Solutions	Marking Scheme	Marks
1. $(a_i + tb_i)^2 \geq 0 \quad \forall t \in R, \ \forall i = 1, \ldots, n$		1M
$\Rightarrow \sum_{i=1}^{n} (a_i + tb_i)^2 \geq 0 \quad \forall t \in R$		
$\Rightarrow \sum_{i=1}^{n} (a_i^2 + 2ta_ib_i + t^2b_i^2) \geq 0 \quad \forall t \in R$		
$\Rightarrow \left(\sum_{i=1}^{n} a_i^2\right) + \left(2\sum_{i=1}^{n} a_ib_i\right)t + \left(\sum_{i=1}^{n} b_i^2\right)t^2 \geq 0 \quad \forall t \in R$		1A
$\Rightarrow 0 \geq \Delta = \left(2\sum_{i=1}^{n} a_ib_i\right)^2 - 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$		1M
$\Rightarrow 4\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq 4\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$		
$\Rightarrow \left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$		
Put $b_i = 1 \quad \forall i = 1, \ldots, n$		1M
we have		
$\left(\sum_{i=1}^{n} a_i \cdot 1\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} 1^2\right)$		
$\Rightarrow \left(\sum_{i=1}^{n} a_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot n$		1A
$\Rightarrow \sum_{i=1}^{n} a_i \leq \sqrt{n\sum_{i=1}^{n} a_i^2}$		
$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} a_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} a_i^2}$		

2. For $n = 1$,

$\text{LHS} = u_1$

$= 1$

$\text{RHS} = \alpha^1 + \beta^1$

$= \alpha + \beta$

$= 1$

$\therefore \quad \text{LHS} = \text{RHS}$ — 1

For $n = 2$,

$\text{LHS} = u_2$

$= 3$

$\text{RHS} = \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= 1^2 - 2(-1)$

$= 3$

$\therefore \quad \text{LHS} = \text{RHS}$ — 1

Assume $u_k = \alpha^k + \beta^k$ and $u_{k-1} = \alpha^{k-1} + \beta^{k-1}$ for some $k \geq 2$. — 1M

Then for $n = k + 1$,

$\text{LHS} = u_{k+1}$

$= u_k + u_{k-1}$

$= \alpha^k + \beta^k + \alpha^{k-1} + \beta^{k-1}$

$= \alpha^k + \alpha^{k-1} + \beta^k + \beta^{k-1}$

$= \alpha^{k-1}(\alpha + 1) + \beta^{k-1}(\beta + 1)$ — 1

$= \alpha^{k-1}\alpha^2 + \beta^{k-1}\beta^2$ — 1

$= \alpha^{k+1} + \beta^{k+1}$

$= \text{RHS}$

By the principle of mathematical induction, $u_n = \alpha^n + \beta^n \ \forall n \geq 1$.

[redacted]

Solutions	Marks

3. (a) Let $x = \alpha$, $y = \beta$, $z = \gamma$ be a solution.

Then $\begin{cases} a\alpha + b\beta + c\gamma = 1 \\ b\alpha + c\beta + a\gamma = 1 \\ c\alpha + a\beta + b\gamma = 1 \\ \alpha + \beta + \gamma = 3 \end{cases}$

$\Rightarrow \begin{cases} (a+b+c)(\alpha+\beta+\gamma) = 3 \text{ and} \\ (\alpha+\beta+\gamma) = 3 \end{cases}$ 1M

$\Rightarrow (a+b+c)\cdot 3 = 3$

$\Rightarrow a+b+c = 1$

(b) (*) is equivalent to

$(*)' \begin{cases} ax + by + cz = 1 \\ bx + cy + az = 1 \\ cx + ay + bz = 1 \end{cases}$

Consider $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ 1M

$= \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix} \quad (\because \; a+b+c = 1)$

$= \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$

$= ac + ab + bc - a^2 - b^2 - c^2$

$= -\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$ 1A

So (*) has a unique solution if and only if a, b, c are not all equal. 1M

(c) We have $\Delta = 0$, and $a = b = c \; (= \frac{1}{3})$. 1A

Thus (*) becomes $x + y + z = 3$.

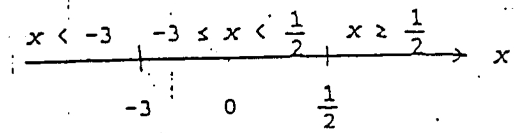
$\therefore$ general solution $= \{(s, t, 3-s-t) : s, t \in R\}$ 1A

	Marks
4. (a) $\lvert 1+z\rvert = \lvert 2-z\rvert$	
$(1+z)\overline{(1+z)} = (2-z)\overline{(2-z)}$	
$1 + z + \overline{z} + z\overline{z} = 4 - 2z - 2\overline{z} + z\overline{z}$	
$3(z+\overline{z}) = 3$	
$z + \overline{z} = 1$	
$\text{Re}(z) = \frac{1}{2}$	1A+1M
(b) Let $z = \frac{1}{2} + it$.	1M
Substitute it into the 1st equation.	
$\lvert \frac{1}{2} + it\rvert^2 - (\frac{1}{2} + it) - \overline{(\frac{1}{2} + it)} + i\left[(\frac{1}{2} + it) - \overline{(\frac{1}{2} + it)}\right] = \frac{1}{2}$	
$(\frac{1}{2} + it)(\frac{1}{2} - it) - (\frac{1}{2} + it) - (\frac{1}{2} - it) + i[(\frac{1}{2} + it) - (\frac{1}{2} - it)] = \frac{1}{2}$	
$(\frac{1}{2} + it)(\frac{1}{2} - it) - 1 + i(2it) = \frac{1}{2}$	
$\frac{1}{4} - (it)^2 - 1 - 2t = \frac{1}{2}$	
$\frac{1}{4} + t^2 - 1 - 2t = \frac{1}{2}$	
$1 + 4t^2 - 4 - 8t = 2$	
$4t^2 - 8t - 5 = 0$	1A
$(2t-5)(2t+1) = 0$	
$t = \frac{5}{2}$ or $-\frac{1}{2}$	
$\therefore\ z = \frac{1}{2} + \frac{5}{2}i$ or $\frac{1}{2} - \frac{1}{2}i$	1A

Solutions	Marks
5. Let $\dfrac{x+4}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$.	1M
Then $x + 4 = A(x+2) + B(x+1)$	
$\Rightarrow$ $x + 4 = (A+B)x + (2A+B)$	1M
$\therefore \begin{cases} A + B = 1 \\ 2A + B = 4 \end{cases}$	
Solving for A , B , we have $A = 3$, $B = -2$.	
$\therefore \dfrac{x+4}{x^2+3x+2} = \dfrac{3}{x+1} - \dfrac{2}{x+2}$	1A
$\sum_{k=2}^{N} \left\{ \dfrac{1}{k-1} - \dfrac{k+4}{k^2+3k+2} \right\}$	
$= \sum_{k=2}^{N} \left(\dfrac{1}{k-1} - \dfrac{3}{k+1} + \dfrac{2}{k+2} \right)$	
$= \sum_{k=2}^{N} \dfrac{1}{k-1} - 3\sum_{k=2}^{N} \dfrac{1}{k+1} + 2\sum_{k=2}^{N} \dfrac{1}{k+2}$	
$= \sum_{k=1}^{N-1} \dfrac{1}{k} - 3\sum_{k=3}^{N+1} \dfrac{1}{k} + 2\sum_{k=4}^{N+2} \dfrac{1}{k}$	1M
$= \left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3}\right) - 3\left(\dfrac{1}{3}\right) - \dfrac{3}{N} - \dfrac{3}{N+1} + \dfrac{2}{N} + \dfrac{2}{N+1} + \dfrac{2}{N+2}$	1A
$\rightarrow \dfrac{5}{6}$ as $N \rightarrow \infty$	1A

Solutions	Marks
6. (a) $\because \det A = \det(A^t)$	
$= \det(-A)$	1M
$= (-1)^3 \det A$	1M
$= -\det A$	
$\therefore \det A = 0$	
(b) $(I-B)^t = \begin{pmatrix} 0 & 2 & -74 \\ -2 & 0 & 67 \\ 74 & -67 & 0 \end{pmatrix}^t$	
$= \begin{pmatrix} 0 & -2 & 74 \\ 2 & 0 & -67 \\ -74 & 67 & 0 \end{pmatrix}$	1M
$= -(I-B)$	
by (a), $\det(I-B) = 0$	1M
<u>Alternatively,</u>	
$\det(I-B) = \det\begin{pmatrix} 0 & 2 & -74 \\ -2 & 0 & 67 \\ 74 & -67 & 0 \end{pmatrix}$	1A
$= -2\begin{vmatrix} -2 & 67 \\ 74 & 0 \end{vmatrix} - 74\begin{vmatrix} -2 & 0 \\ 74 & -67 \end{vmatrix}$	1M
$= -2(74)(67) + 74(2)(67)$	
$= 0$	
Now $(I-B)(I+B+B^2+B^3) = I+B+B^2+B^3-B-B^2-B^3-B^4$	
$= I - B^4$	1M
therefore $\det(I-B^4) = \det((I-B)(I+B+B^2+B^3))$	1M
$= \det(I-B)\det(I+B+B^2+B^3)$	1M
$= 0$	
<u>Alternatively,</u>	
$I - B^4 = (I-B^2)(I+B^2) = (I-B)(I+B)(I+B^2)$	1M
$\therefore \det(I-B^4) = \det((I-B)(I+B)(I+B^2))$	1M
$= \det(I-B)\det((I+B)(I+B^2))$	1M
$= 0$	

