Solutions | Marks

paper IA

1\. (a) $\begin{pmatrix} 1 & t-3 & 5 & 3 \\ -3 & 9 & -15 & s \\ 2 & t & 10 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & t-3 & 5 & 3 \\ 0 & 3t-18 & 0 & s-9 \\ 0 & t-6 & 0 & 0 \end{pmatrix}$ $(R_2' = R_1 \times 3 + R_2,\ R_3' = R_1 \times 2 - R_3)$ 1M

$\sim \begin{pmatrix} 1 & t-3 & 5 & 3 \\ 0 & 0 & 0 & s-9 \\ 0 & t-6 & 0 & 0 \end{pmatrix}$ $(R_2' = R_2 - 3 \times R_3)$

$\sim \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 0 & 0 & s-9 \\ 0 & t-6 & 0 & 0 \end{pmatrix}$ $(R_1' = R_1 - R_3)$ $0 > s+9$ $(t+6)y = 0$

$\therefore$ For consistency, $s = -9$, t = any real number. 1A

(b) Case 1 : $s = -9$, $t = -6$ 1

(*) $\sim \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\therefore$ Solution set $= \{ (\alpha, \beta, \frac{3 - \alpha + 3\beta}{5}) \in R^3 : \alpha, \beta \in R \}$ 1A

Case 2 : $s = -9$, $t \neq -6$ 1

Solution set $= \{ (\alpha, 0, \frac{3 - \alpha}{5}) \in R^3 : \alpha \in R \}$ 1A

2\. (a) Reflexive:

$\forall (x, y) \in R^2$,

$x - x = 0$ 1

$\Rightarrow (x, y) \sim (x, y)$

Symmetric:

$(x_1, y_1) \sim (x_2, y_2) \Rightarrow x_1 - x_2 = n$ for some integer n

$\Rightarrow x_2 - x_1 = -n$ 1

$\Rightarrow (x_2, y_2) \sim (x_1, y_1)$

Transitive:

$(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$

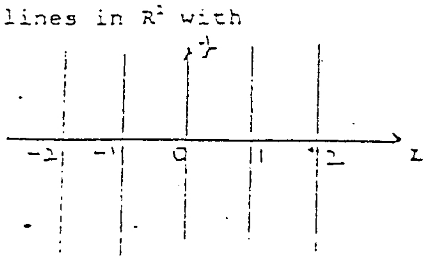
$\Rightarrow x_1 - x_2 = n$ and $x_2 - x_3 = m$ for some integers n, m

$\Rightarrow x_1 - x_3 = n - m$ 1

$\Rightarrow (x_1, y_1) \sim (x_3, y_3)$

(b) $(2, 1)/\sim = \{ (x, y) \in R^2 : x - 2 = n$ for some $n \in Z \}$ 1

= the set of all vertical lines in R^2 with integral x-intercepts 1



1. (a) $B^{-1} = \frac{1}{-2\lambda}\begin{pmatrix}\lambda & 0\\ -1 & -2\end{pmatrix}$ 1A

B^{-1} exists $\Leftrightarrow \lambda \neq 0$ 1

$\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} = B^{-1}AB$

$= -\frac{1}{2\lambda}\begin{pmatrix}\lambda & 0\\ -1 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 3\end{pmatrix}\begin{pmatrix}-2 & 0\\ 1 & \lambda\end{pmatrix}$

$= -\frac{1}{2\lambda}\begin{pmatrix}\lambda & 0\\ -3 & -6\end{pmatrix}\begin{pmatrix}-2 & 0\\ 1 & \lambda\end{pmatrix}$

$= -\frac{1}{2\lambda}\begin{pmatrix}-2\lambda & 0\\ 0 & -6\lambda\end{pmatrix}$

$= \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$\therefore \quad a = 1$ 1A

$b = 3$ 1A

λ can be any non-zero number.

(b) $A^{100} = \left(B\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}B^{-1}\right)^{100}$ 1M

$= B\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}^{100}B^{-1}$

$= B\begin{pmatrix}1 & 0\\ 0 & 3^{100}\end{pmatrix}B^{-1}$ 1M

$= -\frac{1}{2}\begin{pmatrix}-2 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 3^{100}\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & -2\end{pmatrix}$ (choosing $\lambda = 1$)

$= -\frac{1}{2}\begin{pmatrix}-2 & 0\\ 1 & 3^{100}\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & -2\end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix}-2 & 0\\ 1-3^{100} & -3^{100}\cdot 2\end{pmatrix}$

$= \begin{pmatrix}1 & 0\\ \frac{3^{100}-1}{2} & 3^{100}\end{pmatrix}$ 1A

4. $(1-i)^{2n} = \sum_{r=0}^{2n}\binom{2n}{k} i^k$

$= \sum_{r=0}^{n}\binom{2n}{2r} i^{2r} + \sum_{r=0}^{n-1}\binom{2n}{2r+1} i^{2r+1}$

$= \sum_{r=0}^{n}\binom{2n}{2r}(-1)^r - i\sum_{r=0}^{n-1}\binom{2n}{2r+1}(-1)^r$ 1A

On the other hand, $(1-i)^{2n} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right)^{2n}$ 1M

$= 2^n\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$ 1A

> Alternatively:
>
> $(1-i)^{2n} = (2i)^n$ 1M
>
> $= 2^n i^n$ 1A

Therefore, $\sum_{r=0}^{n}(-1)^r C^{2n}_{2r} = 2^n\cos\frac{n\pi}{2}$

$= \begin{cases} 0 & \text{when } n \text{ is odd} \\ 2^n(-1)^{\frac{n}{2}} & \text{when } n \text{ is even} \end{cases}$ 1A

$\sum_{r=0}^{n-1}(-1)^r C^{2n}_{2r+1} = 2^n\sin\frac{n\pi}{2}$

$= \begin{cases} 0 & \text{when } n \text{ is even} \\ 2^n(-1)^{\frac{n-1}{2}} & \text{when } n \text{ is odd} \end{cases}$ 1A

5. To show $2u_n = 2n - 1 - (-1)^n \ \forall n = 1, 2, \ldots$

For $n = 1$, L.H.S. $= 2u_1 = 0$

R.H.S. $= 2 - 1 - (-1) = 0$

Assume $2u_k = 2k - 1 - (-1)^k$ for some k. 1M

For $n = k+1$, L.H.S. $= 2u_{k+1}$

$= 2(2k - u_k)$

$= 2\left(2k - \frac{2k - 1 - (-1)^k}{2}\right)$

$= 4k - 2k + 1 + (-1)^k$

$= 2k + 1 - (-1)^{k+1}$

$= 2(k+1) - 1 - (-1)^{k+1}$ 1

$=$ R.H.S.

To find $\lim_{n\to\infty}\frac{u_n}{n}$:-

$\lim_{n\to\infty}\frac{u_n}{n} = \lim_{n\to\infty}\frac{2n - 1 - (-1)^n}{2n} = \lim_{n\to\infty}\left(1 - \frac{1}{2n} - \frac{(-1)^n}{2n}\right) = 1$ 1A

6. (a) To show f^{-1} is strictly increasing:-

$\forall x, y \in \mathbb{R}$,

if $x < y$,

then $f(x_0) < f(y_0)$ where $x = f(x_0)$, $y = f(y_0)$

$\Rightarrow \quad x_0 < y_0 \quad (\because x_0 \geq y_0 \Rightarrow f(x_0) \geq f(y_0) \Rightarrow x \geq y\ !!)$ 1

$\Rightarrow \quad f^{-1}(x) < f^{-1}(y)$

To show $a_1 < f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right) < a_2$:-

$\because \quad a_1 < a_k \quad (k = 2, \cdots, n)$

$\therefore \quad f(a_1) < f(a_k) \quad (k = 2, \cdots, n)$

$\Rightarrow \quad nf(a_1) < \sum_{k=1}^{n} f(a_k)$

$\Rightarrow \quad f(a_1) < \frac{1}{n}\sum_{k=1}^{n} f(a_k)$ 1

$\Rightarrow \quad f^{-1}(f(a_1)) < f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right)$

$\Rightarrow \quad a_1 < f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right)$

Similarly, we can show that

$f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right) < a_2$. 1

(b) To show $h^{-1}(x) = f^{-1}\left(\frac{x-q}{p}\right)$:-

$h(x) = pf(x) + q$

$\Rightarrow \quad \frac{h(x)-q}{p} = f(x)$

$\Rightarrow \quad x = f^{-1}\left(\frac{h(x)-q}{p}\right)$ 1

$\therefore \quad h^{-1}(x) = f^{-1}\left(\frac{x-q}{p}\right)$

Alternatively:

$h\left(f^{-1}\left(\frac{x-q}{p}\right)\right)$

$= \quad pf\left(f^{-1}\left(\frac{x-q}{p}\right)\right) + q$ 1

$= \quad p\left(\frac{x-q}{p}\right) + q$

$= \quad x$

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(b) To deduce that $h^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(a_k)\right) = f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right)$:-

$$h^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(a_k)\right) = f^{-1}\left(\frac{\frac{1}{n}\sum_{k=1}^{n} h(a_k) - q}{p}\right)$$

$$= f^{-1}\left(\frac{\frac{1}{n}\sum_{k=1}^{n} (pf(a_k) - q) - q}{p}\right)$$

$$= f^{-1}\left(\frac{\frac{1}{n}\sum_{k=1}^{n} pf(a_k) - q - q}{p}\right)$$

$$= f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right)$$

1

	Marks
7. (a) $\dfrac{C_r^n}{n^r} = \dfrac{n!}{(n-r)!\,r!\,n^r}$	1A
$= \dfrac{n(n-1)\cdots(n-r+1)}{r!\,n^r}$	
$= \dfrac{1}{r!}\cdot 1\cdot\left(1-\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)\cdots\left(1-\dfrac{r-1}{n}\right)$	
$\le \dfrac{1}{r!}$ (since $1-\dfrac{k}{n}\in(0,1)$ for $k=1,\cdots,r-1$)	1M
(b) $\left[(1+a_1)(1+a_2)\cdots(1+a_n)\right]^{\frac{1}{n}} \le \dfrac{(1+a_1)+(1+a_2)+\cdots+(1+a_n)}{n}$	
$\Rightarrow (1+a_1)(1+a_2)\cdots(1+a_n) \le \left(\dfrac{n+s}{n}\right)^n$	1A
$= \left(1+\dfrac{s}{n}\right)^n$	
$= 1+s+C_2^n\left(\dfrac{s}{n}\right)^2+C_3^n\left(\dfrac{s}{n}\right)^3+\cdots+C_n^n\left(\dfrac{s}{n}\right)^n$	1
$\le 1+s+\dfrac{1}{2!}s^2+\dfrac{1}{3!}s^3+\cdots+\dfrac{1}{n!}s^n$ (by (a))	
(c) Since $1+\dfrac{1}{2^k}>1$, $\{c_n\}$ is increasing	1M
To show c_n is bounded above:	
Put $a_k=\dfrac{1}{2^k}$ for $k=1,\cdots,n$, then $s=a_1+\cdots+a_n$	
$= \dfrac{1}{2}+\left(\dfrac{1}{2}\right)^2+\cdots+\left(\dfrac{1}{2}\right)^n$	
$\le (-1)+\left(1+\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^2+\cdots\right)$	
$= (-1)+\dfrac{1}{1-\frac{1}{2}} = 1$	1
$\therefore\ c_n = \prod_{k=1}^{n}(1+a_k)$	
$\le 1+s+\dfrac{s^2}{2!}+\cdots+\dfrac{s^n}{n!}$ (by (b))	
$\le 1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots+\dfrac{1}{n!}$	1M
$\le 1+1+\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{(n-1)n}$	
$= 1+1+\left(1-\dfrac{1}{2}\right)+\left(\dfrac{1}{2}-\dfrac{1}{3}\right)+\left(\dfrac{1}{3}-\dfrac{1}{4}\right)+\cdots+\left(\dfrac{1}{n-1}-\dfrac{1}{n}\right)$	1M
$= 1+1+1-\dfrac{1}{n}$	
$= 3-\dfrac{1}{n}$	
≤ 3	

8. (a) $|u + v|^2 = (u + v)\overline{(u + v)}$

$= (u + v)(\bar{u} + \bar{v})$

$= u\bar{u} + u\bar{v} + \bar{u}v + v\bar{v}$

$= |u|^2 + u\bar{v} + \overline{u\bar{v}} + |v|^2$

$= |u|^2 + 2\,\mathrm{Re}(u\bar{v}) + |v|^2$ 1A

$\geq |u|^2 - 2|u\bar{v}| + |v|^2$

$= |u|^2 - 2|u||v| + |v|^2$ 1

$= (|u| - |v|)^2$

(b) (i) If $v = 0$, then $0 \cdot u + 1 \cdot v = 0$ 1

If $v \neq 0$, then $u\bar{v} = \lambda \in R$

$\Rightarrow \dfrac{u\bar{v}v}{v} = \lambda \in R$

$\Rightarrow u|v|^2 = \lambda v, \quad \lambda \in R$ 1

$\Rightarrow |v|^2 u - \lambda v = 0$

$\Rightarrow \alpha u + \beta v = 0$

where $\alpha = |v|^2 \neq 0$, $\beta = -\lambda$, $\alpha, \beta \in R$ 1

(ii) $(|u| - |v|)^2 = |u|^2 - 2|u||v| + |v|^2$

$= u\bar{u} - 2\sqrt{u\bar{u}v\bar{v}} + v\bar{v}$

$= u\bar{u} - 2\sqrt{u\bar{v}\,\overline{u\bar{v}}} + v\bar{v}$

$= u\bar{u} - 2\sqrt{u\bar{v}u\bar{v}} + v\bar{v}$ ($\because u\bar{v} \in R$)

$= u\bar{u} - 2\sqrt{(u\bar{v})^2} + v\bar{v}$ 1A

$= \begin{cases} u\bar{u} - 2u\bar{v} + v\bar{v} & \text{if } u\bar{v} \geq 0 \\ u\bar{u} + 2u\bar{v} + v\bar{v} & \text{if } u\bar{v} < 0 \end{cases}$ 1

$= \begin{cases} u\bar{u} - u\bar{v} - \overline{u\bar{v}} + v\bar{v} & \text{if } u\bar{v} \geq 0 \\ u\bar{u} + u\bar{v} + \overline{u\bar{v}} + v\bar{v} & \text{if } u\bar{v} < 0 \end{cases}$

$= \begin{cases} (u - v)(\bar{u} - \bar{v}) & \text{if } u\bar{v} \geq 0 \\ (u + v)(\bar{u} + \bar{v}) & \text{if } u\bar{v} < 0 \end{cases}$

$= \begin{cases} (u - v)\overline{(u - v)} & \text{if } u\bar{v} \geq 0 \\ (u + v)\overline{(u + v)} & \text{if } u\bar{v} < 0 \end{cases}$ 1

$= \begin{cases} |u - v|^2 & \text{if } u\bar{v} \geq 0 \\ |u + v|^2 & \text{if } u\bar{v} < 0 \end{cases}$

9. (a) For $n = 1$, L.H.S. $= A^1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$ R.H.S. 1

Assume $A^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$ 1M

For $n = k + 1$,

$$A^{k+1} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\sin k\theta\cos\theta - \cos k\theta\sin\theta \\ \sin k\theta\cos\theta + \cos k\theta\sin\theta & \cos k\theta\cos\theta - \sin k\theta\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix}$$ 1

(b) (i). Let $X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $Y = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$.

(I) $$XY = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix} \in M$$ 1A

(II) $$YX = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$ 1A

$\therefore\ XY = YX$

(III) $X \ne \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a \ne 0$ or $b \ne 0$

$\Rightarrow \det X = a^2 + b^2 \ne 0 \quad \therefore X^{-1}$ exists. 1

$$X^{-1} = \frac{1}{a^2 + b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \left(\frac{a}{a^2 + b^2}\right) & -\left(\frac{-b}{a^2 + b^2}\right) \\ \left(\frac{-b}{a^2 + b^2}\right) & \left(\frac{a}{a^2 + b^2}\right) \end{pmatrix} \in M$$ 1A

(ii) Let $X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

<u>Case 1</u> : $a^2 + b^2 \ne 0$

$$X = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$= \sqrt{a^2 + b^2} \cdot \begin{pmatrix} \frac{a}{\sqrt{a^2 + b^2}} & \frac{-b}{\sqrt{a^2 + b^2}} \\ \frac{b}{\sqrt{a^2 + b^2}} & \frac{a}{\sqrt{a^2 + b^2}} \end{pmatrix}$$

$$= r \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ 1

Case 2 : $a^2 + b^2 = 0$

Then $a = b = 0$

$\Rightarrow \quad X = 0 \cdot \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$

$X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad r^n \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $X = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ 1M

$\Rightarrow \quad r^n \cdot \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad \begin{cases} r^n \cos n\theta = 1 \\ r^n \sin n\theta = 0 \end{cases}$

$\Rightarrow \quad r = 1, \ n\theta = 2k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow \quad r = 1, \ \theta = \dfrac{2k\pi}{n}, \quad k \in \mathbb{Z}$

$\therefore \quad X = \begin{pmatrix} \cos \frac{2k\pi}{n} & -\sin \frac{2k\pi}{n} \\ \sin \frac{2k\pi}{n} & \cos \frac{2k\pi}{n} \end{pmatrix}$ where $k = 0, 1, 2, \cdots, n-1$. 1A

(iii) To show that there exist $X \in M$

such that $Y = BX$ and $X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

Case 1 : $\det B = 0$

Then $Y^n = B^n$

$\Rightarrow \quad (\det Y)^n = (\det B)^n$

$\Rightarrow \quad \det Y = 0$

Let $Y = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$

then $a^2 + b^2 = 0$ and $c^2 + d^2 = 0$

$\Rightarrow \quad a = b = c = d = 0$

$\Rightarrow \quad Y = B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ 1

Hence $Y = BI$ where $I \in M$ and $I^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Case 2 : $\det B \neq 0$

Then B^{-1} exists and $Y^n = B^n$

$\Rightarrow \quad (B^{-1})^n Y^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad (B^{-1}Y)^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (by (b)(i)) 1

Hence $Y = B(B^{-1}Y)$ where $(B^{-1}Y)^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B^{-1}Y \in M$

Therefore $Y = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{2k\pi}{n} & -\sin\frac{2k\pi}{n} \\ \sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix}$

$= \begin{pmatrix} \cos\frac{2k\pi}{n} + 2\sin\frac{2k\pi}{n} & 2\cos\frac{2k\pi}{n} - \sin\frac{2k\pi}{n} \\ -2\cos\frac{2k\pi}{n} - \sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} + 2\sin\frac{2k\pi}{n} \end{pmatrix}$ 1

