.$$ 1A

Let $t=\tan\frac{x}{2}$ $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x) = \lim_{t\to 1}\left(\frac{1+t^2}{1-t^2}-\frac{2t}{1-t^2}\right)$

$= \lim_{t\to 1}\frac{(1-t)^2}{1-t^2}$ 1A

$= \lim_{t\to 1}\frac{1-t}{1+t}$

$= 0$ 1A

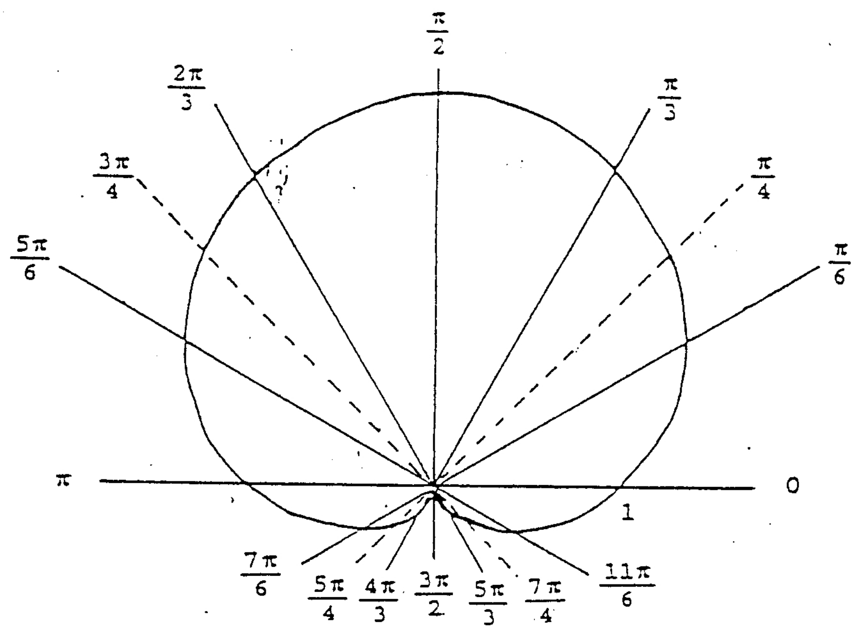
(4)

Solution	Marks
2. (a) $\int \tan^3 x\,dx = \int \tan x(\sec^2 x - 1)\,dx$	for $1 + \sec^2 x$ =
$= \int \tan x\, d(\tan x) - \int \dfrac{d(\cos x)}{\cos x}$	
$= \dfrac{1}{2}\tan^2 x + \ln\lvert\cos x\rvert + c$ 3A for correct any without steps.	1A + 1A accept: without c, without absolute
(b) $\displaystyle\int \frac{x^2-x+2}{x(x-2)^2}\,dx = \int\left[\frac{1}{2x} + \frac{1}{2(x-2)} + \frac{2}{(x-2)^2}\right]dx$	1M for $\frac{A}{x} + \frac{B}{x-2} + \ldots$ or $\frac{A}{x} + \frac{Bx+C}{(x-2)\ldots}$
$= \dfrac{1}{2}\ln\lvert x\rvert + \dfrac{1}{2}\ln\lvert x-2\rvert - \dfrac{2}{x-2} + c$	2A for all terms, 1A for any 2 terms, accept: without c
	(6)
3. Let the direction ratios of the line be (a:b:c), then	for the dot product
$\begin{cases} a+b+c=0 \\ a+2b=0 \end{cases}$	1M + 1A
$\therefore$ a:b:c = −2:1:1 (or 2:−1:−1 etc.)	1A
The equations of the line are	
$\dfrac{x-4}{-2} = \dfrac{y-2}{1} = \dfrac{z+3}{1}$.	1A
	(4)
D.R. of the line $= (\vec{i}+\vec{j}+\vec{k}) \times (\vec{i}+2\vec{j})$ 1M	
$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}$ 1A	
$= -2\vec{i} + \vec{j} + \vec{k}$ 1A.	