Solutions	Marks
7. Substitute y into the 1st inequality.	
$\lvert 2x-1\rvert > \lvert x+3\rvert + 1$	1A
Divide the real line into 3 regions as follows:	1M



When $x < -3$,

$$\begin{aligned} &\lvert 2x-1\rvert > \lvert x+3\rvert + 1 \\ \Rightarrow\ & -2x+1 > -x-3+1 \\ \Rightarrow\ & x < 3 \end{aligned}$$

$\therefore \begin{cases} x = t \\ y = -t-3 \end{cases}$ where $t < -3$ — 1A

When $-3 \le x < \frac{1}{2}$

$$\begin{aligned} &\lvert 2x-1\rvert > \lvert x+3\rvert + 1 \\ \Rightarrow\ & -2x+1 > x+3+1 \\ \Rightarrow\ & -3 > 3x \\ \Rightarrow\ & x < -1 \end{aligned}$$

$\therefore \begin{cases} x = t \\ y = t+3 \end{cases}$ where $-3 \le t < -1$ — 1A

When $x \ge \frac{1}{2}$

$$\begin{aligned} &\lvert 2x-1\rvert > \lvert x+3\rvert + 1 \\ \Rightarrow\ & 2x-1 > x+3+1 \\ \Rightarrow\ & x > 5 \end{aligned}$$

$\therefore \begin{cases} x = t \\ y = t+3 \end{cases}$ where $t > 5$ — 1A

$\therefore$ answer is $\begin{cases} x = t \\ y = t+3 \end{cases}$ where $t > 5$

or $\begin{cases} x = t \\ y = t+3 \end{cases}$ where $-3 \le t < -1$

or $\begin{cases} x = t \\ y = -t-3 \end{cases}$ where $t < -3$. — 1A

	Marks
8. (a) If $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = 0$, then the following system of linear equations has non-zero solution:	1M
$(*) \begin{cases} u_1 x + v_1 y + w_1 z = 0 \\ u_2 x + v_2 y + w_2 z = 0 \\ u_3 x + v_3 y + w_3 z = 0 \end{cases}$	
$\Rightarrow$ $\exists \alpha, \beta, \gamma$, not all zero, such that $\begin{cases} u_1 \alpha + v_1 \beta + w_1 \gamma = 0 \\ u_2 \alpha + v_2 \beta + w_2 \gamma = 0 \\ u_3 \alpha + v_3 \beta + w_3 \gamma = 0 \end{cases}$	1M
$\Rightarrow$ $\alpha(u_1\ u_2\ u_3) + \beta(v_1\ v_2\ v_3) + \gamma(w_1\ w_2\ w_3) = 0$	
$\Rightarrow$ u, v, w are linearly dependent	1M
$\Rightarrow$ a contradiction	1M
	4
(b) Let $\mathbf{s} = (s_1, s_2, s_3)$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$.	
Then $\begin{cases} u_1 s_1 + u_2 s_2 + u_3 s_3 = 0 \\ v_1 s_1 + v_2 s_2 + v_3 s_3 = 0 \\ w_1 s_1 + w_2 s_2 + w_3 s_3 = 0 \end{cases}$	1
by (a), $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \neq 0$	1M
$\Rightarrow$ unique solution exists for (*).	
$\Rightarrow$ $\mathbf{s} = 0$	1
	3
(c) $u \times (v \times w) = 0$	
$\Rightarrow$ $u = \mu(v \times w)$ for some $\mu \in R$	1
$\Rightarrow$ $u \cdot v = \mu(v \times w)v = \mu 0 = 0$	1
$u \cdot w = \mu(v \times w)w = \mu 0 = 0$	1
Similarly, $(u \times v) \times w = 0$	
$\Rightarrow$ $w = \lambda(u \times v)$ for some $\lambda \in R$	
$\Rightarrow$ $w \cdot v = \lambda((u \times v) \cdot v) = \lambda 0 = 0$	1
	4

Solutions	Marks
(d) Let $\mathbf{r} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ for some $\alpha, \beta, \gamma \in R$	1M
then $\mathbf{r}\cdot\mathbf{u} = (\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w})\cdot\mathbf{u}$	1
$= \alpha\mathbf{u}\cdot\mathbf{u} + \beta\mathbf{v}\cdot\mathbf{u} + \gamma\mathbf{w}\cdot\mathbf{u}$	
$= \alpha\mathbf{u}\cdot\mathbf{u} + 0 + 0$	
$\Rightarrow \alpha = \dfrac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} \quad (\because \mathbf{u}\cdot\mathbf{u} \neq 0)$	1
Similarly, we can show that	
$\beta = \dfrac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$	1
$\gamma = \dfrac{\mathbf{r}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$	
hence $\mathbf{r} = \dfrac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \dfrac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \dfrac{\mathbf{r}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$	
	4
Alternatively, consider	
$\mathbf{s} = \mathbf{r} - \left(\dfrac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \dfrac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \dfrac{\mathbf{r}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}\right)$	1M
Since $\mathbf{s}\cdot\mathbf{u} = \mathbf{r}\cdot\mathbf{u} - \dfrac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u} = 0$	1
$\mathbf{s}\cdot\mathbf{v} = \mathbf{r}\cdot\mathbf{v} - \dfrac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\cdot\mathbf{v} = 0$	1
$\mathbf{s}\cdot\mathbf{w} = \mathbf{r}\cdot\mathbf{w} - \dfrac{\mathbf{r}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}\cdot\mathbf{w} = 0$	
by (b), $\mathbf{s} = 0$	1
$\Rightarrow \mathbf{r} = \dfrac{\mathbf{r}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} + \dfrac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \dfrac{\mathbf{r}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$	
	4

Solutions	Marks
9. (a) (i) $f(0) = f(0+0)$	
$= f(0) + f(0)$	
$\Rightarrow f(0) = 0$	1
(ii) $f(-x) + f(x) = f(-x+x)$	
$= f(0)$	
$= 0$	1
$\Rightarrow f(-x) = -f(x)$	
(iii) By (i), we need only to show $f(nx) = nf(x)$ for $n = \pm 1, \pm 2, \ldots$.	
Case 1: $n > 0$	
We shall use mathematical induction to show that $f(nx) = nf(x)$.	
For $n = 1$,	
$f(1 \cdot x) = f(x)$	
$= 1 \cdot f(x)$	1
Assume $f(kx) = kf(x)$.	
Then $f((k+1)x) = f(kx + x)$	
$= f(kx) + f(x)$	
$= kf(x) + f(x)$	1
$= (k+1)f(x)$	
Case 2: $n < 0$	
$f(nx) = f((-n)(-x))$	
$= (-n)f(-x)$ (by Case 1)	1
$= -n(-f(x))$ (by (ii))	
$= nf(x)$	
	5
(b) If $\exists x_0 \in R$ such that $f(x_0) \neq 0$, then $f(x_0) > 0$ or $f(x_0) < 0$.	
Case 1: $f(x_0) > 0$	
Then we can choose a positive integer n such that	1M
$nf(x_0) > K$	
$\Rightarrow f(nx_0) > K$	
$\Rightarrow$ contradicting the fact that $f(x) < K$ for all $x \in R$.	1
Case 2: $f(x_0) < 0$	
Replace x_0 by $-x_0$ and use the same arguments in Case 1.	1
	3

Solutions	Marks
(c) (i) $g(x+y) = f(x+y) - f(1)(x+y)$	
$= f(x) + f(y) - f(1)x - f(1)y$	1
$= (f(x) - f(1)x) + (f(y) - f(1)y)$	
$= g(x) + g(y)$	
(ii) $g(x+1) = f(x+1) - f(1)(x+1)$	
$= f(x) + f(1) - f(1)x - f(1)$	1
$= f(x) - f(1)x$	
$= g(x)$	
(iii) $\forall x \in \mathbb{R}$, there exists $h \in [0, 1)$ such that $x - h$ is an integer.	1
By (b)(ii), $g(x) = g(h)$	1
$= f(h) - f(1)h$	
$< K - f(1)h$	1
$< K + \lvert f(1) \rvert \quad (\because\ 0 \le h < 1)$	1
By (b), $g(x) = 0 \quad \forall x \in \mathbb{R}$	1
$\Rightarrow f(x) - f(1)x = 0 \quad \forall x \in \mathbb{R}$	
$\Rightarrow f(x) = f(1)x \quad \forall x \in \mathbb{R}$	
	7

