92-AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1992

PURE MATHEMATICS PAPER I

9.00 am-12.00 noon (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B. Answer BOTH sections.
2. Section A: Answer ALL questions, using AL(C1) answer book.
3. Section B: Answer any FOUR questions, using AL(C2) answer book.

Section A (40 marks)
Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. Consider the following system of linear equations:

$$(*)\quad \begin{cases} x + (t+3)y + 5z = 3 \\ -3x + 9y - 15z = s \\ 2x + ty + 10z = 6 \end{cases}$$

(a) If (*) is consistent, find s and t.

(b) Solve (*) when it is consistent.

(6 marks)

2. A relation $\sim$ is defined on $\mathbf{R}^2$ as follows:

$(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 - x_2 = n$ for some integer n.

(a) Prove that $\sim$ is an equivalence relation.

(b) Sketch the equivalence class containing (2, 1).

(5 marks)

3. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}$.

(a) If B^{-1} exists and $B^{-1}AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, find λ, a and b.

(b) Hence find A^{100}.

(7 marks)

4. By considering $(1 + i)^{2n}$, or otherwise, evaluate $\sum_{r=0}^{n} (-1)^r C_{2r}^{2n}$ and $\sum_{r=0}^{n-1} (-1)^r C_{2r+1}^{2n}$, where n is a positive integer.

(5 marks)

5. Consider the sequence $\{u_n\}$ in which

$$u_1 = 0, \quad u_{n+1} = 2n - u_n \quad \text{for } n = 1, 2, \cdots .$$

Using mathematical induction or otherwise, show that

$$2u_n = 2n - 1 + (-1)^n \quad \text{for } n = 1, 2, \cdots .$$

Hence find $\lim_{n \to \infty} \frac{u_n}{n}$.

(4 marks)

6. Let $f: \mathbf{R} \to \mathbf{R}$ be bijective and $a_1 < a_2 < \cdots < a_n$, where $n \geq 2$.

(a) Suppose f is strictly increasing. Prove that its inverse f^{-1} is also strictly increasing and deduce that

$$a_1 < f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right) < a_n .$$

(b) Define $h(x) = pf(x) + q$, where $p, q \in \mathbf{R}$ and $p \neq 0$.

Show that $h^{-1}(x) = f^{-1}\left(\frac{x-q}{p}\right)$

and deduce that $h^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} h(a_k)\right) = f^{-1}\left(\frac{1}{n}\sum_{k=1}^{n} f(a_k)\right)$.

(5 marks)

7. (a) Prove that $\frac{C_r^n}{n^r} \leq \frac{1}{r!}$, where n, r are positive integers and $n \geq r$.

(b) If $a_1, a_2, \cdots, a_n$ are positive real numbers and $s = a_1 + a_2 + \cdots + a_n$, using "A.M. $\geq$ G.M." and (a), or otherwise, prove that

$$(1 + a_1)(1 + a_2)\cdots(1 + a_n) \leq 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^n}{n!} .$$

(c) Let $c_n = \prod_{k=1}^{n} \left(1 + \frac{1}{2^k}\right)$. Using (b) or otherwise, show that the sequence $\{c_n\}$ converges.

(8 marks)

SECTION B (60 marks)
Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. Let $u, v \in \mathbf{C}$.

(a) Show that

$$|u| + |v| \geq |u + v| .$$

(3 marks)

(b) Suppose $u\bar{v} \in \mathbf{R}$.

Prove that

(i) there exist real numbers α and β, not both zero, such that $\alpha u + \beta v = 0$.

(ii) $$|u| + |v| = \begin{cases} |u + v| & \text{if } u\bar{v} \geq 0 \\ |u - v| & \text{if } u\bar{v} < 0 \end{cases}$$

(6 marks)

(c) Suppose $u\bar{v} \notin \mathbf{R}$.

Given $z \in \mathbf{C}$, show that there exist unique $\alpha, \beta \in \mathbf{R}$ such that

$$z = \alpha u + \beta v .$$

(6 marks)

9. (a) Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Prove by mathematical induction that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \quad \text{for } n = 1, 2, \cdots .$$

(3 marks)

(b) Let $M = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbf{R} \right\}$ and n be a positive integer.

(i) For any $X, Y \in M$, show that

(I) $XY \in M$,
(II) $XY = YX$,
(III) if $X \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then X^{-1} exists and $X^{-1} \in M$.