4. (a) $r = 1 + \sin\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
r	1	1.5	1.7	1.87	2	1.87	1.7	1.5

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
r	1	0.5	0.3	0.13	0	0.13	0.3	0.5



2

(b) Area $= \frac{1}{2}\int_0^{2\pi} r^2 d\theta \quad \left(\text{ or } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 d\theta \right)$

$= \frac{1}{2}\int_0^{2\pi} (1+\sin\theta)^2 d\theta$ 1M

$= \frac{1}{2}\int_0^{2\pi} \left(1+2\sin\theta+\frac{1-\cos2\theta}{2}\right) d\theta$ 1A

$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{\sin2\theta}{4}\right]_0^{2\pi}$ 1A

$= \frac{3\pi}{2}$

(5)

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Solution	Marks
5. $s_n = \sum_{k=1}^{n} 3^{k-1}\sin^3\left(\frac{\theta}{3^k}\right)$	
$= \sum_{k=1}^{n}\left[\frac{3^k}{4}\sin\left(\frac{\theta}{3^k}\right) - \frac{3^{k-1}}{4}\sin\left(\frac{\theta}{3^{k-1}}\right)\right]$	1M for a triple form
$= \frac{3^n}{4}\sin\left(\frac{\theta}{3^n}\right) - \frac{1}{4}\sin\theta$	1A
$\therefore \lim_{n\to\infty} s_n = \left[\lim_{n\to\infty}\frac{\theta}{4}\frac{\sin\left(\frac{\theta}{3^n}\right)}{\left(\frac{\theta}{3^n}\right)}\right] - \frac{1}{4}\sin\theta$	1M
$= \frac{\theta}{4} - \frac{1}{4}\sin\theta$ (or $\frac{1}{4}(\theta - \sin\theta)$)	1A
	(4)
6. If $x=0$, the equality clearly holds.	
If $x\neq 0$, let $y = xt$, then $dy = x\,dt$ and	1M for change varia
$\int_0^1 xf(xt)\,dt = \int_0^x f(y)\,dy = \int_0^x f(t)\,dt$	1
Let $F(x) = \int_0^x f(t)\,dt$, then $F'(x) = f(x)$.	1 for $\frac{d}{dx}\int_0^x f = f$
If $\int_0^1 f(xt)\,dt = 0$ for all $x\in\mathbb{R}$,	
then $F(x) = \int_0^1 xf(xt)\,dt = x\int_0^1 f(xt)\,dt = 0$	1 for $\int_0^x f$ prove $x\int_0^1 f$
$\therefore$ $f(x) = F'(x) = 0$ for all $x\in\mathbb{R}$.	1
	(5)

$\int_0^1 f(xt)\,dt = 0 \Rightarrow x\int_0^1 f(xt)\,dt = 0$

$\Rightarrow \int_0^x f(t)\,dt = 0.$ 1

$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = 0$) (on By Fundamental Theorem of Integral Calculus

$\Rightarrow f(x) = 0.$ 1 $\therefore f(x) = 0$) } 2

no reason no mark

Solution	Marks
7. (a) $\because f'(x) = \sin(\cos x) > 0 \quad \forall x \in [0, \frac{\pi}{2})$	1 (for $f'(x) = \cdots$ only)
$\therefore$ f is strictly increasing on $[0, \frac{\pi}{2})$.	1
$\therefore$ f is injective on $[0, \frac{\pi}{2})$	1
(b) $\because f(g(x))=x$	
$\therefore f'(g(x))g'(x)=1$	1A (for relation between $\frac{dy}{dx} \cdot \frac{dx}{dy}$)
$g'(x) = \dfrac{1}{f'(g(x))}$	
$\because f(1)=0, \therefore g(0)=1$	1A
Hence $g'(0) = \dfrac{1}{\sin(\cos(g(0)))} = \dfrac{1}{\sin(\cos 1)}$. $(=1.944)$	1A (-1 for correct ans without steps)
Alternatively	
$\because g(f(x)) = x$	
$\therefore g'(f(x))f'(x)=1$	1A
$\because f(1)=0$	
$\therefore g'(0)f'(1)=1$	1A
$g'(0) = \dfrac{1}{f'(1)} = \dfrac{1}{\sin(\cos 1)}$	1A
	(6)