Solutions	Marks
10. (a) (Reflexive)	
$\forall A \in M$,	
$A = IAI^{-1}$	1M
$\therefore A \sim A$	
(Symmetric)	
$\forall A, B \in M$,	
$A \sim B$	
$\Rightarrow A = PBP^{-1}$ for some P	
$\Rightarrow P^{-1}AP = B$	1M
$\Rightarrow B = P^{-1}A(P^{-1})^{-1}$	
$\Rightarrow B \sim A$	
(Transitive)	
$\forall A, B, C \in M$,	
$A \sim B$ and $B \sim C$	
$\Rightarrow A = PBP^{-1}$ and $B = QCQ^{-1}$ for some P, Q	
$\Rightarrow A = P(QCQ^{-1})P^{-1}$	
$\Rightarrow A = (PQ)C(PQ)^{-1}$	1M
$\Rightarrow A \sim C$	3
(b) If $A \sim B$	
then $A = PBP^{-1}$ for some P	1M
$\Rightarrow A^k = (PBP^{-1})^k$	
$= (PBP^{-1}) \dots (PBP^{-1})$	1M
$= PB(P^{-1}P)B \dots B(P^{-1}P)BP^{-1}$	
$= PBIB \dots BIBP^{-1}$	
$= PB^kP^{-1}$	
$\Rightarrow A^k \sim B^k$	2
(c) (i) If $C \sim 0$	1
then $C = POP^{-1}$ for some P	
$= 0$	

(ii) Consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1

and $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Then $AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 1

and $BA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= 0$

$\therefore$ $\forall$ non-singular P, $P(BA)P^{-1} = 0$ 1M

$\neq AB$ $(\because AB \neq 0)$

$\Rightarrow AB \neq BA$

4

(d) $(\Rightarrow)$

If $A \sim \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

then $\exists P$ such that $A = P\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}P^{-1}$

$\Rightarrow AP = P\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ 1

Let $P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$

then $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ and $(x_3\ y_3\ z_3)$ are linearly independent

($\because$ P is non-singular) 1

Moreover,

$$A\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Solutions	Marks
$\Rightarrow \left(A\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \; A\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \; A\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right) = \left(\lambda_1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \; \lambda_2\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \; \lambda_3\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right)$	1
$\Rightarrow A\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = \lambda_i\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad i = 1, 2, 3.$	
$(\Rightarrow)$	
Consider $P = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$	
Since $(x_1\ y_1\ z_1)$, $(x_2\ y_2\ z_2)$ and $(x_3\ y_3\ z_3)$ are linearly independent, P is non-singular.	1
Furthermore,	
$AP = A\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$	
$= \left(A\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \; A\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \; A\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right)$	
$= \left(\lambda_1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \; \lambda_2\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \; \lambda_3\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \right)$	1
$= \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$	
$= P\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$	1
$\Rightarrow A = P\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}P^{-1}$ ($\because$ P is non-singular)	
$\Rightarrow A \sim \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$	
	6

Solutions	Marks
11. (a) $\Delta Z_1 Z_2 Z_3 \sim \Delta W_1 W_2 W_3$	
$\Leftrightarrow \angle Z_3 Z_1 Z_2 = \angle W_3 W_1 W_2$ and $\dfrac{Z_3 Z_1}{Z_2 Z_1} = \dfrac{W_3 W_1}{W_2 W_1}$	1
$\Leftrightarrow arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = arg\left(\dfrac{w_3 - w_1}{w_2 - w_1}\right)$ and $\dfrac{\|z_3 - z_1\|}{\|z_2 - z_1\|} = \dfrac{\|w_3 - w_1\|}{\|w_2 - w_1\|}$	1
$\Leftrightarrow arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = arg\left(\dfrac{w_3 - w_1}{w_2 - w_1}\right)$ and $\left\|\dfrac{z_3 - z_1}{z_2 - z_1}\right\| = \left\|\dfrac{w_3 - w_1}{w_2 - w_1}\right\|$	1
$\Leftrightarrow \dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{w_3 - w_1}{w_2 - w_1}$	1
	4
(b) Let E_1, E_2, E_3 be the points representing 1, ϵ, ϵ^2 respectively.	1
Then, $\Delta Z_1 Z_2 Z_3$ is equilateral	
$\Leftrightarrow \Delta Z_1 Z_2 Z_3 \sim \Delta E_1 E_2 E_3$	1M
$\Leftrightarrow (z_3 - z_1)(\epsilon - 1) = (z_2 - z_1)(\epsilon^2 - 1)$ (by (a))	1M
$\Leftrightarrow (z_3 - z_1)(\epsilon - 1) = (z_2 - z_1)(\epsilon - 1)(\epsilon + 1)$	
$\Leftrightarrow z_3 - z_1 = (z_2 - z_1)(\epsilon + 1)$	
$\Leftrightarrow z_3 - z_1 = z_2(\epsilon + 1) - z_1(\epsilon + 1)$	
$\Leftrightarrow (\epsilon + 1 - 1) z_1 - (\epsilon + 1) z_2 + z_3 = 0$	
$\Leftrightarrow \epsilon z_1 - (1 + \epsilon) z_2 + z_3 = 0$	
$\Leftrightarrow \epsilon z_1 + \epsilon^2 z_2 + z_3 = 0$ ($\because$ $1 + \epsilon + \epsilon^2 = 0$)	1
$\Leftrightarrow \epsilon^2(\epsilon z_1 + \epsilon^2 z_2 + z_3) = 0$	
$\Leftrightarrow \epsilon^3 z_1 + \epsilon^4 z_2 + \epsilon^2 z_3 = 0$	
$\Leftrightarrow z_1 + \epsilon z_2 + \epsilon^2 z_3 = 0$ ($\because$ $\epsilon^3 = 1$)	1
	5

Solutions	Marks
(c) Let Z_1, Z_2, Z_3 be distinct points representing $z_j = a_j + ib_j$ with a_j, $b_j \in \mathbb{Z}$ ($j = 1, 2, 3$).	
If $\triangle Z_1Z_2Z_3$ is equilateral, then by (b),	
$(a_1 + b_1 i) + \epsilon(a_2 + b_2 i) + \epsilon^2(a_3 + b_3 i) = 0$	1A
$\Rightarrow (a_1 + b_1 i) + (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)(a_2 + b_2 i) + (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)(a_3 + b_3 i) =$	1A
$\Rightarrow \begin{cases} a_1 - \frac{a_2}{2} - \frac{\sqrt{3}}{2} b_2 - \frac{a_3}{2} + \frac{\sqrt{3}}{2} b_3 = 0 \\ b_1 + \frac{\sqrt{3}}{2} a_2 - \frac{b_2}{2} - \frac{\sqrt{3}}{2} a_3 - \frac{b_3}{2} = 0 \end{cases}$	1
$\Rightarrow \begin{cases} (a_1 - \frac{a_2}{2} - \frac{a_3}{2}) + \frac{\sqrt{3}}{2}(b_3 - b_2) = 0 \\ (b_1 - \frac{b_2}{2} - \frac{b_3}{2}) + \frac{\sqrt{3}}{2}(a_2 - a_3) = 0 \end{cases}$	1
$\Rightarrow \begin{cases} a_1 - \frac{a_2}{2} - \frac{a_3}{2} = 0 & \text{..........(1)} \\ b_3 - b_2 = 0 & \text{..........(2)} \\ b_1 - \frac{b_2}{2} - \frac{b_3}{2} = 0 & \text{..........(3)} \\ a_2 - a_3 = 0 & \text{..........(4)} \end{cases}$ ($\because$ a_i, $b_i \in \mathbb{Z}$)	1
from (2), $b_2 = b_3$; from (4), $a_2 = a_3$ $\Rightarrow z_2 = z_3$	
$\Rightarrow$ a contradiction ($\because$ Z_2, Z_3 are distinct)	1
	6

Solutions	Marks
12. (a) $\forall p \in A$,	
Case 1 $p = 0$	
then $r \mid p$	1
Case 2 $p \neq 0$	
then by Euclidean Algorithm,	
$p = qr + s$ where $s = 0$ or $\deg s < \deg r$.	1
Now $s = p - qr$	
$= (mf + ng) - q(m'f + n'g)$ where $p = mf + ng$ and $r = m'f + n'g$	
$= (m - qm')f + (n - qn')g$	
$\in A$	1
$\therefore$ $\deg r \leq \deg s$ (by the property of r)	
$\Rightarrow$ $\deg s \nless \deg r$	
$\Rightarrow$ $s = 0$	1
Hence $p = qr$	
$\Rightarrow$ $r \mid p$	
$f = 1 \cdot f + 0 \cdot g \in A$	
$\therefore$ $r \mid f$	
$g = 0 \cdot f + 1 \cdot g \in A$	1
$\therefore$ $r \mid g$	
Thus r divides both f and g.	
If $\begin{cases} f = th \\ g = wh \end{cases}$ for some $t, w \in P$	
then $r = m'f + n'g$	
$= m'th + n'wh$	1
$= (m't + n'w)h$	
$\Rightarrow$ $h \mid r$	
Hence r is a G.C.D. of f and g.	
	6