$k = 0, 1, \cdots, n-1$

10. (a) $\sum_{i=1}^{k} a_i(b_i - b_{i-1}) = \sum_{i=1}^{k}(a_i b_i - a_i b_{i-1})$

$= \sum_{i=1}^{k} a_i b_i - \sum_{i=1}^{k} a_i b_{i-1}$

$= a_k b_k + \sum_{i=1}^{k-1} a_i b_i - \sum_{i=1}^{k} a_i b_{i-1}$ 1M

$= a_k b_k + \sum_{i=1}^{k-1} a_i b_i - \sum_{i=2}^{k} a_i b_{i-1} \quad (\because b_0 = 0)$ 1

$= a_k b_k + \sum_{i=1}^{k-1} a_i b_i - \sum_{i=1}^{k-1} a_{i+1} b_i$ 1

$= a_k b_k + \sum_{i=1}^{k-1} (a_i b_i - a_{i+1} b_i)$ 1

$= a_k b_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i$

(b) $\left|\sum_{i=1}^{k} a_i(b_i - b_{i-1})\right| = \left|a_k b_k + \sum_{i=1}^{k-1}(a_i - a_{i+1}) b_i\right|$ 1

$\le |a_k||b_k| + \left|\sum_{i=1}^{k-1}(a_i - a_{i+1}) b_i\right|$ 1

$\le |a_k||b_k| + \sum_{i=1}^{k-1}|a_i - a_{i+1}||b_i|$ 1

$\le |a_k|K + \sum_{i=1}^{k-1}|a_i - a_{i+1}|K \quad (\because b_i \le K)$ 1

$= K\left\{|a_k| + \sum_{i=1}^{k-1}|a_i - a_{i+1}|\right\}$

$= K\left\{|a_k| + \sum_{i=1}^{k-1}(a_i - a_{i+1})\right\} \quad (\because a_i \ge a_{i+1})$ 1

$= K\left\{|a_k| + \sum_{i=1}^{k-1} a_i - \sum_{i=2}^{k} a_i\right\}$

$= K\{|a_k| + a_1 - a_k\}$ 1

$\le K\{|a_k| + |a_1| + |a_k|\}$

$= K\{|a_1| + 2|a_k|\}$

(c) $\left|\sum_{i=0}^{p-1} \frac{(-1)^i}{i}\right| = \left|\sum_{j=1}^{p-1} \frac{(-1)^{n-1-j}}{(n-1+j)}\right|$ $(i = n-1-j)$ 1M

$= \left|\sum_{j=1}^{p-1} (-1)^{n-1} \frac{(-1)^j}{(n-1+j)}\right|$

$= \left|\sum_{j=1}^{p-1} \frac{(-1)^j}{(n-1+j)}\right|$ 1A

$= \left|\sum_{j=1}^{p-1} a_j (b_j - b_{j-1})\right|$

where $a_j = \frac{1}{n-1-j}$, $b_j = \frac{1}{2}(-1)^j$ 1M

$\le \frac{1}{2}\cdot\{|a_1| + 2|a_{p-1}|\}$ (by (b)) 1M

$= \frac{1}{2}\left(\frac{1}{n} + 2\left(\frac{1}{n-p}\right)\right)$ 1A

$\le \frac{1}{2}\left(\frac{1}{n} + \frac{2}{n}\right)$

$= \frac{3}{2n}$

11. (a) $y^2 - 2ya^n\cos n\theta + a^{2n} = 0$

$(y - a^n(\cos n\theta + i\sin n\theta))(y - a^n(\cos n\theta - i\sin n\theta)) = 0$ 1

$\therefore \quad y = a^n(\cos n\theta + i\sin n\theta)$ or $a^n(\cos n\theta - i\sin n\theta)$ 1A

$x^{2n} - 2x^n a^n\cos n\theta + a^{2n}$

$= (x^n - a^n(\cos n\theta + i\sin n\theta))(x^n - a^n(\cos n\theta - i\sin n\theta))$ 1

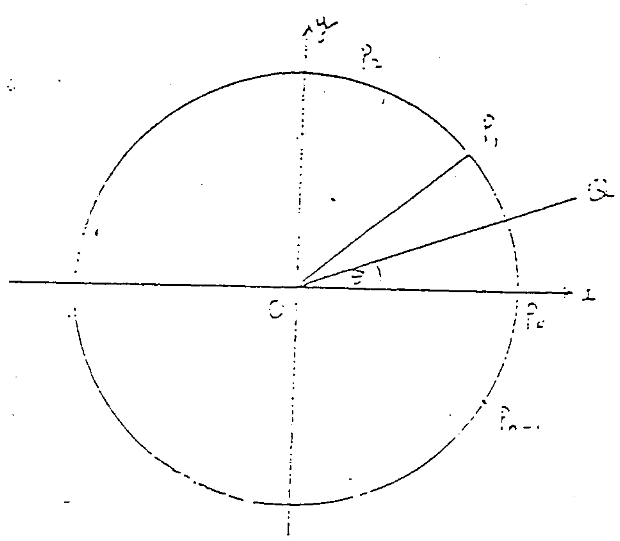
$= \left\{\prod_{r=0}^{n-1}\left(x - a\left(\cos\left(\frac{n\theta + 2r\pi}{n}\right) + i\sin\left(\frac{n\theta + 2r\pi}{n}\right)\right)\right)\right\}$ 1

$\left\{\prod_{r=0}^{n-1}\left(x - a\left(\cos\left(\frac{n\theta + 2r\pi}{n}\right) - i\sin\left(\frac{n\theta + 2r\pi}{n}\right)\right)\right)\right\}$ 1

$= \prod_{r=0}^{n-1}\left[\left(x - a\left(\cos\left(\theta + \frac{2r\pi}{n}\right) + i\sin\left(\theta + \frac{2r\pi}{n}\right)\right)\right)\left(x - a\left(\cos\left(\theta + \frac{2r\pi}{n}\right) - i\sin\left(\theta + \frac{2r\pi}{n}\right)\right)\right)\right]$ 1

$= \prod_{r=0}^{n-1}\left(x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right)$

(b) (i)



$\prod_{r=0}^{n-1} d_r^2 = \prod_{r=0}^{n-1}\left(OQ^2 + OP_r^2 - 2(OQ)(OP_r)\cos\left(\frac{2r\pi}{n} - \theta\right)\right)$ (cosine rule) 1M

$= \prod_{r=0}^{n-1}\left(x^2 + a^2 - 2xa\cos\left(\frac{2r\pi}{n} - \theta\right)\right)$ 1

$= x^{2n} - 2x^n a^n\cos n(-\theta) + a^{2n}$ (by (a)) 1

$= x^{2n} - 2x^n a^n\cos n\theta + a^{2n}$

(ii) Q lies on the positive real axis $\Rightarrow \theta = 0$ 1A

$\therefore \quad \prod_{r=0}^{n-1} d_r^2 = x^{2n} - 2x^n a^n \cos 0 + a^{2n}$ 1M

$= x^{2n} - 2x^n a^n + a^{2n}$

$= (x^n - a^n)^2$ 1

$\therefore \quad \prod_{r=0}^{n-1} d_r = |x^n - a^n|$

(iii) $\theta = \frac{2\pi}{2n} = \frac{\pi}{n}$ 1A

$\therefore \quad \prod_{r=0}^{n-1} d_r^2 = x^{2n} - 2x^n a^n \cos\left(n \cdot \frac{\pi}{n}\right) + a^{2n}$ 1M

$= x^{2n} + 2x^n a^n + a^{2n}$

$= (x^n + a^n)^2$

$\Rightarrow \quad \prod_{r=0}^{n-1} d_r = x^n + a^n$

12. (a) $\vec c \cdot \vec a = 0$

$\Rightarrow (\alpha\vec a + \beta\vec b)\cdot\vec a = 0$

$\Rightarrow \alpha\vec a\cdot\vec a + \beta\vec b\cdot\vec a = 0$ ....(1) 1A

$\vec c\cdot\vec b = 1$

$\Rightarrow (\alpha\vec a + \beta\vec b)\cdot\vec b = 1$

$\Rightarrow \alpha\vec a\cdot\vec b + \beta\vec b\cdot\vec b = 1$ ....(2) 1A

from (1) and (2), we have

$$\alpha = \frac{\begin{vmatrix} 0 & \vec b\cdot\vec a \\ 1 & \vec b\cdot\vec b \end{vmatrix}}{\begin{vmatrix} \vec a\cdot\vec a & \vec b\cdot\vec a \\ \vec a\cdot\vec b & \vec b\cdot\vec b \end{vmatrix}}$$

$$= \frac{-\vec b\cdot\vec a}{(\vec a\cdot\vec a)(\vec b\cdot\vec b) - (\vec a\cdot\vec b)^2}$$ 1A

$$\beta = \frac{\begin{vmatrix} \vec a\cdot\vec a & 0 \\ \vec a\cdot\vec b & 1 \end{vmatrix}}{\begin{vmatrix} \vec a\cdot\vec a & \vec b\cdot\vec a \\ \vec a\cdot\vec b & \vec b\cdot\vec b \end{vmatrix}}$$

$$= \frac{\vec a\cdot\vec a}{(\vec a\cdot\vec a)(\vec b\cdot\vec b) - (\vec a\cdot\vec b)^2}$$ 1A

(b) (i) $(\vec x - \vec c)\cdot\vec a = \vec x\cdot\vec a - \vec c\cdot\vec a$ 1M

$= 0 - 0$

$= 0$

$\therefore \vec x - \vec c \perp \vec a$

$(\vec x - \vec c)\cdot\vec b = \vec x\cdot\vec b - \vec c\cdot\vec b$

$= 1 - 1$ 1M

$= 0$

$\therefore \vec x - \vec c \perp \vec b$

(ii) $\vec x - \vec c \parallel \vec a\times\vec b$ ($\because \vec x - \vec c \perp \vec a, \vec b$) 1M

$\Rightarrow \vec x - \vec c = \lambda(\vec a\times\vec b)$ for some $\lambda\in\mathbb{R}$ 1M

$\Rightarrow \vec x = \vec c + \lambda(\vec a\times\vec b)$ for some $\lambda\in\mathbb{R}$

(iii) By (ii), $\vec x = \vec c + \lambda(\vec a\times\vec b)$ for some $\lambda\in\mathbb{R}$.