(b) $y = \int_1^x \sin(\cos x)\,dt$

$\frac{dy}{dx} = \sin(\cos x)$

$\frac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ 1A

$= \dfrac{1}{\sin(\cos x)}$

$y=0 \Rightarrow x=1.$ 1M

$g'(0) = \dfrac{1}{\sin(\cos 1)}$ 1A

(a) Let $f(x_1) = f(x_2)$, where $x_1 > x_2$

$\Rightarrow \int_1^{x_1} \sin(\cos x)dx = \int_1^{x_2} \sin(\cos x)dx$ 1M.

$\Rightarrow \int_{x_2}^{x_1} \sin(\cos x)dx = 0$ 1A

$\because \sin(\cos x) > 0$ for $x \in [0, \frac{\pi}{2})$ 1

$\therefore x_1 = x_2$

Solution	Marks
8. (a) If $a<y$, by the mean value theorem,	1 for using the Mean Value Theorem
$\exists\ \xi\in(a,y)$ such that $\dfrac{e^y-e^a}{y-a}=e^{\xi}>e^a$	1 for without distinguishing $y>a$, $y<a$
$\because\ y-a>0\quad\therefore\ e^y-e^a>e^a(y-a)$	1 for Case $y>$ ... $y<$
If $a>y$, by the mean value theorem,	(accept without mentioning $y=$
$\exists\ \zeta\in(y,a)$ such that $\dfrac{e^y-e^a}{y-a}=e^{\zeta}<e^a$	
$\because\ y-a<0\quad\therefore\ e^y-e^a>e^a(y-a)$	
If $a=y$, the equality holds.	

Alternatively

$\forall\ a\in\mathbb{R}$, let $f(y)=e^y-e^a-e^a(y-a)$

$f'(y)=e^y-e^a$ — 1M

$\because\ e^x$ is increasing on $\mathbb{R}$

$\therefore\ f'(y)<0\quad\forall\ y<a$

$f'(y)>0\quad\forall\ y>a$ — 1

or $f'(y)=0\Rightarrow y=a$

$f''(y)=e^y$

$f''(a)=e^a>0.$

Hence $f(y)\ge f(a)=0\quad\forall\ y\in\mathbb{R}$

i.e. $e^y-e^a\ge e^a(y-a)$ — 1 for using $f(a)$ is min.

(b) Put $y=x^2$ and $a=\frac{1}{3}$ in (a), we have — 1A for $a=$

$$e^{x^2}-e^{\frac{1}{3}}\ge e^{\frac{1}{3}}\left(x^2-\frac{1}{3}\right)$$

$$\int_0^1 e^{x^2}dx-e^{\frac{1}{3}}\int_0^1 dx\ge e^{\frac{1}{3}}\left(\int_0^1 x^2dx-\frac{1}{3}\int_0^1 dx\right)$$ — 1M for putting y and $\int_0^1\cdots$ on inequality

$$\int_0^1 e^{x^2}dx-e^{\frac{1}{3}}\ge e^{\frac{1}{3}}\left(\left[\frac{x^3}{3}\right]_0^1-\frac{1}{3}\right)$$

$$\int_0^1 e^{x^2}dx\ge e^{\frac{1}{3}}.$$ — 1

(6)

(4)