Solutions	Marks
(b) $\forall p \in A$,	
$p = mf + ng$ for some $m, n \in \rho$	
$\Rightarrow p = m(m'r) + n(n'r)$ for some $m', n' \in \rho$ ($\because r$ divides both f and g)	1
$= (mm' + nn')r$	1
$\in B$	
$\therefore A \subset B$.	
$\forall p \in B$, $p = hr$	
$= h(mf + ng)$ ($\because r \in A$)	1
$= (hm)f + (hn)g$	1
$\in A$	
$\therefore B \subset A$	
Therefore $A = B$.	
	4
(c) r is a non-zero constant	
$\Rightarrow r = m'f + n'g$ for some $m', n' \in \rho$ ($\because r \in A$)	1
$\Rightarrow 1 = m_0 f + n_0 g$ where $m_0 = \dfrac{m'}{r}$, $n_0 = \dfrac{n'}{r} \in \rho$	1
By (b), $A = B$	1
$= \{hr : h \in \rho\}$	1
$= \{k : k \in \rho\}$ ($\because r$ is a non-zero constant)	1
$= \rho$	
	5

Solutions	Marks
13. (a) (i) If $z \in G_p \cap H_p$	
then $z^p = 1$ and $z^p = -1$,	1
a contradiction.	
(ii) $z \in G_p \cup H_p \Rightarrow z \in G_p$ or $z \in H_p$	
$\Rightarrow z^p = 1$ or $z^p = -1$	1
$\Rightarrow z^{2p} = 1$	
$\Rightarrow z \in G_{2p}$	
	2
(b) If $z \in H_p \cap H_q$	
then $z \in H_p$ and $z \in H_q$	
$\Rightarrow z^p = -1$ and $z^q = -1$	1
we have $z^{pq} = (z^p)^q = (-1)^q = 1$ ($\because q$ is even)	1
and $z^{pq} = (z^q)^p = (-1)^p = -1$ ($\because p$ is odd)	
$\Rightarrow$ a contradiction	
	2
(c) (i) $z \in G_q$	
$\Rightarrow z^q = 1$	
$\Rightarrow z^p = z^{mq}$	
$= (z^q)^m$	1
$= 1^m$	
$= 1$	
$\Rightarrow z \in G_p$	
(ii) $\because z \in H_q \Rightarrow z^p = z^{mq}$	
$= (z^q)^m$	
$= (-1)^m$ ($\because z \in H_q$)	1
$= -1$ ($\because m$ is odd)	1
$\Rightarrow z \in H_p$	

Solutions	Marks
(ii) To show $G_pH_p = H_p$:	
$z \in G_pH_p \Rightarrow z = st$ where $s \in G_p$, $t \in H_p$	
$\Rightarrow z^p = (st)^p = s^p t^p = 1 \cdot (-1) = -1$	1
$\Rightarrow z \in H_p$	
$z \in H_p \Rightarrow z = s(\frac{z}{s})$ where $s \in G_p$	
$\Rightarrow z \in G_pH_p$ (because $(\frac{z}{s})^p = \frac{z^p}{s^p} = \frac{-1}{1} = -1$)	1
To show $G_pH_p = H_pG_p$:	
$G_pH_p = \{z \in C : z = st$ for some $s \in G_p$, $t \in H_p\}$	
$= \{z \in C : z = ts$ for some $t \in H_p$, $s \in G_p\}$	1
$= H_pG_p$	6

(iii) $z \in H_q \Rightarrow z^p = z^{mq}$

$= (z^q)^m$

$= (-1)^m \quad (\because z \in H_q)$ 1

$= 1 \quad (\because m \text{ is even})$ 1

$\Rightarrow z \in G_p$ 5

(d) (i) To show $G_pG_p = G_p$:

$z \in G_pG_p \Rightarrow z = st$ where $s, t \in G_p$

$\Rightarrow z^p = (st)^p = s^pt^p = 1 \cdot 1 = 1$

$\Rightarrow z \in G_p$ 1

$z \in G_p \Rightarrow z^p = 1$

$\Rightarrow z = 1 \cdot z$ where $1^p = z^p = 1$

$\Rightarrow z = 1 \cdot z$ where $1, z \in G_p$

$\Rightarrow z \in G_pG_p$ 1

To show $H_pH_p = G_p$:

$z \in H_pH_p \Rightarrow z = st$ where $z, t \in H_p$

$\Rightarrow z^p = (st)^p = s^pt^p = (-1)(-1) = 1$

$\Rightarrow z \in G_p$

$z \in G_p \Rightarrow z = (\frac{z}{t})(t)$ where $t \in H_p$

$\Rightarrow z \in H_pH_p$ (because $(\frac{z}{t})^p = \frac{z^p}{t^p} = \frac{1}{-1} = -1$) 1

Solutions | Marks

1. (a) $\lim_{x\to\frac{\pi}{2}} \frac{x-\frac{\pi}{2}}{\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{-\sin x}$ 1M

$= -1$ 1A

(b) $\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0} \frac{nm(1+mx)^{n-1}-nm(1+nx)^{m-1}}{2x}$ 1M

$= \lim_{x\to 0} \frac{n(n-1)m(1+mx)^{n-2}-nm(m-1)(1+nx)^{m-2}}{2}$ 1M

$= \frac{n(n-1)m-nm(m-1)}{2}$

$= \frac{nm}{2}[(n-1)-(m-1)]$

$= \frac{nm(n-m)}{2}$ 1A

Alternatively

$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$

$= \lim_{x\to 0} \frac{\left(1+nmx+m^2x^2\frac{n(n-1)}{1\cdot 2}+\cdots\right)-\left(1+nmx+n^2x^2\frac{m(m-1)}{1\cdot 2}+\cdots\right)}{x^2}$ 1M

$= \lim_{x\to 0} \left(\frac{1}{2}(m^2n(n-1)-n^2m(m-1))+\cdots\right)$ 1M

$= \frac{1}{2}mn(m(n-1)-n(m-1))$

$= \frac{nm(n-m)}{2}$ 1A

Solutions	Marks
2. Let the equation of the plane be	
$k(x + y + z - 1) + (x + 4y + 3z) = 0$	1M
$(k + 1)x + (k + 4)y + (k + 3)z - k = 0$	
Since it is parallel to $\dfrac{x-1}{3} = \dfrac{y}{1} = \dfrac{z+1}{1}$ which has direction ratio (3, 1, 1), we have	1A
$(k + 1)\cdot 3 + (k + 4)\cdot 1 + (k + 3)\cdot 1 = 0$	1M
$3k + 3 + k + 4 + k + 3 = 0$	
$5k + 10 = 0$	
$k = -2$	1A
$\therefore$ the equation is	
$(-2 + 1)x + (-2 + 4)y + (-2 + 3)z - (-2) = 0$	
i.e. $-x + 2y + z + 2 = 0$	1A

Solutions	Marks
3. Since Q lies on OP, we may let $P = (r, \theta)$ and $Q = (s, \theta)$ with $r, s \geq 0$.	1M
Now $OP \cdot OQ = a^2$	
$\Rightarrow \quad r \cdot s = a^2$ ....(1)	1A
But P lies on C,	
$\therefore \quad r = 2a\cos\theta$ ....(2)	1A
Eliminating r from (1) and (2), we have	
$2sa\cos\theta = a^2$	1A
$\Rightarrow \quad s\cos\theta = \frac{a}{2}$	
$\Rightarrow \quad x = \frac{a}{2} \quad (\because \; x = s\cos\theta)$	1M

Solutions	Marks
4. $\frac{dx}{dt} = 3\sin^2 t\cos t$	
$\frac{dy}{dt} = -3\cos^2 t\sin t$	1
$\therefore \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$	
$= \sqrt{9\sin^2 t\cos^2 t}\,dt$	
$= 3\sin t\cos t\,dt$	1
Surface area $= 2\int_0^{\frac{\pi}{2}} (2\pi y)(3\sin t\cos t)\,dt$	1M
$= 12\pi\int_0^{\frac{\pi}{2}} \cos^4 t\sin t\,dt$	
$= -12\pi\int_0^{\frac{\pi}{2}} \cos^4 t\,d(\cos t)$	1M
$= -12\pi\left[\frac{\cos^5 t}{5}\right]_0^{\frac{\pi}{2}}$	
$= -\frac{12\pi}{5}(0-1)$	
$= \frac{12\pi}{5}$	1A

5. $\int e^{2x}(\sin x + \cos x)^2\,dx = \int e^{2x}(\sin^2 x + \cos^2 x + 2\sin x\cos x)\,dx$