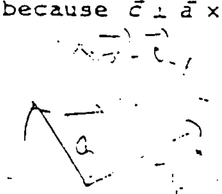
$\Rightarrow |\vec x|^2 = |\vec c + \lambda(\vec a\times\vec b)|^2$ 1A

$= |\vec c|^2 + |\lambda(\vec a\times\vec b)|^2$ 1M

(by Pythagora's theorem, because $\vec c \perp \vec a\times\vec b$)

$\geq |\vec c|^2$

$\therefore |\vec x| \geq |\vec c|$



(c) Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$.

$a_1b_2 \neq a_2b_1 \Rightarrow \vec{a}, \vec{b}$ are linearly independent

Consider $\vec{x} = \left(\dfrac{-a_2}{a_1b_2 - a_2b_1}\right)\vec{i} + \left(\dfrac{a_1}{a_1b_2 - a_2b_1}\right)\vec{j} + 0\vec{k}$ 1M

then $\vec{x}\cdot\vec{a} = \dfrac{-a_2a_1}{a_1b_2 - a_2b_1} + \dfrac{a_1a_2}{a_1b_2 - a_2b_1} + 0 = 0$ 1M

and $\vec{x}\cdot\vec{b} = \dfrac{-a_2b_1}{a_1b_2 - a_2b_1} + \dfrac{a_1b_2}{a_1b_2 - a_2b_1} + 0$

$= \dfrac{-a_2b_1 + a_1b_2}{a_1b_2 - a_2b_1} = 1$ 1M

by (b)(iii), $|\vec{x}| \geq |\vec{c}| \Rightarrow |\vec{x}|^2 \geq |\vec{c}|^2$

Now $|\vec{x}|^2 = \left(\dfrac{-a_2}{a_1b_2 - a_2b_1}\right)^2 + \left(\dfrac{a_1}{a_1b_2 - a_2b_1}\right)^2$ 1A

it remains to show $|\vec{c}|^2 = \dfrac{\sum_{r=1}^{2} a_r^2}{\left(\sum_{r=1}^{2} a_r^2\right)\left(\sum_{r=1}^{2} b_r^2\right) - \left(\sum_{r=1}^{2} a_rb_r\right)^2}$:

$|\vec{c}|^2 = \vec{c}\cdot\vec{c}$

$= \vec{c}\cdot(\alpha\vec{a} + \beta\vec{b})$

$= \alpha\vec{c}\cdot\vec{a} + \beta\vec{c}\cdot\vec{b}$

$= \alpha 0 + \beta 1$

$= \beta$

$= \dfrac{\vec{a}\cdot\vec{a}}{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b}) - (\vec{a}\cdot\vec{b})^2}$ (by (a)) 1M

$= \dfrac{\sum_{r=1}^{3} a_r^2}{\left(\sum_{r=1}^{3} a_r^2\right)\left(\sum_{r=1}^{3} b_r^2\right) - \left(\sum_{r=1}^{3} a_rb_r\right)^2}$

12

Solutions | Marks

1\. (a) $\lim_{x\to 0}\frac{\tan x - x}{x^2\sin x} = \lim_{x\to 0}\frac{\sec^2 x - 1}{2x\sin x + x^2\cos x}$

$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{2\sin x + 2x\cos x + 2x\cos x - x^2\sin x}$

$= \lim_{x\to 0}\frac{2\sec^2 x\sin x}{2\sin x + 4x\cos x - x^2\sin x}$

$= \lim_{x\to 0}\frac{2(\sec^2 x)\left(\frac{\sin x}{x}\right)}{2\left(\frac{\sin x}{x}\right) + 4\cos x - x\sin x}$ 1M

$= \frac{2(1)(1)}{2(1)+4(1) - (0)}$

$= \frac{2}{6}$

$= \frac{1}{3}$ 1A

(b) $\left|x\sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right|$

$\leq |x|\cdot 1$ 1

$= |x| \quad \forall x \neq 0$

$-|x| \leq x\sin\frac{1}{x} \leq |x|$

$\because \lim_{x\to 0}|x| = 0$

by Squeezing theorem,

$\lim_{x\to 0} x\sin\frac{1}{x} = 0$ 1

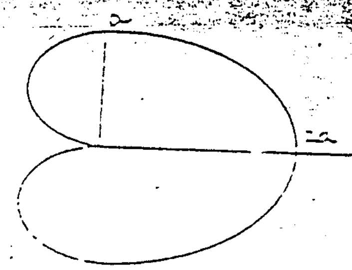
Hence, $\lim_{x\to 0}\frac{\frac{1}{x}+\sin\frac{1}{x}}{\frac{1}{x}-\sin\frac{1}{x}} = \lim_{x\to 0}\frac{1 + x\sin\frac{1}{x}}{1 - x\sin\frac{1}{x}}$

$= \frac{1+0}{1-0}$

$= 1$ 1A

Solutions	Marks

2.



2

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta$$ 1

$$= \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \frac{\cos 2\theta + 1}{2}\,d\theta$$ 1M

$$= \frac{a^2}{4}\int_0^{2\pi} 3 + 4\cos\theta + \cos 2\theta\,d\theta$$

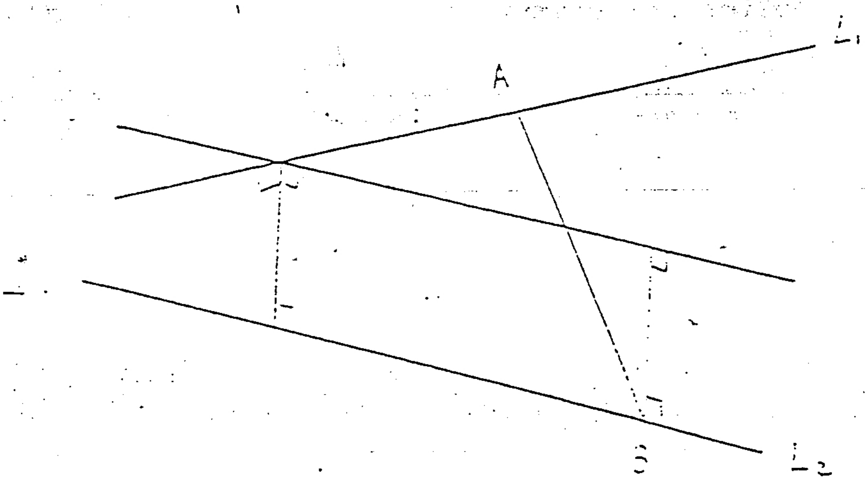
$$= \frac{a^2}{4}\left\{ [3\theta+4\sin\theta]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}\cos 2\theta\,d(2\theta)\right\}$$

$$= \frac{a^2}{4}\left\{ [3\theta+4\sin\theta]_0^{2\pi} + \frac{1}{2}[\sin 2\theta]_0^{2\pi}\right\}$$

$$= \frac{a^2}{4}(6\pi + 0)$$

$$= \frac{3}{2}\pi a^2$$ 1A

Solutions	Marks
3. Let the lines be (L_1) and (L_2)	
$(L_1) \perp (L_2)$	
$\Rightarrow \quad (\hat{i} + p\hat{j} + \hat{k}) \cdot (\hat{i} - \hat{j} + q\hat{k}) = 0$	1M
$\Rightarrow \quad 1 - p + q = 0$ ....(1)	1A



$A = (2, 4, 4) \in (L_1)$

$B = (0, 3, 2) \in (L_2)$

Shortest distance between (L_1) and (L_2)

$= \vec{AB} \cdot (\hat{i} + p\hat{j} + \hat{k}) \times (\hat{i} - \hat{j} + q\hat{k}) / \sqrt{(2 + p^2)(2 + q^2)}$ 1M

$= \begin{vmatrix} 2 & 1 & 2 \\ 1 & p & 1 \\ 1 & -1 & q \end{vmatrix} \div \sqrt{(2 + p^2)(2 + q^2)}$ 1

$= (2pq - 2 + 1 - 2p - q - 2) / \sqrt{(2 + p^2)(2 + q^2)}$

$= (2pq - 2p - q + 1) / \sqrt{(2 + p^2)(2 + q^2)}$

$\because \quad (L_1)$ and (L_2) are coplanar,

$\therefore \quad 2pq - 2p - q + 1 = 0$ ....(2) 1A

3.