1A

1A

=0

Solution	Marks
(a) Sub. $y=mx+c$ into $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have	
$(a^2m^2+b^2)x^2+2a^2cmx+a^2(c^2-b^2)=0$.	1A
If $y=mx+c$ is a tangent to (E), then $\Delta=0$.	1M
i.e. $4a^4c^2m^2-4a^2(c^2-b^2)(a^2m^2+b^2)=0$	
$c^2-b^2-a^2m^2=0$	1A
$\because P(h,k)$ lies on $y=mx+c$	
$\therefore c=k-mh$.	
$(k-mh)^2-b^2-a^2m^2=0$	
$(h^2-a^2)m^2-2hkm+k^2-b^2=0$.	1
	(4)
(b) (i) Let $A=(x_1,y_1)$ and $B=(x_2,y_2)$.	
(1) the tangent at A, T_1, has equation	
$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$;	
(2) the tangent at B, T_2, has equation	
$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$;	1
$\because P(h,k)$ lies on both T_1 and T_2 ,	
$\therefore \frac{hx_1}{a^2}+\frac{ky_1}{b^2}=1$ and $\frac{hx_2}{a^2}+\frac{ky_2}{b^2}=1$.	1
Thus the equation of the line AB is	
$\frac{hx}{a^2}+\frac{ky}{b^2}=1$.	1A
(ii) Solving $\begin{cases}\frac{hx}{a^2}+\frac{ky}{b^2}=1\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\end{cases}$, we have	
$(a^2k^2+b^2h^2)x^2-2a^2b^2hx+a^4(b^2-k^2)=0$.	1A
$\therefore$ the x-coordinate of the mid-pt. of AB	
$=\frac{1}{2}$ (sum of roots)	
$=\frac{ha^2b^2}{a^2k^2+b^2h^2}$.	1A
Similarly, the y-coordinate of the mid-pt. of AB	
$=\frac{ka^2b^2}{a^2k^2+b^2h^2}$.	1A
	(6)

Solution	Marks
9. (c) (1) If one of the tangents from P to (E) is vertical,	1
then $h=\pm a$ or $h^2-a^2=0$.	
The tangents are perpendicular	
iff $P=(\pm a,\pm b)$	1
iff P lies on $x^2+y^2=a^2+b^2$.	
(2) If the tangents from P to (E) can be written as $y=mx+c$,	1
then $h^2-a^2\neq 0$.	
By (a), the slopes of the two tangents are the roots of	1
the equation $(h^2-a^2)m^2-2hkm+k^2-b^2=0$.	
The tangents are perpendicular	
iff product of roots $=-1$	
iff $\dfrac{k^2-b^2}{h^2-a^2}=-1$	
iff $h^2+k^2=a^2+b^2$	1
	(5)

Solution	Marks

0\. (a) (i) $f'(x) = \dfrac{2(1-2x^2)}{3\sqrt[3]{x}(x^2+1)^2}$ $(x\neq 0)$ — 1A

$\because$ $\dfrac{f(0+h)-f(0)}{h} = \dfrac{1}{\sqrt[3]{h}(h^2+1)} \to \pm\infty$ as $h\to\pm 0$

$\therefore$ $f'(0)$ does not exist. — 1

(ii) $f'(x)=0$ when $x=\pm\dfrac{1}{\sqrt{2}}$ — 1A

x	$(-\infty,-\frac{1}{\sqrt{2}})$	$-\frac{1}{\sqrt{2}}$	$(-\frac{1}{\sqrt{2}},0)$	0	$(0,\frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}},\infty)$
f'	+	0	−	$\nexists$	+	0	−
f	↗	R.Max.	↘	R.Min.	↗	R.Max.	↘

f is increasing on $(-\infty,-\frac{1}{\sqrt{2}}] \cup [0,\frac{1}{\sqrt{2}}]$. — 1A

f is decreasing on $[-\frac{1}{\sqrt{2}},0] \cup [\frac{1}{\sqrt{2}},\infty)$. — 1A

(iii) From (a) (ii), (0, 0) is the relative minimum pt. — 1A

$(-\frac{1}{\sqrt{2}},\frac{\sqrt[3]{2^2}}{3})$ and $(\frac{1}{\sqrt{2}},\frac{\sqrt[3]{2^2}}{3})$ are the relative maximum pts. — 1A + 1A

(8)

(b) (i) $f''(x) = \dfrac{2(14x^4-23x^2-1)}{9x\sqrt[3]{x}(x^2+1)^3}$ — 1A

$$= \frac{2\left[x^2-\frac{23+\sqrt{585}}{28}\right]\left[x^2-\frac{23-\sqrt{585}}{28}\right]}{9x\sqrt[3]{x}(x^2+1)^3}$$

$f''(x)=0$ when $x=\pm x_0$, where $x_0=\sqrt{\dfrac{23+\sqrt{585}}{28}}$ (≈ 1.298)

x	$(-\infty,-x_0)$	$-x_0$	$(-x_0,0)\cup(0,x_0)$	x_0	(x_0,∞)
f''	+	0	−	0	+

$\therefore$ $(-x_0,f(-x_0))$ and $(x_0,f(x_0))$ — 1M + 1A
[approx. (−1.298, 0.443) and (1.298, 0.443)]
are the points of inflexion.