$= \int e^{2x}(1 + 2\sin x\cos x)\,dx$ 1A

$= \int e^{2x}dx + 2\int e^{2x}\sin x\cos x\,dx$

$= \frac{1}{2}e^{2x} + \int e^{2x}\sin 2x\,dx$ 1A

$= \frac{1}{2}e^{2x} + \frac{1}{2}\int e^{y}\sin y\,dy \quad (y = 2x)$

$= \frac{1}{2}e^{2x} + \frac{1}{2}I$ where $I = \int e^{y}\sin y\,dy$ 1A

now $I = \int e^{y}\sin y\,dy$

$= \int \sin y\,d(e^{y})$

$= e^{y}\sin y - \int e^{y}d(\sin y)$ 1M

$= e^{y}\sin y - \int e^{y}\cos y\,dy$

$= e^{y}\sin y - \int \cos y\,d(e^{y})$

$= e^{y}\sin y - \left\{ e^{y}\cos y - \int e^{y}d(\cos y) \right\}$

$= e^{y}\sin y - \left\{ e^{y}\cos y + \int e^{y}\sin y\,dy \right\}$

$= e^{y}\sin y - e^{y}\cos y - I$ 1A

$\therefore \quad 2I = e^{y}\sin y - e^{y}\cos y$

$I = \frac{1}{2}e^{y}(\sin y - \cos y) + c'$ 1A

Therefore, $\int e^{2x}(\sin x + \cos x)^2\,dx = \frac{1}{2}e^{2x} + \frac{1}{4}e^{2x}(\sin 2x - \cos 2x) + c$ 1A

 Marks

6. (a) $\alpha > \beta \geq 0 \Rightarrow \alpha\beta + \alpha > \alpha\beta + \beta \geq 0$

$\Rightarrow \alpha(\beta + 1) > \beta(\alpha + 1) \geq 0$ 1

$\Rightarrow \dfrac{\alpha}{\alpha + 1} > \dfrac{\beta}{\beta + 1} \geq 0$ 1

$\Rightarrow \sqrt{\dfrac{\alpha}{\alpha + 1}} > \sqrt{\dfrac{\beta}{\beta + 1}}$

(b) $u_{n+1} - u_n = \sum_{m=1}^{n+1} \dfrac{1}{2^m}\sqrt{\dfrac{n+1-m}{n+1-m+1}} - \sum_{m=1}^{n} \dfrac{1}{2^m}\sqrt{\dfrac{n-m}{n-m+1}}$

$= \sum_{m=1}^{n} \dfrac{1}{2^m}\left\{\sqrt{\dfrac{n+1-m}{n+1-m+1}} - \sqrt{\dfrac{n-m}{n-m+1}}\right\} + \dfrac{1}{2^{n+1}}\sqrt{\dfrac{0}{1}}$ 1M

> 0 (by (a)) 1M

Also, $u_n = \sum_{m=1}^{n} \dfrac{1}{2^m}\sqrt{\dfrac{n-m}{n-m+1}}$

$< \sum_{m=1}^{n} \dfrac{1}{2^m}$ 1

$< \sum_{m=1}^{\infty} \dfrac{1}{2^m}$ 1

$= \dfrac{1}{2} \cdot \dfrac{1}{\left(1 - \dfrac{1}{2}\right)}$

$= 1$

Since $\{u_n\}$ is increasing and bounded above, $\lim u_n$ exists. 1M

	Marks
7. (a) By Leibniz's Theorem, $y^{(n)} = \sum_{r=0}^{n}\binom{n}{r}u^{(r)}(x)\,(e^{qx})^{(n-r)}$	1M
$= \sum_{r=0}^{n}\binom{n}{r}u^{(r)}(x)\,q^{n-r}e^{qx}$	1A
(b) $u^{(r)}(x) = p^{r}e^{px}$	1A
$\therefore\ y^{(n)} = \sum_{r=0}^{n}\binom{n}{r}p^{r}q^{n-r}e^{(p+q)x}$	1
On the other hand,	
$y^{(n)} = (u(x)\,e^{qx})^{(n)} = (e^{px}e^{qx})^{(n)} = (e^{(p+q)x})^{(n)}$	
$= (p+q)^{n}e^{(p+q)x}$	1
Hence $(p+q)^{n} = \sum_{r=0}^{n}\binom{n}{r}p^{r}q^{n-r}$ $\quad(\because e^{(p+q)x} \neq 0)$	1M

8. (a) $f(x) = (x^2 - x^3)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}}(2x - 3x^2)$$

$$= \frac{2 - 3x}{3x^{\frac{1}{3}}(1 - x)^{\frac{2}{3}}}$$ 1A

$$f''(x) = -\frac{2}{9}(x^2 - x^3)^{-\frac{5}{3}}(2x - 3x^2)^2 + \frac{1}{3}(x^2 - x^3)^{-\frac{2}{3}}(2 - 6x)$$

$$= \frac{-2x^2}{9(x^2 - x^3)^{\frac{5}{3}}}$$

$$= \frac{-2}{9(1 - x)(x^2 - x^3)^{\frac{2}{3}}}$$ 1A

2

(b) $$\frac{f(x) - f(0)}{x} = \frac{1}{x}(x^2 - x^3)^{\frac{1}{3}}$$

$$= \left(\frac{1}{x} - 1\right)^{\frac{1}{3}}$$ 1

$\to \pm\infty$ as $x \to \pm 0$

$\therefore$ $f'(0)$ does not exist.

$$\frac{f(x) - f(1)}{x - 1} = \frac{1}{x - 1}(x^2 - x^3)^{\frac{1}{3}}$$

$$= \frac{x^{\frac{2}{3}}(1 - x)^{\frac{1}{3}}}{x - 1}$$

$$= \frac{-x^{\frac{2}{3}}}{(x - 1)^{\frac{2}{3}}}$$

$\to -\infty$ as $x \to 1$ 1

$\therefore$ $f'(1)$ does not exist.

2

(c) (i) $f'(x) = 0$ — $x = \frac{2}{3}$

(ii) $f'(x) > 0$ — $0 < x < \frac{2}{3}$

(iii) $f'(x) < 0$ — $x < 0$, $\frac{2}{3} < x < 1$, $1 < x$

(iv) $f''(x) \neq 0$ for all x

(v) $f''(x) > 0$ — $x > 1$

(vi) $f''(x) < 0$ — $x < 0$, $0 < x < 1$

($\frac{1}{2}$ mark each) 3

3

(d)

x	$x<0$	0	$0<x<\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}<x<1$	1	$x>1$
$f'(x)$	$-$	undefined	$+$	0	$-$	undefined	$-$
$f''(x)$	$-$	undefined	$-$	$-$	$-$	undefined	$+$
$f(x)$	↘	0	↗	$\frac{1}{3}\cdot\sqrt[3]{4}$	↘	0	↘

$\therefore$ relative maximum is at $\left(\frac{2}{3}, \frac{1}{3}\sqrt[3]{4}\right)$, 1A

the point of inflexion at (1, 0), 1A

relative minimum is at (0, 0). 1A

3

(e) Let the oblique asymptote be $y = mx + c$

$$m = \lim_{x\to\infty} \frac{y}{x}$$

$$= \lim_{x\to\infty} \frac{\sqrt[3]{x^2 - x^3}}{x}$$

$$= \lim_{x\to\infty} \sqrt[3]{\frac{1}{x} - 1}$$ 1A

$$= -1$$

$$c = \lim_{x\to\infty} (\sqrt[3]{x^2 - x^3} + x)$$

$$= \lim_{x\to\infty} \left(x \cdot \sqrt[3]{\frac{1}{x} - 1} + x\right)$$

$$= \lim_{x\to\infty} \frac{x \cdot \left\{(\frac{1}{x} - 1) + 1\right\}}{(\frac{1}{x} - 1)^{\frac{2}{3}} - (\frac{1}{x} - 1)^{\frac{1}{3}} + 1}$$ 1M

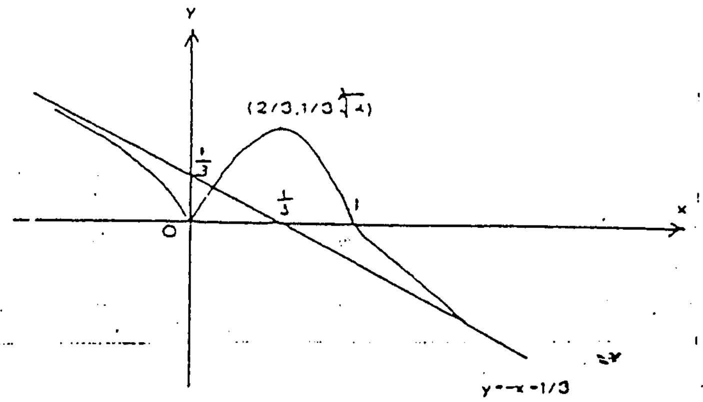
$$= \lim_{x\to\infty} \frac{1}{(\frac{1}{x} - 1)^{\frac{2}{3}} - (\frac{1}{x} - 1)^{\frac{1}{3}} + 1}$$

$$= \frac{1}{3}$$ 1A

$\therefore$ The oblique asymptote is $y = -x + \frac{1}{3}$.