Alternatively:

Let $(\alpha, \beta, \gamma) \in (L_1) \cap (L_2)$

then $\begin{cases} \dfrac{\alpha-2}{1} = \dfrac{\beta-4}{p} = \dfrac{\gamma-4}{1} = s \\ \dfrac{\alpha}{1} = \dfrac{\beta-3}{-1} = \dfrac{\gamma-2}{q} = t \end{cases}$ for some $s, t \in R$

$\Rightarrow \begin{cases} \alpha = 2+s = t \\ \beta = 4+ps = 3-t \\ \gamma = 4+s = 2+qt \end{cases}$ 1

$\Rightarrow \begin{cases} s-t+2=0 \\ ps+t+1=0 \\ s-qt+2=0 \end{cases}$

$\Rightarrow \begin{cases} x-y+z=0 \\ px+y+z=0 \\ x-qy+2z=0 \end{cases}$ has non-trivial solution 1M

$\Rightarrow \begin{vmatrix} 1 & -1 & 2 \\ p & 1 & 1 \\ 1 & -q & 2 \end{vmatrix} = 0$

$\Rightarrow$ $2pq - 2p - q + 1 = 0$ ....(2) 1A

From (2), $2p(q-1) - (q-1) = 0$

$(2p-1)(q-1) = 0$

$p = \frac{1}{2}$ or $q = 1$

at $p = \frac{1}{2}$, (1) $\Rightarrow q = -\frac{1}{2}$

at $q = 1$, (1) $\Rightarrow p = 2$

Therefore, $\begin{cases} p = 2 \\ q = 1 \end{cases}$ or $\begin{cases} p = \frac{1}{2} \\ q = -\frac{1}{2} \end{cases}$ 1A

Marks

4. $\int_0^2 xe^{|x-1|}dx = \int_0^1 xe^{1-x}dx + \int_1^2 xe^{x-1}dx$

$= e\int_0^1 xe^{-x}dx + e^{-1}\int_1^2 xe^{x}dx$ 1M+1M

$= e\int_0^1 (-x)e^{-x}d(-x) + \frac{1}{e}\int_1^2 xe^{x}dx$

$= e\int_0^{-1} ye^{y}dy + \frac{1}{e}\int_1^2 xe^{x}dx$

Now $\int xe^{x}dx = \int x\,d(e^{x})$

$= xe^{x} - \int e^{x}dx$

$= xe^{x} - e^{x} + c$

$\therefore \int_0^{-1} ye^{y}dy = [ye^{y} - e^{y}]_0^{-1}$ 1A

$= (-\frac{1}{e} - \frac{1}{e}) - (0 - 1)$

$= 1 - \frac{2}{e}$

$\int_1^2 xe^{x}dx = [xe^{x} - e^{x}]_1^2$

$= (2e^2 - e^2) - (e - e)$

$= e^2$

$\therefore$ answer $= e(1 - \frac{2}{e}) + \frac{1}{e}(e^2)$

$= e - 2 + e$

$= 2e - 2$ 1A

5. $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2n^2 + k^2}{n^3 + k^3} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{2 + (\frac{k}{n})^2}{1 + (\frac{k}{n})^3}\cdot\frac{1}{n}$ 1

$= \int_0^1 \frac{2 + x^2}{1 + x^3}dx$ 1M+1A

$= \int_0^1 \frac{1}{1 + x} + \frac{1}{x^2 - x + 1}dx$ 1A

$= [\ln(1 + x)]_0^1 + \int_0^1 \frac{1}{(x - \frac{1}{2})^2 + \frac{3}{4}}dx$ 2

$= \ln 2 + \frac{1}{\sqrt{\frac{3}{4}}}\int_0^1 \frac{1}{1 + \left(\sqrt{\frac{4}{3}}(x - \frac{1}{2})\right)^2}d\left(\sqrt{\frac{4}{3}}(x - \frac{1}{2})\right)$

$= \ln 2 + \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{\frac{4}{3}}(x - \frac{1}{2})\right]_0^1$

$= \ln 2 + \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$

$= \ln 2 + \frac{2}{\sqrt{3}}\frac{\pi}{3}$ 1A

Solutions	Marks
6. Let $P = (c, 2a)$	
$QR: a^2 = cx + 2ay$	
$\Rightarrow$ slope of $QR = -\dfrac{c}{2a}$	1A
Since $QR \perp OM$, where $M(p, q)$ is the mid-point of QR, we have	
$\left(-\dfrac{c}{2a}\right)\left(\dfrac{q}{p}\right) = -1$	1M
$\Rightarrow c = \dfrac{2ap}{q}$ ....(1)	1
Let $Q = (x_1, y_1)$ and $R = (x_2, y_2)$.	
Then $a^2 = cx_1 + 2ay_1$	
and $a^2 = cx_2 + 2ay_2$	
$\Rightarrow 2a^2 = c(x_1 + x_2) + 2a(y_1 + y_2)$	
$2a^2 = 2cp + 4aq$ ....(2)	1
Substituting (1) into (2),	
$2a^2 = 2\left(\dfrac{2ap}{q}\right)p + 4aq$	
$\Rightarrow aq = 2p^2 + 2q^2$	
$\therefore$ M lies on the circle $2x^2 + 2y^2 - ay = 0$.	
$x^2 + y^2 - \dfrac{a}{2}y = 0$	
$x^2 + \left(y - \dfrac{a}{4}\right)^2 = \left(\dfrac{a}{4}\right)^2$	
$\therefore$ centre $= \left(0, \dfrac{a}{4}\right)$	1A
radius $= \dfrac{a}{4}$	1A

Solutions	Marks

7\. (a) $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$ 1

Now $f(0) = f(0 + 0)$

$= f(0) + f(0) + 0$

$= 2f(0)$

$\Rightarrow f(0) - 2f(0) = 0$

$\Rightarrow -f(0) = 0$

$\Rightarrow f(0) = 0$ 1

$\therefore f'(0) = \lim_{h \to 0} \frac{f(h) - 0}{h}$

$= \lim_{h \to 0} \frac{f(h)}{h}$

> Alternatively,
>
> $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$ 1
>
> $= \lim_{h \to 0} \frac{f(h) + f(0) - 0 - f(0)}{h}$ 1
>
> $= \lim_{h \to 0} \frac{f(h)}{h}$

(b) $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{f(x) + f(h) + 3xh(x + h) - f(x)}{h}$ 1

$= \lim_{h \to 0} \frac{f(h)}{h} + \lim_{h \to 0} 3x(x + h)$ 1

$= f'(0) + 3x^2$

Integrating, $f(x) = f'(0)x + x^3 + c$ 1M

Put $x = 0$, $0 = f(0) = 0 + 0 + c \Rightarrow c = 0$ 1

$\therefore f(x) = f'(0)x + x^3$.

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Solutions	Marks
8. (a) $f'(x) = \frac{d}{dx}(xe^{-x^2})$	
$= e^{-x^2} + xe^{-x^2}\frac{d}{dx}(-x^2)$	
$= e^{-x^2} - 2x^2e^{-x^2}$	
$= e^{-x^2}(1 - 2x^2)$	1A
$f''(x) = \frac{d}{dx}(e^{-x^2}(1 - 2x^2))$	
$= -4xe^{-x^2} + (1 - 2x^2)e^{-x^2}(-2x)$	
$= -4xe^{-x^2} - 2x(1 - 2x^2)e^{-x^2}$	
$= -2xe^{-x^2}(2 + 1 - 2x^2)$	
$= -2xe^{-x^2}(3 - 2x^2)$	1A
(b) (i) $f'(x) = 0$	
$e^{-x^2}(1 - 2x^2) = 0$	
$x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$	$\frac{1}{2}$A
(ii) $f'(x) > 0$	$\frac{1}{2}$A
$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	
(iii) $f'(x) < 0$	
$x < -\frac{1}{\sqrt{2}}$ or $x > \frac{1}{\sqrt{2}}$	$\frac{1}{2}$A
(iv) $f''(x) = 0$	
$-2xe^{-x^2}(3 - 2x^2) = 0$	
$x = 0$ or $\frac{\sqrt{3}}{\sqrt{2}}$ or $-\frac{\sqrt{3}}{\sqrt{2}}$	$\frac{1}{2}$A
(v) $f''(x) > 0$	
$-\frac{\sqrt{3}}{\sqrt{2}} < x < 0$ or $x > \frac{\sqrt{3}}{\sqrt{2}}$	$\frac{1}{2}$A
(vi) $f''(x) < 0$	
$x < -\frac{\sqrt{3}}{\sqrt{2}}$ or $0 < x < \frac{\sqrt{3}}{\sqrt{2}}$	$\frac{1}{2}$A

Solutions	Marks

(c)

x	$(-\infty, -\frac{3}{\sqrt{2}})$	$-\frac{3}{\sqrt{2}}$	$(-\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$-\frac{1}{\sqrt{2}}$	$(-\frac{1}{\sqrt{2}}, 0)$	0	$(0, \frac{1}{\sqrt{2}})$	$\frac{1}{\sqrt{2}}$	$(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}})$	$\frac{3}{\sqrt{2}}$	$(\frac{3}{\sqrt{2}}, \infty)$
f'	−	−	−	0	+	−	−	0	−	−	−
f''	−	0	−	−	+	0	−	−	−	0	−
f	↘	Pt of inflexion	↘	min	↗	0	↗	max	↘	Pt of inflexion	↘

From the table, we have

$$\text{minimum point} = \left(-\frac{1}{\sqrt{2}}, f(-\frac{1}{\sqrt{2}})\right)$$

$$= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}\right)$$ 1A

$$\text{maximum point} = \left(\frac{1}{\sqrt{2}}, f(\frac{1}{\sqrt{2}})\right)$$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}e^{-\frac{1}{2}}\right)$$ 1A