(ii) $\because$ $f(x)\to 0$ as $x\to\pm\infty$
$\therefore$ The x-axis ($y=0$) is an asymptote. — 1
There is no vertical asymptotes.

(4)

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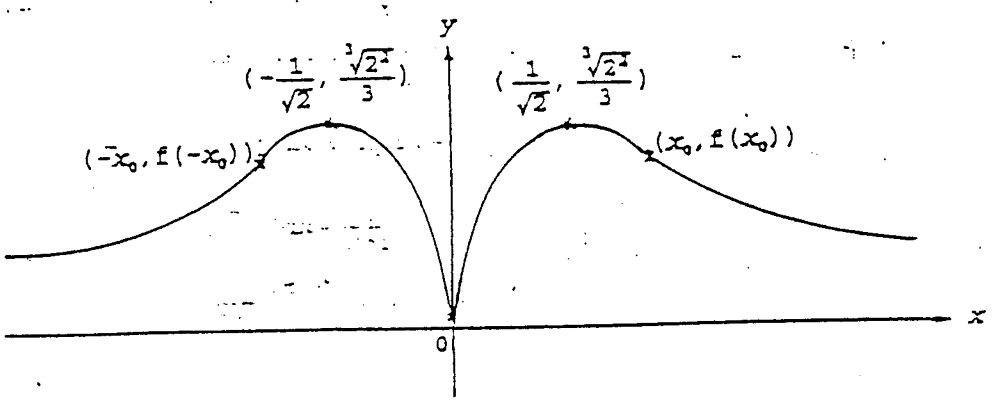
Solution	Marks
10. (c)	3



(3)

Solution	Marks
11. (a) (i) $\because \quad x \le \tan x \le \frac{4x}{\pi} \quad$ for $\quad x \in [0, \frac{\pi}{4}]$	
$\therefore \quad x^n \le \tan^n x \le \left(\frac{4}{\pi}\right)^n x^n \quad$ for $\quad x \in [0, \frac{\pi}{4}],\ n \in N$	1
$\int_0^{\frac{\pi}{4}} x^n dx \le I_n \le \left(\frac{4}{\pi}\right)^n \int_0^{\frac{\pi}{4}} x^n dx$	1
$\because \quad \int_0^{\frac{\pi}{4}} x^n dx = \frac{1}{n+1}[x^{n+1}]_0^{\frac{\pi}{4}} = \frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1}$	
$\therefore \quad \frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} \le I_n \le \frac{1}{n+1}\left(\frac{\pi}{4}\right)$.	1
(ii) $\because \quad I_n \ge 0$ and $\lim_{n\to\infty} \frac{1}{n+1}\left(\frac{\pi}{4}\right) = 0$,	1
by (a)(i) and the sandwich rule,	
$\lim_{n\to\infty} I_n = 0$.	1A
(iii) $I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2} x)\, dx$	
$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\tan^2 x + 1)\, dx$	
$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\, dx$	1M
$= \frac{1}{n-1}[\tan^{n-1} x]_0^{\frac{\pi}{4}}$	1A
$= \frac{1}{n-1} \qquad$ for $n = 2, 3, 4, \ldots$	1
	(8)
(b) (i) $a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$	
$= \sum_{k=1}^{n} (-1)^{k+1} (I_{2k} + I_{2k-2})$	1M
$= \sum_{k=1}^{n} (-1)^{k+1} I_{2k} + \sum_{k=0}^{n-1} (-1)^{k+2} I_{2k}$	1A
$= \sum_{k=1}^{n} (-1)^{k+1} I_{2k} - \sum_{k=0}^{n-1} (-1)^{k+1} I_{2k}$	1A
$= (-1)^{n+1} I_{2n} + I_0$	1A
$= (-1)^{n+1} I_{2n} + \frac{\pi}{4}$	1A
(ii) By (a) (ii), $\lim_{n\to\infty} I_{2n} = 0$.	1M
$\therefore \quad \lim_{n\to\infty} a_n = \frac{\pi}{4}$	1A
	(7)