3

(f)



2

2

9. (a) (i) $\frac{[\frac{1}{2}a(t+\frac{1}{t})]^2}{a^2} - \frac{[\frac{1}{2}b(t-\frac{1}{t})]^2}{b^2}$

$= \frac{1}{4}(t+\frac{1}{t})^2 - \frac{1}{4}(t-\frac{1}{t})^2$

$= \frac{1}{4}[(t+\frac{1}{t})+(t-\frac{1}{t})][(t+\frac{1}{t})-(t-\frac{1}{t})]$

$= \frac{1}{4}[2t][2(\frac{1}{t})]$ 1

$= 1$

(ii) Differentiating both sides of the equation of H with respect to x, we have

$\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}1$

$\Rightarrow \frac{2}{a^2}x - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}(\frac{b}{a})^2$ 1

By slope point form, equation of tangent at P is

$\frac{y - \frac{1}{2}b(t-\frac{1}{t})}{x - \frac{1}{2}a(t+\frac{1}{t})} = \frac{\frac{1}{2}a(t+\frac{1}{t})}{\frac{1}{2}b(t-\frac{1}{t})}(\frac{b}{a})^2$ 1

$\frac{y - \frac{1}{2}b(t-\frac{1}{t})}{x - \frac{1}{2}a(t+\frac{1}{t})} = \frac{b(t+\frac{1}{t})}{a(t-\frac{1}{t})}$

$a(t-\frac{1}{t})y - \frac{1}{2}ab(t-\frac{1}{t})^2 = b(t+\frac{1}{t})x - \frac{1}{2}ab(t+\frac{1}{t})^2$

$b(t+\frac{1}{t})x - a(t-\frac{1}{t})y = \frac{ab}{2}\left[(t+\frac{1}{t})^2 - (t-\frac{1}{t})^2\right]$

$= \frac{ab}{2}\left[(t+\frac{1}{t})+(t-\frac{1}{t})\right]\left[(t+\frac{1}{t})-(t-\frac{1}{t})\right]$

$= \frac{ab}{2}(2t)(\frac{2}{t})$

$= 2ab$

i.e. $\frac{x}{2a}(t+\frac{1}{t}) - \frac{y}{2b}(t-\frac{1}{t}) = 1$ ....(*) 1A

4

Solutions	Marks

(b) (i) Asymptotes : $y = \pm \frac{b}{a}x$ **1A**

Substituting $y = \frac{b}{a}x$ into (*), we have

$$\frac{x}{2a}\left(t + \frac{1}{t}\right) - \frac{x}{2a}\left(t - \frac{1}{t}\right) = 1$$

$$\frac{x}{at} = 1$$

$$x = at$$

$$\Rightarrow \quad y = bt$$

$$\Rightarrow \quad S = (at,\ bt)$$ **1A**

Substituting $y = -\frac{b}{a}x$ into (*), we have

$$\frac{x}{2a}\left(t + \frac{1}{t}\right) + \frac{x}{2a}\left(t - \frac{1}{t}\right) = 1$$

$$\frac{xt}{a} = 1$$

$$\Rightarrow \quad x = \frac{a}{t}$$

$$\Rightarrow \quad y = -\frac{b}{t}$$

$$\Rightarrow \quad T = \left(\frac{a}{t},\ -\frac{b}{t}\right)$$ **1A**

Let the equation of circle OST be

$$x^2 + y^2 + 2gx + 2fy + c = 0 .$$

Substituting (0, 0) into the equation, we have $c = 0$. **1A**

Substituting S , T into the equation, we have

$$\begin{cases} a^2t^2 + b^2t^2 + 2gat + 2fbt = 0 \\ \dfrac{a^2}{t^2} + \dfrac{b^2}{t^2} + \dfrac{2ga}{t} - \dfrac{2fb}{t} = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} a^2t^2 + b^2t^2 + 2gat + 2fbt = 0 \\ a^2 + b^2 + 2gat - 2fbt = 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} 4gat = -(a^2 + b^2)(1 + t^2) \\ 4fbt = -(a^2 + b^2)(t^2 - 1) \end{cases}$$

$$\Rightarrow \quad \begin{cases} g = -\dfrac{1}{4at}(a^2 + b^2)(1 + t^2) \\ f = -\dfrac{1}{4bt}(a^2 + b^2)(t^2 - 1) \end{cases}$$

Thus the centre of the circle is given by

$$\begin{cases} x = \dfrac{1}{4at}(a^2+b^2)(1+t^2) \\ y = \dfrac{1}{4bt}(a^2+b^2)(t^2-1) \end{cases}$$ 1A

Eliminate t :

$$\begin{cases} x = \dfrac{1}{4a}(a^2+b^2)(\dfrac{1}{t}+t) \\ y = \dfrac{1}{4b}(a^2+b^2)(t-\dfrac{1}{t}) \end{cases}$$

$$\begin{cases} \dfrac{4ax}{a^2+b^2} = t+\dfrac{1}{t} \\ \dfrac{4by}{a^2+b^2} = t-\dfrac{1}{t} \end{cases}$$

$$\therefore \quad \left(\frac{4ax}{a^2+b^2}\right)^2 - \left(\frac{4by}{a^2+b^2}\right)^2 = (t+\frac{1}{t})^2 - (t-\frac{1}{t})^2 = 4$$ 1A

$$\text{or} \quad \frac{x^2}{\left(\dfrac{a^2+b^2}{2a}\right)^2} - \frac{y^2}{\left(\dfrac{a^2+b^2}{2b}\right)^2} = 1$$

(ii) $OS \cdot OT = [(at)^2+(bt)^2]^{\frac{1}{2}} \cdot \left[(\frac{a}{t})^2+(\frac{b}{t})^2\right]^{\frac{1}{2}}$ 1

$= a^2+b^2$

Let $S' = (-at, bt)$.

Hence $S'OT$ are collinear and 1

$OS' \cdot OT = OS \cdot OT = a^2+b^2$

Let F , F' be the foci, so

$OF \cdot OF' = a^2+b^2 = OS' \cdot OT$ 1

Therefore $\Delta OFT \sim \Delta OS'F'$ and the points 1

F , F' , S' , T are concyclic. Since the centre 1

of such circle lies on the y-axis, S , T and the

foci are concyclic.

11

Solutions	Marks
10. (a) $x^3 + y^3 = 3axy$	
$\Rightarrow 1 + (\frac{y}{x})^3 = 3(\frac{a}{x^2})(\frac{y}{x})$	
Let $\lim_{x\to\infty} \frac{y}{x} = m$	
Taking $x \to \infty$, we have	
$1 + m^3 = 0 \cdot m$ $\quad (\because \lim_{x\to\infty} \frac{a}{x^2} = 0)$	1
$\Rightarrow m = -1$	1A
Now $x^3 + y^3 = 3axy$	
$\Rightarrow x + y = \frac{3axy}{x^2 - xy + y^2}$	
$= \frac{3a(\frac{y}{x})}{1 - \frac{y}{x} + (\frac{y}{x})^2}$	1
$\to \frac{3a(-1)}{1 - (-1) + (-1)^2} = -a$ as $x \to \infty$	
$\therefore$ Equation of L is $x + y + a = 0$	3
(b) Putting $x = r\cos\theta$ and $y = r\sin\theta$,	1M
we have equation of Γ written as	
$r^3\cos^3\theta + r^3\sin^3\theta = 3ar^2\cos\theta\sin\theta$	
i.e. $r = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$	1A
and equation of L written as	
$r\cos\theta + r\sin\theta + a = 0$	
i.e. $r = \frac{-a}{\cos\theta + \sin\theta}$	1A
	3

Marks

(c) Area of loop $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\,d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2\cos^2\theta\sin^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\,d\theta$ 1A

$= \frac{9}{2}a^2\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta$

$= \frac{9}{2}a^2\int_1^{\infty} \frac{dw}{3w^2}$ where $w = 1 + \tan^3\theta$ 1

$= \frac{3}{2}a^2\left[-\frac{1}{w}\right]_1^{\infty}$

$= \frac{3}{2}(0+1)a^2$

$= \frac{3}{2}a^2$ 1A

3

(d) $A_\phi = \int_\phi^\pi \frac{1}{2}r_1^2 - \frac{1}{2}r_2^2\,d\theta$ where $r_1 = \frac{-a}{\cos\theta + \sin\theta}$ 1M

and $r_2 = \frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

$= \frac{a^2}{2}\left\{\int_\phi^\pi \frac{d\theta}{(\sin\theta+\cos\theta)^2} - \int_\phi^\pi \frac{9\sin^2\theta\cos^2\theta}{(\sin^3\theta+\cos^3\theta)^2}\,d\theta\right\}$ 1A

Now $\int_\phi^\pi \frac{1}{(\sin\theta+\cos\theta)^2}\,d\theta = \int_\phi^\pi \frac{\sec^2\theta\,d\theta}{(1+\tan\theta)^2}$

$= -\left[\frac{1}{1+\tan\theta}\right]_\phi^\pi$

$= \frac{1}{1+\tan\phi} - 1$ 1A

and $\int_\phi^\pi \frac{9\sin^2\theta\cos^2\theta\,d\theta}{(\sin^3\theta+\cos^3\theta)^2} = 9\cdot\left[\frac{1}{-3(1+\tan^3\theta)}\right]_\phi^\pi$

$= 3\left(\frac{1}{1+\tan^3\phi} - 1\right)$ 1A

$\therefore A_\phi = \frac{a^2}{2}\left\{\left(\frac{1}{1+\tan\phi} - 1\right) - 3\left(\frac{1}{1+\tan^3\phi} - 1\right)\right\}$