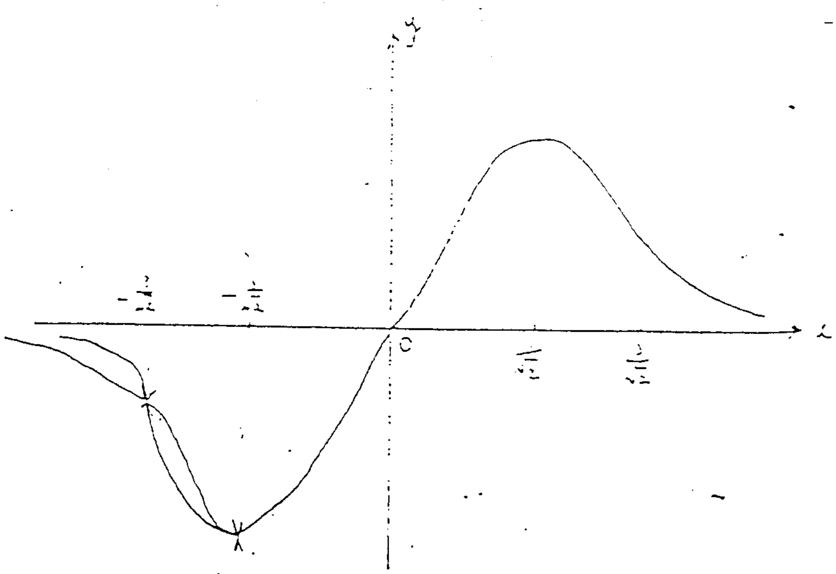
Points of inflexion are

$$\left(-\frac{3}{\sqrt{2}}, f(-\frac{3}{\sqrt{2}})\right), \left(\frac{3}{\sqrt{2}}, f(\frac{3}{\sqrt{2}})\right) \text{ and } (0, f(0))$$

$$= \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}e^{-\frac{9}{2}}\right), \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}e^{-\frac{9}{2}}\right) \text{ and } (0, 0)$$ 1A

(d) asymptote : $y = 0$ 1A

(e)



3

Solutions | Marks

(f) Consider $\begin{cases} x' = (x - y)\left(\frac{1}{\sqrt{2}}\right) \\ y' = (x + y)\left(\frac{1}{\sqrt{2}}\right) \end{cases}$

then $x + y = (x - y)\, e^{-\frac{1}{2}(x-y)^2}$

→ $y' = x' e^{-x'^2}$ which has the same graph as in (e) | 1M

On the other hand,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. Each (x, y) is transformed to (x', y') by a rotation of 45° anticlockwise. | 1M

→ the graph $x + y = (x - y)\, e^{-\frac{1}{2}(x-y)^2}$ is obtained by rotating the graph in (e) 45° clockwise.

$x + y = (x-y)e^{-\frac{1}{2}(x-y)^2}$

Put $u = \frac{1}{\sqrt{2}}(x-y)$

$\therefore x - y = \sqrt{2}u$

Put $\sqrt{2}v = x + y$

$\therefore$ it becomes

$\sqrt{2}v = \sqrt{2}ue^{-u^2}$

i.e. $v = ue^{-u^2}$

$\begin{cases} u = \frac{1}{\sqrt{2}}(x-y) = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\ v = \frac{1}{\sqrt{2}}(x+y) = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{cases}$

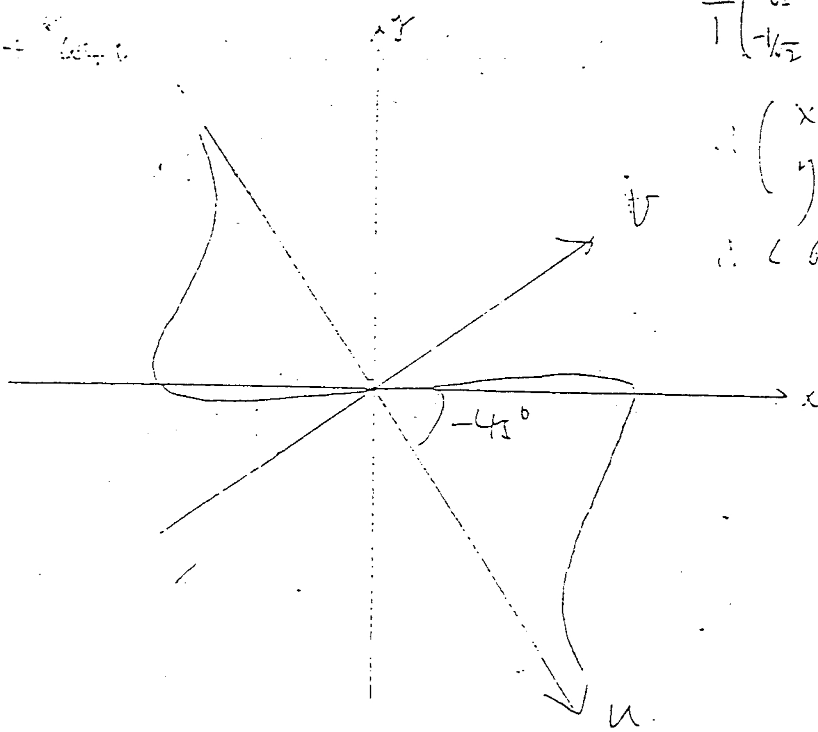
i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{1}{1}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

$\therefore \angle$ of rotation = −45°

$\begin{cases} x = X\cos\theta - Y\sin\theta \\ y = X\sin\theta + Y\cos\theta \end{cases}$



Solution | Marks

9. (a) $\int_0^x (x-t)^p g'(t)\,dt = \int_0^x (x-t)^p d(g(t))$ 1M

$= [(x-t)^p g(t)]_0^x - \int_0^x g(t)\,d((x-t)^p)$ 1M

$= -x^p g(0) + p\int_0^x g(t)(x-t)^{p-1}\,dt$

(b) $n = 1$, 1

L.H.S. $= e^x$

R.H.S. $= 1 + \frac{1}{0!}\int_0^x e^t\,dt$

$= e^x$

Assume $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} e^t\,dt$ 1

Then $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!\,n}\left\{n\int_0^x (x-t)^{n-1} e^t\,dt\right\}$

$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{n!}\left\{x^n e^0 + \int_0^x (x-t)^n e^t\,dt\right\}$ 1
(by (a))

$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \frac{1}{n!}\int_0^x (x-t)^n e^t\,dt$

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(2n)!} + \frac{1}{(2n)!}\int_0^1 (1-t)^{2n} e^t\,dt$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(2n)!} + \frac{1}{(2n)!}\int_0^{-1} (-1-t)^{2n} e^t\,dt$ 1A

Adding, 1M

$e + \frac{1}{e}$

$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2n)!}\right) + \frac{1}{(2n)!}\left\{\int_0^1 (1-t)^{2n} e^t\,dt + \int_0^{-1} (-1-t)^{2n} e^t\,dt\right\}$

$\therefore \quad \left|\left(e + \frac{1}{e}\right) - 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2n)!}\right)\right|$

$= \frac{1}{(2n)!}\left|\int_0^1 (1-t)^{2n} e^t\,dt - \int_{-1}^0 (-1-t)^{2n} e^t\,dt\right|$

$\le \frac{1}{(2n)!}\left\{\int_0^1 |1-t|^{2n} e^t\,dt + \int_{-1}^0 |-1-t|^{2n} e^t\,dt\right\}$

$\le \frac{1}{(2n)!}\left\{\int_0^1 e^t\,dt + \int_{-1}^0 e^t\,dt\right\}$ 1M

$= \frac{1}{(2n)!}\left\{[e^t]_0^1 + [e^t]_{-1}^0\right\}$

$= \frac{1}{(2n)!}\left[(e-1) + \left(1 - \frac{1}{e}\right)\right]$

$< \frac{e}{(2n)!}$ 1A

$< \frac{3}{(2n)!}$

(c) (i) $f_n(x) = \int_0^x f_{n-1}(t)\,dt$

$= 1\int_0^x (x-t)^{1-1} f_{n-1}(t)\,dt$

$= x^1 f_{n-1}(0) + \int_0^x (x-t)^1 \frac{d}{dt} f_{n-1}(t)\,dt$ (by (a)) 1M

$= \int_0^x (x-t)^1 f_{n-2}(t)\,dt$

$= \frac{1}{2}\left\{2\int_0^x (x-t)^1 f_{n-2}(t)\,dt\right\}$

$= \frac{1}{2}\left\{x^2 f_{n-2}(0) + \int_0^x (x-t)^2 \frac{d}{dt} f_{n-2}(t)\,dt\right\}$ (by (a)) 1M

$= \frac{1}{2}\int_0^x (x-t)^2 f_{n-3}(t)\,dt$

$= (\frac{1}{3})(\frac{1}{2})\int_0^x (x-t)^3 f_{n-4}(t)\,dt$ (by similar arguments) 1M

$= \dots$

$= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f_0(t)\,dt$

(ii) Let $f_0(x) = |\sin(x^2)|$, a continuous function

Define $f_n(x) = \int_0^x f_{n-1}(t)\,dt$ for $n = 1, 2, \cdots$ 1M

by (c)(i), $f_n(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}|\sin(t^2)|\,dt$

$\Rightarrow$ $f_{100}(x) = \frac{1}{99!}\int_0^x (x-t)^{99}|\sin(t^2)|\,dt$

$\Rightarrow$ $f_{99}(x) = \frac{d}{dx} f_{100}(x) = \frac{1}{99!}\frac{d}{dx}\int_0^x (x-t)^{99}|\sin(t^2)|\,dt$ 1M