Solution	Marks
12. (a) (i) $f'(x+1) = \lim_{h\to 0} \frac{f(x+1+h)-f(x+1)}{h}$	1
$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ by cond. B	
$= f'(x) \quad \forall x \in \mathbb{R}$.	1
Alternatively	
$f'(x) = \frac{d}{dx}f(x)$	
$= \frac{d}{dx}f(x+1)$	1
$= f'(x+1)\frac{d}{dx}(x+1)$	
$= f'(x+1)$	1
	(2)
(ii) $\because g(x) = \frac{f'(x)}{f(x)} \quad \forall x \in \mathbb{R}$.	1
$\therefore g(x+1) = \frac{f'(x+1)}{f(x+1)}$	
$= \frac{f'(x)}{f(x)}$ by (a) and cond. B	
$= g(x) \quad \forall x \in \mathbb{R}$.	1
(iii) Using cond. C and differentiate w.r.t. x on both sides,	
$\frac{1}{4}f'(\frac{x}{4})f(\frac{x+1}{4}) + \frac{1}{4}f(\frac{x}{4})f'(\frac{x+1}{4}) = f'(x)$	1
$\frac{1}{4}\left[\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} + \frac{f'(\frac{x+1}{4})}{f(\frac{x+1}{4})}\right] = \frac{f'(x)}{f(\frac{x}{4})f(\frac{x+1}{4})}$	1
$= \frac{f'(x)}{f(x)}$	1
$\frac{1}{4}[g(\frac{x}{4}) + g(\frac{x+1}{4})] = g(x) \quad \forall x \in \mathbb{R}$.	1
	(6)

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Solution	Marks
12. (b) (i) $\because g(x)=g(x+1) \quad \forall x\in\mathbb{R}$,	
$\therefore |g(x)|\le M \quad \forall x\in[0,1] \Rightarrow |g(x)|\le M \quad \forall x\in\mathbb{R}$	1
By (a)(iii), $|g(x)| \le \frac{1}{4}\left[\left|g(\frac{x}{4})\right| + \left|g(\frac{x+1}{4})\right|\right]$	
$\le \frac{1}{4}(M+M)$	
$= \frac{M}{2} \quad \forall x\in\mathbb{R}$	1
Similarly, if $|g(x)| \le \frac{M}{2^k} \quad \forall x\in\mathbb{R}$	
then $|g(x)| \le \frac{M}{2^{k+1}} \quad \forall x\in\mathbb{R}$	1
Inductively, we have	
$|g(x)| \le \frac{M}{2^n} \quad \forall n\in\mathbb{N},\ x\in\mathbb{R}$.	1
$\because \lim_{n\to\infty} \frac{M}{2^n} = 0$	
$\therefore g(x) = 0 \quad \forall x\in\mathbb{R}$	1
(ii) By (b)(i),	
$\ln f(x) = c$ for some constant c	
$\Rightarrow f(x) = e^c$ which is also a constant.	1
By cond. C, $(f(x))^2=f(x) \quad \forall x\in\mathbb{R}$	
$\Rightarrow f(x)=1 \quad \forall x\in\mathbb{R}$	1
	(7)