$= \frac{a^2}{2}\left\{\frac{1}{1+\tan\phi} - \frac{3}{1+\tan^3\phi} + 2\right\}$

Solutions	Marks
$\displaystyle \lim_{\phi \to -\frac{3\pi}{4}} A_\phi = \lim_{\phi \to -\frac{3\pi}{4}} \frac{a^2}{2}\left\{ \frac{1}{1+\tan\phi} - \frac{3}{1+\tan^3\phi} + 2 \right\}$	
$\displaystyle = a^2 + \frac{a^2}{2}\lim_{\phi \to -\frac{3\pi}{4}}\left\{ \frac{1}{1+\tan\phi} - \frac{3}{1+\tan^3\phi} \right\}$	
$\displaystyle = a^2 + \frac{a^2}{2}\lim_{\phi \to -\frac{3\pi}{4}} \frac{1-\tan\phi+\tan^2\phi-3}{1+\tan^3\phi}$	
$\displaystyle = a^2 + \frac{a^2}{2}\lim_{\phi \to -\frac{3\pi}{4}} \frac{\tan^2\phi-\tan\phi-2}{1+\tan^3\phi}$	
$\displaystyle = a^2 + \frac{a^2}{2}\lim_{\phi \to -\frac{3\pi}{4}} \frac{(\tan\phi+1)(\tan\phi-2)}{1+\tan^3\phi}$	
$\displaystyle = a^2 + \frac{a^2}{2}\lim_{\phi \to -\frac{3\pi}{4}} \frac{\tan\phi-2}{1-\tan\phi+\tan^2\phi}$	1M
$\displaystyle = a^2 + \frac{a^2}{2}\left(\frac{(-1)-2}{1-(-1)+(-1)^2} \right)$	
$\displaystyle = a^2 + \frac{a^2}{2}\left(\frac{-3}{3} \right)$	
$\displaystyle = \frac{a^2}{2}$	1A
	6

11. (a) (i) By Mean Value Theorem, $f(t) = f(n) + f'(\zeta)(t-n)$ for some $\zeta \in (t, n)$ where $t \in (n, n+1]$.

As $f'' \geq 0$, f' is increasing, $\therefore$ $f'(n) \leq f'(\zeta) \leq f'(n+1)$ — 1

$\therefore$ $f(n) + f'(n)(t-n) \leq f(t) \leq f(n) + f'(n+1)(t-n)$

and obviously it holds for $t = n$. — 1

(ii) $\because$ $f(n) + f'(n)(t-n) \leq f(t) + f'(n+1)(t-n)$ $\forall t \in [n, n+1]$,

$\therefore$ $\int_n^{n+1} f(n) + f'(n)(t-n)\,dt \leq \int_n^{n+1} f(t)\,dt \leq \int_n^{n+1} f(n) + f'(n+1)(t-n)$ — 1M

$\Rightarrow$ $f(n) + f'(n)\left[\frac{(t-n)^2}{2}\right]_n^{n+1} \leq \int_n^{n+1} f(t)\,dt \leq f(n) + f'(n+1)\left[\frac{(t-n)^2}{2}\right]_n^{n+1}$

$\Rightarrow$ $f(n) + \frac{f'(n)}{2} \leq \int_n^{n+1} f(t)\,dt \leq f(n) + \frac{f'(n+1)}{2}$

$\Rightarrow$ $f(n) + \frac{f'(n)}{2} - \left(\frac{f(n) + f(n+1)}{2}\right) \leq \int_n^{n+1} f(t)\,dt - \left(\frac{f(n) + f(n+1)}{2}\right)$

$\leq f(n) + \frac{f'(n+1)}{2} - \left(\frac{f(n) + f(n+1)}{2}\right)$

$\Rightarrow$ $\frac{f'(n)}{2} + \frac{f(n) - f(n+1)}{2} \leq \int_n^{n+1} f(t)\,dt - \left(\frac{f(n) + f(n+1)}{2}\right)$

$\leq \frac{f'(n+1)}{2} + \frac{f(n) - f(n+1)}{2}$ — 2A

Now $f(n) - f(n+1) = -(f(n+1) - f(n))$

$= -f'(\eta)$ for some $\eta \in (n, n+1)$ — 1

since f' is increasing, we have $-f'(n+1) \leq f(n) - f(n+1) \leq -f'(n)$ — 1

hence $\frac{f'(n)}{2} - \frac{f'(n+1)}{2} \leq \int_n^{n+1} f(t)\,dt - \left(\frac{f(n) + f(n+1)}{2}\right)$

$\leq \frac{f'(n+1)}{2} - \frac{f'(n)}{2}$

$\Rightarrow$ $-\frac{1}{2}(f'(n+1) - f'(n)) \leq \int_n^{n+1} f(t)\,dt - \left(\frac{f(n) - f(n+1)}{2}\right)$

$\leq \frac{1}{2}(f'(n+1) - f'(n))$ — 1

$\Rightarrow$ $\left|\int_n^{n+1} f(t)\,dt - \left(\frac{f(n) + f(n+1)}{2}\right)\right| \leq \frac{f'(n+1) - f'(n)}{2}$

Solutions	Marks

(iii) $\left|\int_n^{n^2} f(t)\,dt - \sum_{j=n}^{n^2-1} \frac{f(j)+f(j+1)}{2}\right| g(n^2)$

$\le g(n^2)\cdot \sum_{j=n}^{n^2-1}\left|\int_j^{j+1} f(t)\,dt - \left(\frac{f(j)+f(j+1)}{2}\right)\right|$ — 1

$\le g(n^2)\cdot \sum_{j=n}^{n^2-1} \frac{f'(j+1)-f'(j)}{2}$

$= g(n^2)\cdot \frac{f'(n^2)-f'(n)}{2}$

$= \frac{1}{2} g(n^2) f''(\zeta_n)$ for some $\zeta_n \in (n, n^2)$. — 1M

(*)

$\le \frac{1}{2} g(\zeta_n) f''(\zeta_n)$ as g is decreasing.

As $n \to \infty$, $\zeta_n \to \infty$. $\therefore \lim_{n\to\infty} g(\zeta_n) f''(\zeta_n) = 0$ (by Condition C)

$\therefore \lim_{n\to\infty}\left|\int_n^{n^2} f(t)\,dt - \sum_{j=n}^{n^2-1} \frac{f(j)+f(j+1)}{2}\right| g(n^2) = 0$

$\Rightarrow \lim_{n\to\infty}\left\{\int_n^{n^2} f(t)\,dt - \sum_{j=n}^{n^2-1} \frac{f(j)+f(j+1)}{2}\right\} g(n^2) = 0$

— 9

* *Note: marks for this part have been reallocated*

(b) Put $f(t) = \frac{1}{\ln t}$ and $g(t) = \frac{\ln t}{t}$

then $g'(t) = \frac{1-\ln t}{t^2}$

$\le 0 \quad \forall t \in (e, \infty)$

$\therefore$ $g(t)$ is decreasing on (e, ∞). — 1

$\Rightarrow$ Condition A is satisfied.

Now $f'(t) = -\frac{1}{(\ln t)^2}\cdot\frac{1}{t}$

$f''(t) = \frac{2}{(\ln t)^3}\cdot\frac{1}{t^2} + \frac{1}{(\ln t)^2}\cdot\frac{1}{t^2}$

≥ 0 on (e, ∞)

$g(t) \ge 0$ on (e, ∞)

$\Rightarrow$ Condition B is satisfied. — 1

Also, $\lim_{t\to\infty} g(t) f''(t) = \lim_{t\to\infty} \frac{\ln t}{t}\left\{\frac{2}{(\ln t)^3}\cdot\frac{1}{t^2} + \frac{1}{(\ln t)^2}\cdot\frac{1}{t^2}\right\}$

$= \lim_{t\to\infty}\left\{\frac{2}{(\ln t)^2 t^3} + \frac{1}{(\ln t)\, t^3}\right\}$

$= 0$

$\Rightarrow$ Condition C is satisfied. — 1

Then by (a),

$$\lim_{n\to\infty}\left\{\int_n^{n^2}\frac{1}{\ln t}\,dt-\sum_{j=n}^{n^2-1}\frac{1}{2}\left(\frac{1}{\ln j}+\frac{1}{\ln(j+1)}\right)\right\}\frac{\ln n^2}{n^2}=0$$ 1

$$\Rightarrow \lim_{n\to\infty}\left\{\int_n^{n^2}\frac{1}{\ln t}\,dt-\sum_{j=n}^{n^2-1}\frac{1}{2}\left(\frac{1}{\ln j}+\frac{1}{\ln(j+1)}\right)\right\}\left(\frac{\ln n}{n^2}\right)=0$$

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{\ln n}{n^2}\int_n^{n^2}\frac{1}{\ln t}\,dt-\frac{\ln n}{n^2}\sum_{j=n}^{n^2-1}\frac{1}{2}\left(\frac{1}{\ln j}+\frac{1}{\ln(j+1)}\right)\right\}=0$$

$$\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n^2}\sum_{j=n}^{n^2-1}\frac{1}{2}\left(\frac{1}{\ln j}+\frac{1}{\ln(j+1)}\right)=\frac{1}{2}$$ 1