$\Rightarrow$ $f_{98}(x) = \frac{d}{dx} f_{99}(x) = \frac{1}{99!}\frac{d^2}{dx^2}\int_0^x (x-t)^{99}|\sin(t^2)|\,dt$

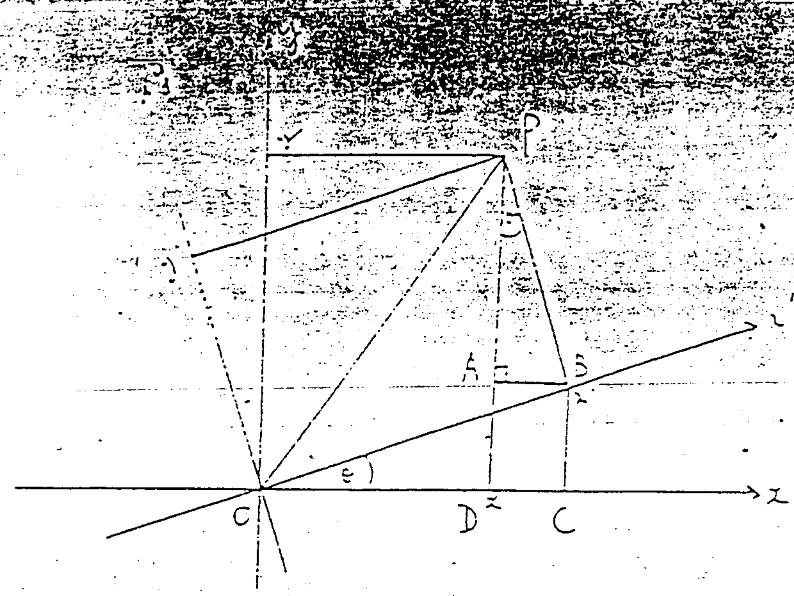
$\vdots$

$\Rightarrow$ $f_0(x) = \frac{d}{dx} f_1(x) = \frac{1}{99!}\frac{d^{100}}{dx^{100}}\int_0^x (x-t)^{99}|\sin(t^2)|\,dt$

$\therefore$ $\frac{d^{100}}{dx^{100}}\int_0^x (x-t)^{99}|\sin(t^2)|\,dt = 99!\,f_0(x)$

$= 99!\,|\sin(x^2)|$ 1A

10. (a) (i)



$y = PA + BC$

$\quad = PB\cos\theta + BO\sin\theta$

$\quad = y'\cos\theta + x'\sin\theta$ 1A

$x = CO - DC$

$\quad = OB\cos\theta - PB\sin\theta$

$\quad = x'\cos\theta - y'\sin\theta$ 1A

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'\cos\theta - y'\sin\theta \\ x'\sin\theta + y'\cos\theta \end{pmatrix}$ 1

$\quad = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$

(ii) Now $V = MV'$

$\therefore \quad (MV')^t A (MV') = C$ 1M

$\Rightarrow \quad V'^t (M^t A M) V' = C$

$\Rightarrow \quad V'^t A' V' = C \quad$ where $A' = M^t A M$ 1

also, $\det A' = \det(M^t A M)$

$\quad = \det M^t \det A \det M$

$\quad = 1 \cdot \det A \cdot 1$ 1

$\quad = \det A$

$A' = M^t A M$

$= \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ $(c = \cos\theta,\ s = \sin\theta)$

$= \begin{pmatrix} ac + sh & ch + bs \\ -sa + ch & -sh + cb \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$= \begin{pmatrix} ac^2 + sch + sch + bs^2 & -acs - s^2h + c^2h + cbs \\ -sac + c^2h - s^2h + scb & s^2a - sch - csh + c^2b \end{pmatrix}$ 1

$\therefore$ A' is diagonal when $-sac + c^2h - s^2h + scb = 0$ 1

i.e when $(b - a)cs = -(c^2 - s^2)h$

$\Rightarrow$ $\dfrac{b-a}{2}\sin 2\theta = -h\cos 2\theta$ ....(*) 1

if $b - a = 0$, the equation (*) is satisfied if $\cos 2\theta = 0$

if $b - a \neq 0$, the equation (*) is satisfied if $\tan 2\theta = -\dfrac{2h}{b-a}$ 1

$\therefore$ $\exists\, \theta \in R$ such that A' is diagonal

(b) The conic section is represented by

$(x\ y)\begin{pmatrix} 7 & h \\ h & 13 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 16$ in Γ

and $(x'\ y')\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 16$ in Γ'

where $\det\begin{pmatrix} 7 & h \\ h & 13 \end{pmatrix} = \det\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

i.e. $91 - h^2 = \lambda\mu$ 1A

(i) If the conic section is an ellipse, then

$91 - h^2 = \lambda\mu > 0$

$\Rightarrow$ $91 > h^2$

$\Rightarrow$ $-\sqrt{91} < h < \sqrt{91}$ 1A

(ii) If the conic section is a hyperbola, then

$91 - h^2 = \lambda\mu < 0$

$\Rightarrow$ $91 < h^2$

$\Rightarrow$ $h > \sqrt{91}$ or $h < -\sqrt{91}$ 1A

(iii) If the conic section is a pair of straight lines, then

$91 - h^2 = \lambda\mu = 0$

$\Rightarrow$ $h = \pm\sqrt{91}$ 1A

(iv) $\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $\Rightarrow$ $\lambda = 4,\ \mu = 16$

$\therefore$ $91 - h^2 = 64$

$\Rightarrow$ $h^2 = 27$

$\Rightarrow$ $h = \pm\sqrt{27} = \pm 3\sqrt{3}$ 1A

Solutions	Marks
11. (a) Let $\deg p(x) = m \ge n \ge 1$	
Divide $p(x)$ by $(x-a)^m$, we have	1M
$p(x) = c_0(x-a)^m + q_1(x)$ for some $c_0 \in R\backslash\{0\}$ and $\deg q_1(x) < m$	1A
Divide $q_1(x)$ by $(x-a)^{m-1}$, we have	
$q_1(x) = c_1(x-a)^{m-1} + q_2(x)$ for some $c_1 \in R$ and $\deg q_2(x) < m-1$	
Thus	
$p(x) = c_0(x-a)^m + c_1(x-a)^{m-1} + q_2(x)$ for some $q_2(x)$	1A
such that $\deg q_2(x) < m-1$	
Repeating the same arguments, we have	
$p(x) = c_0(x-a)^m + c_1(x-a)^{m-1} + \cdots + c_{m-1}(x-a) + c_m$, where $c_i \in R$	1
Putting $x = a$, we have $0 = p(a) = c_m \Rightarrow c_m = 0$	1M
Differentiating once and putting $x = a$, we have	1M
$0 = p'(a) = c_{m-1} \Rightarrow c_{m-1} = 0$	
Differentiating twice and putting $x = a$, we have	
$0 = p''(a) = 2c_{m-2} \Rightarrow c_{m-2} = 0$	
By the same arguments, since $p^{(k)}(a) = 0$ for $k = 0, 1, \cdots, n-1$,	1
we have $c_{m-k} = 0$ for $k = 0, 1, \cdots, n-1$.	
Therefore,	
$p(x) = c_0(x-a)^m + c_1(x-a)^{m-1} + \cdots + c_{m-n}(x-a)^{m-(m-n)}$	
$= c_0(x-a)^m + c_1(x-a)^{m-1} + \cdots + c_{m-n}(x-a)^n$	1
$= (x-a)^n[c_0(x-a)^{m-n} + \cdots + c_{m-n}]$	
$\therefore$ $p(x)$ is divisible by $(x-a)^n$.	

Solutions	Marks

Alternatively: (use M.I.)

For $n = 1$, let $p(x) = c_0(x - a) + r$ 1

put $x = a$, we have $r = 0$

$\therefore \quad (x - a) \mid p(x)$

Assume $p^{(k)}(a) = 0$ for $k = 0, \cdots, n - 1 \Rightarrow (x - a)^n \mid p(x)$ 1

Now if $p^{(k)}(a) = 0$ for $k = 0, \cdots, n$, 1

then by induction assumption, $p(x) = (x - a)^n q(x)$ for some $q(x)$ 1

differentiating n times and putting $x = a$, 1

we have $0 = p^{(n)}(a) = n!\,q(a)$

$\Rightarrow \quad q(a) = 0$ 1

$\Rightarrow \quad q(x) = (x - a)\, q_1(x)$ for some $q_1(x)$ 1

$\therefore \quad p(x) = (x - a)^{n+1} q_1(x)$ 1

i.e. $\quad (x - a)^{n+1} \mid p(x)$

(b) by (a), since $\deg F(x) \geq 4$, we need only to show

$F(1) = F'(1) = F''(1) = F'''(1) = 0$ 1M

To show $F(1) = 0$:

$$F(1) = \left(\int_1^1 pr\,dt\right)\left(\int_1^1 qs\,dt\right) - \left(\int_1^1 pq\,dt\right)\left(\int_1^1 rs\,dt\right) = 0$$ 1

To show $F'(1) = 0$:

$$F'(x) = pr\int_1^x qs\,dt + qs\int_1^x pr\,dt - pq\int_1^x rs\,dt - rs\int_1^x pq\,dt$$

$\therefore \quad F'(1) = 0$ 1

$$F''(x) = (pr)'\int_1^x qs\,dt + pqrs + (qs)'\int_1^x pr\,dt + pqrs$$

$$- (pq)'\int_1^x rs\,dt - pqrs - (rs)'\int_1^x pq\,dt - pqrs$$

$$= (pr)'\int_1^x qs\,dt + (qs)'\int_1^x pr\,dt - (pq)'\int_1^x rs\,dt - (rs)'\int_1^x pq\,dt$$ 1