Solution	Marks
13. (a) For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $0 \le \cos x \le 1$,	
$\therefore$ $0 \le \cos^n x \le 1$.	1
Hence $L_n \le \left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\right\}^{\frac{1}{n}}$	1
$\le \pi^{\frac{1}{n}}$.	1
	(3)
(b) From the graph of $\cos x$,	
$\because$ $\cos x$ increases on $[-\frac{\pi}{2}, 0]$ and decreases on $[0, \frac{\pi}{2}]$	1
$\therefore$ $\cos x \ge r_n$ iff. $\cos x \ge \cos \frac{1}{2n}$	
iff $-\frac{1}{2n} \le x \le \frac{1}{2n}$	1
Thus	
$L_n \ge \left\{\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \cos^n x\, dx\right\}^{\frac{1}{n}}$	1
$\ge \left\{\int_{-\frac{1}{2n}}^{\frac{1}{2n}} (r_n)^n dx\right\}^{\frac{1}{n}}$	1
$= r_n \left(\frac{1}{n}\right)^{\frac{1}{n}}$	1
	(5)
(c) (i) Let $y = x^{\frac{1}{x}}$, then $\ln y = \frac{\ln x}{x}$.	1
$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0$	1
$\therefore$ $y \to 1$ as $x \to \infty$ $\Rightarrow$ $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$	1
(ii) From (a) and (b), $r_n \left(\frac{1}{n}\right)^{\frac{1}{n}} \le L_n \le \pi^{\frac{1}{n}}$.	1
$\because$ (1) $\lim_{n \to \infty} r_n = \lim_{n \to \infty} \cos \frac{1}{2n} = 1$	
$\lim_{n \to \infty} r_n \left(\frac{1}{n}\right)^{\frac{1}{n}} = \left(\lim_{n \to \infty} r_n\right)\left(\lim_{n \to \infty} \frac{1}{n^{\frac{1}{n}}}\right) = 1$	1
and (2) $\lim_{n \to \infty} \pi^{\frac{1}{n}} = 1$ ($\because 1 \le \pi^{\frac{1}{n}} \le n^{\frac{1}{n}}$ for $n \ge 3$)	1
$\therefore$ by the sandwich rule, $\lim_{n \to \infty} L_n = 1$.	1
	(7)

Solution	Marks
14. (a) (i) Using the Fundamental Theorem of Calculus, we have	
$g'(t) = b - f(t)$.	1A
$g'(t) = 0$ when $t = f^{-1}(b)$.	
$\because$ $f(x)$ is strictly increasing on $[0, c]$,	
$\therefore$ (1) $g'(t) > 0$ when $0 < t < f^{-1}(b)$.	
i.e. $g(t)$ is increasing on $[0, f^{-1}(b)]$.	1
(2) $g'(t) < 0$ when $f^{-1}(b) < t < c$.	
i.e. $g(t)$ is decreasing on $[f^{-1}(b), c]$.	1
Hence $g(t)$ is maximum when $t = f^{-1}(b)$,	
$\Rightarrow$ $g(t) \le g(f^{-1}(b))$ $\forall\ t \in [0, c]$.	1
(ii) $\int_0^b f^{-1}(y)\,dy = \int_0^{f^{-1}(b)} x\,df(x)$	1A
$= [xf(x)]_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} f(x)\,dx$	1A
$= [f^{-1}(b)]\,b - \int_0^{f^{-1}(b)} f(x)\,dx$	1A
$= g(f^{-1}(b))$	1
(iii) From (a)(i) and (ii), we have	
$g(a) \le \int_0^b f^{-1}(y)\,dy$	1M
$\Rightarrow$ $ab - \int_0^a f(x)\,dx \le \int_0^b f^{-1}(y)\,dy$	
$\Rightarrow$ $\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \ge ab$	1
	(10)
(b) $\because$ $\frac{1}{p} + \frac{1}{q} = 1$, either $p = q = 2$ or one of p, q must be greater than 2.	1
w.l.g. let $p \ge 2$.	
Let $f(x) = x^{p-1}$,	1M
then $f^{-1}(x) = x^{\frac{1}{p-1}}$.	
By (a)(iii), we have	
$\int_0^a x^{p-1}dx + \int_0^b x^{\frac{1}{p-1}}\,dx \ge ab$	1A
$\Rightarrow$ $\frac{1}{p}a^p + \frac{p-1}{p}b^{\frac{p}{p-1}} \ge ab$	1A
$\therefore$ $\frac{1}{p}a^p + \frac{1}{q}b^q \ge ab$ ..	1
	(5)