$\left(\because \text{ given that } \lim_{n\to\infty}\frac{\ln n}{n^2}\int_n^{n^2}\frac{1}{\ln t}\,dt=\frac{1}{2}\right)$

$$\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n^2}\left\{\sum_{j=n}^{n^2-1}\frac{1}{\ln(j+1)}+\frac{1}{2\ln n}-\frac{1}{2\ln n^2}\right\}=\frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{\ln n}{n^2}\sum_{j=n}^{n^2-1}\frac{1}{\ln(j+1)}+\frac{1}{2n^2}-\frac{1}{4n^2}\right\}=\frac{1}{2}$$ 1

$$\Rightarrow \lim_{n\to\infty}\left\{\frac{\ln n}{n^2}\sum_{j=n}^{n^2-1}\frac{1}{\ln(j+1)}-\frac{1}{4n^2}\right\}=\frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\ln n}{n^2}\sum_{j=n}^{n^2-1}\frac{1}{\ln(j+1)}=\frac{1}{2}\quad\left(\because \lim_{n\to\infty}\frac{1}{n^2}=0\right)$$

6

Solutions | Marks

12. (a) Let $S = 1 + (-t^2) + (-t^2)^2 + \cdots + (-t^2)^{n-1}$ — 1

then $(-t^2)S = (-t^2) + (-t^2)^2 + \cdots + (-t^2)^{n-1} + (-t^2)^n$

hence $S - (-t^2)S = 1 - (-t^2)^n$

$\Rightarrow \quad (1 + t^2)S = 1 - (-1)^n t^{2n}$ — 1

$\Rightarrow \quad S = \dfrac{1}{1 + t^2} - \dfrac{(-1)^n t^{2n}}{1 + t^2}$

$\Rightarrow \quad \dfrac{1}{1 + t^2} = S + \dfrac{(-1)^n t^{2n}}{1 + t^2}$

$\Rightarrow \quad \dfrac{1}{1 + t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{n-1} t^{2n-2} + \dfrac{(-1)^n t^{2n}}{1 + t^2}$

Integrating both sides from $t = 0$ to $t = x$, — 1M

we have $\displaystyle\int_0^x \frac{1}{1 + t^2}\,dt = \int_0^x 1 - t^2 + t^4 - \cdots + (-1)^{n-1} t^{2n-2} + \frac{(-1)^n t^{2n}}{1 + t^2}\,dt$

$\Rightarrow \quad \displaystyle[\tan^{-1} t]_0^x = \int_0^x 1 - t^2 + t^4 - \cdots + (-1)^{n-1} t^{2n-2}\,dt + \int_0^x \frac{(-1)^n t^{2n}}{1 + t^2}\,dt$ — 1

$\Rightarrow \quad \displaystyle\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1} x^{2n-1} + \int_0^x \frac{(-1)^n t^{2n}}{1 + t^2}\,dt$

4

(b) $\displaystyle\left|\tan^{-1} x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1} x^{2n-1}\right)\right|$

$\displaystyle = \left|\int_0^x \frac{(-1)^n t^{2n}}{1 + t^2}\,dt\right|$ (by (a)) — 1

$\displaystyle \le \int_0^x \left|\frac{(-1)^n t^{2n}}{1 + t^2}\right| dt$ — 1

$\displaystyle = \int_0^x \frac{t^{2n}}{1 + t^2}\,dt \quad (\because \ x \ge 0)$

$\displaystyle \le \int_0^x t^{2n}\,dt$ — 1

$\displaystyle = \frac{1}{2n + 1}[t^{2n+1}]_0^x$

$\displaystyle = \frac{1}{2n + 1} x^{2n+1}$

Putting $x = 1$, we have — 1M

$\displaystyle\left|\tan^{-1} 1 - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1}\right)\right| \le \frac{1}{2n + 1}$

$\displaystyle\Rightarrow \quad \left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1}\right)\right| \le \frac{1}{2n + 1}$

Taking $n \to \infty$, we have $\displaystyle\lim_{n \to \infty}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{(-1)^{n-1}}{2n - 1}\right) = \frac{\pi}{4}$ — 1M

$\displaystyle\Rightarrow \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} = \frac{\pi}{4}$ — 1A

6

(c) $\tan\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right) = \dfrac{\tan(\tan^{-1}\frac{1}{2}) + \tan(\tan^{-1}\frac{1}{3})}{1 - \tan(\tan^{-1}\frac{1}{2})\tan(\tan^{-1}\frac{1}{3})}$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$$

$$= \frac{\frac{5}{6}}{1-\frac{1}{6}}$$

$$= 1$$

$$= \tan\frac{\pi}{4} \qquad 1$$

Since $0 < \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} < \frac{\pi}{2}$, we have 1

$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ because tan is one-one on $(0, \frac{\pi}{2})$.

$$\left|\frac{\pi}{4} - \left[\left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{2^{2n-1}}+\frac{1}{3^{2n-1}}\right)\right]\right|$$

$$= \left|\left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}\right) - \left[\left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{2^{2n-1}}+\frac{1}{3^{2n-1}}\right)\right]\right|$$

$$= \left|\left[\tan^{-1}\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{2}\right)^{2n-1}\right)\right] + \left[\tan^{-1}\frac{1}{3} - \left(\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{3}\right)^{2n-1}\right)\right]\right|$$

$$\le \left|\tan^{-1}\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{2}\right)^{2n-1}\right)\right| + \left|\tan^{-1}\frac{1}{3} - \left(\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \cdots + \frac{(-1)^{n-1}}{2n-1}\left(\frac{1}{3}\right)^{2n-1}\right)\right|$$

$$\le \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n+1} + \frac{\left(\frac{1}{3}\right)^{2n-1}}{2n+1} \quad \text{(by (b))} \qquad 1$$

$$\le \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n+1} + \frac{\left(\frac{1}{2}\right)^{2n-1}}{2n+1} \qquad 1$$

$$= \frac{2}{(2n+1)(2^{2n-1})}$$

$$\le \frac{2}{(2n)(2^{2n-1})} \qquad 1$$

$$\le \frac{1}{n\cdot 2^{2n-1}} \qquad 5$$

Solutions	Marks
13. (a) If $f'(a) \le 0$,	
then $f'(x) \le 0 \ \forall x \in (a, b)$ ($\because f'$ is decreasing) ......(*)	1
But by Mean Value Theorem, $\exists \zeta \in (a, b)$	
such that $f'(\zeta) = \dfrac{f(b) - f(a)}{b - a}$	
> 0 ($\because f(b) > 0$, $f(a) < 0$, $b > a$)	1
contradicting (*).	
Hence $f'(a) > 0$.	2
(b) By (a), $x_1 = a - \dfrac{f(a)}{f'(a)} > a$	1
Rewrite $f(a) = -f'(a)(x_1 - a)$ ....(**)	
By MVT, $f(b) - f(a) = f'(c)(b - a)$ for some $c \in (a, b)$	
$< f'(a)(b - a)$ ($\because f'$ is decreasing)	
$\Rightarrow 0 < f(b) < f'(a)(b - a) - f'(a)(x_1 - a)$	
$= f'(a)(b - x_1)$ by (**)	1
$\Rightarrow x_1 < b$ since $f'(a) > 0$ by (a).	
By MVT, $f(x_1) - f(a) = f'(c)(x_1 - a)$ for some $c \in (a, x_1) \subset (a, b)$	
$< f'(a)(x_1 - a)$ ($\because f'$ is strictly decreasing)	1
$\Rightarrow f(x_1) < 0$ by (**)	1
If $f'(x_1) \le 0$, then $f(b) - f(x_1) = f'(\eta)(b - x)$ for some $\eta \in (x_1, b)$	
< 0 ($\because f'$ is decreasing)	1
$\Rightarrow f(b) < f(x_1) \le 0$	1
$\Rightarrow$ a contradiction	6
(c) We shall use mathematical induction to show that	1M
$x_n \in (a, b)$, $f(x_n) < 0$ and $f'(x_n) > 0 \ \forall n = 1, 2, \ldots$.	
For $n = 1$,	1
by (a), the result follows.	
Assume $x_k \in (a, b)$, $f(x_k) < 0$ and $f'(x_k) > 0$ for some $k \ge 1$.	1M
Using the same arguments in (b), we can show that	1
$x_{k+1} \in (a, b)$, $f(x_{k+1}) < 0$ and $f'(x_{k+1}) > 0$.	4

(d) Since $\{x_n\}$ is increasing and bounded above, $\lim_{n\to\infty} x_n$ exists. **1**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow 0 > f(x_n) = -(x_{n+1} - x_n) f'(x_n)$$

$$> -(x_{n+1} - x_n) f'(a)$$ **1**

$$\to 0 \text{ as } n \to \infty$$

$$\therefore \lim_{n\to\infty} f(x_n) = 0$$

$$\Rightarrow f(\lim_{n\to\infty} x_n) = 0 \quad (\because f \text{ is continuous})$$ **1**

3