$\therefore \quad F''(1) = 0$

$$F''(x) = (pr)''\int_1^x qs\,dt + (pr)'(qs)' + (qs)''\int_1^x pr\,dt + (pr)'(qs)'$$

$$- (pq)''\int_1^x rs\,dt - (ps)'(rs)' - (rs)''\int_1^x pq\,dt - (pq)'(rs)'$$

$$= (pr)''\int_1^x qs\,dt + (qs)''\int_1^x pr\,dt + (pqrs)''$$

$$- (pq)''\int_1^x rs\,dt - (rs)''\int_1^x pq\,dt - (pqrs)''$$ 1

$$= (pr)''\int_1^x qs\,dt + (qs)''\int_1^x pr\,dt - (pq)''\int_1^x rs\,dt - (rs)''\int_1^x pq\,dt$$ 1

$\therefore \quad F''(1) = 0$

12\. (a) Since $x > 0$, $\frac{1}{t^2} < \frac{1}{t} < 1 \quad \forall t \in (1, 1+x)$ — 1M

$$\therefore \int_1^{1+x} \frac{1}{t^2}\,dt < \int_1^{1+x} \frac{1}{t}\,dt < \int_1^{1+x} 1\,dt$$ — 1M

$$\Rightarrow \left[-\frac{1}{t}\right]_1^{1+x} < [\ln t]_1^{1+x} < [t]_1^{1+x}$$

$$\Rightarrow 1 - \frac{1}{1+x} < \ln(1+x) < x$$

$$\Rightarrow \frac{x}{1+x} < \ln(1+x) < x$$

Replace x by $\frac{1}{x}$, we have — 1M

$$\Rightarrow \frac{\frac{1}{x}}{1+\frac{1}{x}} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$

$$\Rightarrow \frac{1}{x+1} < \ln\left(1+\frac{1}{x}\right) < \frac{1}{x}$$

(b) $\ln f = x\ln\left(1+\frac{1}{x}\right)$

$$\Rightarrow \frac{1}{f}f' = \ln\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow f' = \left(1+\frac{1}{x}\right)^x\left\{\ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right\}$$ — 1M

> 0 (by (a))

Hence f is strictly increasing.

$\forall x > 0$, $f(x) = \left(1+\frac{1}{x}\right)^x$

$> (1+0)^x$ — 1M

$= 1$

Since $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ — 1M

$= e$

and f is strictly increasing, — 1M

we have $f(x) < e \quad \forall x > 0$.

(c) (i) $F_a' = f' - \frac{d}{dx}\int_x^a \frac{1}{t^2 f(t)}\,dt$

$= f' + \frac{d}{dx}\int_a^x \frac{1}{t^2 f(t)}\,dt$

$= f' + \frac{1}{x^2 f(x)}$

$> 0 \quad (\because f' > 0,\ x^2 > 0 \text{ and } f > 0)$ — 1M

$\therefore$ F_n is increasing.

Now $F_n(n) = f(n) - f(n-1) - \int_n^n \frac{1}{t^2 f(t)}\,dt$

$= f(n) - f(n-1) > 0$ ($\because f$ is increasing) 1M

and $F_n(n-1) = f(n-1) - f(n-1) - \int_{n-1}^n \frac{1}{t^2 f(t)}\,dt$

$= -\int_{n-1}^n \frac{1}{t^2 f(t)}\,dt < 0$ 1M

Since F_n is increasing and F_n has opposite signs at $n-1$, n, $\exists$ unique $\alpha_n \in R$ such that $F_n(\alpha_n) = 0$.

From the above argument, $\alpha_n \in (n-1, n)$

$\Rightarrow$ $\alpha_n > n-1$

$\therefore$ $\lim_{n\to\infty} \alpha_n = \infty$ 1M

(ii) $\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty}\left\{ f(x) - f(n-1) - \int_x^n \frac{1}{t^2 f(t)}\,dt \right\}$

$= f(x) - \lim_{n\to\infty} f(n-1) - \lim_{n\to\infty}\int_x^n \frac{1}{t^2 f(t)}\,dt$

$= f(x) - e - \lim_{n\to\infty}\int_x^n \frac{1}{t^2 f(t)}\,dt$ 1A

It remains to show $\lim_{n\to\infty}\int_x^n \frac{1}{t^2 f(t)}\,dt$ exists : 1A

Now $\int_x^n \frac{1}{t^2 f(t)}\,dt$ increases with n ($\because t^2, f > 0$) 1M

and $\int_x^n \frac{1}{t^2 f(t)}\,dt < \int_x^n \frac{1}{t^2}\,dt = \left[-\frac{1}{t}\right]_x^n$

$= \frac{1}{x} - \frac{1}{n}$

$< \frac{1}{x}$

13. (a) To show $a_{k+1} \le 2a_k$ for $k = 0, 1, 2, \cdots$:

For $k = 0$,

$$\begin{aligned} a_{k+1} &= a_1 \\ &= 1 \\ &\le 2 \cdot 1 \\ &= 2 \cdot a_0 \\ &= 2a_k \end{aligned}$$

For $k > 0$,

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \\ &\le a_k + a_k \quad (\because\ a_k > 0 \Rightarrow (a_k) \text{ is increasing}) \\ &= 2a_k \end{aligned}$$

1

1

To show $a_k \le 2^k$ for $k = 0, 1, 2, \cdots$:

For $k = 0$,

$$\begin{aligned} a_k &= a_0 \\ &= 1 \\ &\le 2^0 \\ &= 2^k \end{aligned}$$

For $k > 0$,

$$\begin{aligned} a_k &= \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \\ &\le 2 \cdot 2 \cdots 2 \quad (\because\ a_{k+1} \le 2a_k) \\ &= 2^k \end{aligned}$$

1M

To show $S_n(x) < 3$ for $n = 0, 1, 2, \cdots$:

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{n} a_k x^k \\ &< \sum_{k=0}^{n} 2^k \left(\frac{1}{3}\right)^k \quad (\because\ 0 < a_k \le 2^k \text{ and } x < \tfrac{1}{3}) \\ &= \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k \\ &< \frac{1}{1 - \frac{2}{3}} \\ &= 3 \end{aligned}$$

1M

(b) To show $\lim_{n\to\infty} S_n(x)$ exists:

Case 1 : $x \ge 0$

$\because$ $S_n(x)$ is increasing and bounded above 1M

$\therefore$ $\lim_{n\to\infty} S_n(x)$ exists.

Case 2 : $x < 0$

write $y = -x$

then $S_n(x) = \sum_{k=0}^{n} a_k(-y)^k$

$= U_n(x) - V_n(x)$ where $U_n(x) = \sum_{k=1,3,\ldots} a_k y^k$ 1M

and $V_n(x) = \sum_{k=0,2,4,\ldots} a_k y^k$

Since both $U_n(x)$ and $V_n(x)$ are monotonic increasing and bounded above, both $\lim_{n\to\infty} U_n(x)$ and $\lim_{n\to\infty} V_n(x)$ exist. 1M

$\therefore$ $\lim_{n\to\infty} S_n(x)$ exists.

To find $\lim_{n\to\infty} S_n(x)$:

$S_n(x) = \sum_{k=0}^{n} a_k x^k$

$= a_0 - a_1 x + \sum_{k=2}^{n} a_k x^k$

$= 1 - x + \sum_{k=2}^{n} (a_{k-1} + a_{k-2}) x^k$

$= 1 - x + \sum_{k=2}^{n} a_{k-1} x^k + \sum_{k=2}^{n} a_{k-2} x^k$

$= 1 - x + x \sum_{k=2}^{n} a_{k-1} x^{k-1} - x^2 \sum_{k=2}^{n} a_{k-2} x^{k-2}$

$= 1 - x + x\left(\sum_{k=0}^{n-1} a_k x^k - 1\right) - x^2 \sum_{k=0}^{n-2} a_k x^k$ 1M

Take $n \to \infty$, we have

$s = 1 + x - x(s - 1) - x^2 s$, where $s = \lim_{n\to\infty} S_n(x)$ 1A

$s = 1 + x - xs - x - x^2 s$

$s(1 - x - x^2) = 1$

$s = \dfrac{1}{1 - x^2 - x^2}$

Solutions	Marks
(c) (i) $\sum_{k=0}^{\infty} a_k \left(\frac{1}{5}\right)^k = \dfrac{1}{1-\frac{1}{5}-\frac{1}{25}}$	
$= \dfrac{25}{19}$	1A
(ii) $\sum_{k=0}^{\infty} (-1)^k a_k \left(\frac{1}{5}\right)^k = \dfrac{1}{1+\frac{1}{5}-\frac{1}{25}}$	
$= \dfrac{25}{29}$	1A
(iii) $\dfrac{25}{19}+\dfrac{25}{29} = \sum_{k=0}^{\infty} a_k \left(\frac{1}{5}\right)^k + \sum_{k=0}^{\infty} a_k \left(-\frac{1}{5}\right)^k$	1M
$= \sum_{k=0}^{\infty} a_{2k}\left[\left(\frac{1}{5}\right)^{2k} + \left(\frac{1}{5}\right)^{2k}\right]$	
$= 2\sum_{k=0}^{\infty} a_{2k}\left(\frac{1}{25}\right)^k$	
$\therefore \sum_{k=0}^{\infty} a_{2k-1}\left(\frac{1}{25}\right)^k = \frac{1}{2}\left[\frac{25}{19}+\frac{25}{29}\right]$	1A