(ii) For any $X \in M$, show that there exist $r \geq 0$ and $\theta \in \mathbf{R}$ such that $X = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Hence find all $X \in M$ such that $X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(iii) If $Y, B \in M$ and $Y^n = B^n$, show that there exists $X \in M$ such that $X^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Y = BX$.

Hence find all $Y \in M$ such that $Y^n = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^n$.

(12 marks)

10. Let $\{a_1, a_2, \cdots\}$, $\{b_1, b_2, \cdots\}$ be two sequences of real numbers, and $b_0 = 0$.

(a) Show that

$$\sum_{i=1}^{k} a_i(b_i - b_{i-1}) = a_k b_k + \sum_{i=1}^{k-1} (a_i - a_{i+1}) b_i, \quad k = 2, 3, \cdots .$$

(4 marks)

(b) Suppose $\{a_i\}$ is decreasing and $|b_i| \le K$ for all i, where K is a constant.

Show that

$$\left| \sum_{i=1}^{k} a_i(b_i - b_{i-1}) \right| \le K\{|a_1| + 2|a_k|\}, \quad k = 1, 2, \cdots .$$

(6 marks)

(c) Using (b), or otherwise, show that for any positive integers n and p,

$$\left| \sum_{i=n}^{n+p} \frac{(-1)^i}{i} \right| \le \frac{3}{2n} .$$

(5 marks)

11. Let a be a positive real number and n a positive integer.

(a) Solve the quadratic equation $y^2 - 2ya^n \cos n\theta + a^{2n} = 0$ where $\theta \in \mathbf{R}$.

Hence show that the polynomial $x^{2n} - 2x^n a^n \cos n\theta + a^{2n}$ can be factorized as $\prod_{r=0}^{n-1} \left\{ x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2 \right\}$.

(6 marks)

(b) Let $P_0, P_1, P_2, \cdots, P_{n-1}$ be the n points in the Argand plane representing the nth roots of a^n, arranged anti-clockwise, with P_0 on the positive real axis. Let Q be the point representing $x(\cos\theta + i\sin\theta)$ where $x > 0$. For $r = 0, 1, 2, \cdots, n-1$, denote the length of the segment $\overline{QP_r}$ by d_r.

(i) Show that $\prod_{r=0}^{n-1} d_r^2 = x^{2n} - 2x^n a^n \cos n\theta + a^{2n}$.

(ii) If Q lies on the positive real axis, show that

$$\prod_{r=0}^{n-1} d_r = |x^n - a^n| .$$

(iii) If OQ bisects $\angle P_0OP_1$, where O is the origin, show that

$$\prod_{r=0}^{n-1} d_r = x^n + a^n .$$

(9 marks)

12. Let $\mathbf{a}$ and $\mathbf{b}$ be linearly independent vectors in $\mathbf{R}^3$. Let $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ for some $\alpha, \beta \in \mathbf{R}$ such that $\mathbf{c}\cdot\mathbf{a} = 0$ and $\mathbf{c}\cdot\mathbf{b} = 1$.

(a) Find α and β in terms of $\mathbf{a}\cdot\mathbf{a}$, $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{b}\cdot\mathbf{b}$.

(4 marks)

(b) For any $\mathbf{x} \in \mathbf{R}^3$ such that $\mathbf{x}\cdot\mathbf{a} = 0$ and $\mathbf{x}\cdot\mathbf{b} = 1$, prove that

(i) $\mathbf{x} - \mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$,

(ii) $\mathbf{x} = \mathbf{c} + \lambda(\mathbf{a} \times \mathbf{b})$ for some $\lambda \in \mathbf{R}$,

(iii) $|\mathbf{c}| \le |\mathbf{x}|$.
(Note: $|\mathbf{v}|$ represents the length of the vector $\mathbf{v}$.)

(6 marks)

(c) For any real numbers $a_1, a_2, a_3, b_1, b_2, b_3$ such that $a_1b_2 \neq a_2b_1$, use (a) and (b), or otherwise, to show that

$$\frac{\sum_{r=1}^{3} a_r^2}{\left(\sum_{r=1}^{3} a_r^2\right)\left(\sum_{r=1}^{3} b_r^2\right) - \left(\sum_{r=1}^{3} a_rb_r\right)^2} \le \frac{a_1^2 + a_2^2}{(a_1b_2 - a_2b_1)^2}.$$

(5 marks)

13. Let M be the set of all 2×2 matrices. For any $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M$, define $\operatorname{tr}(A) = a_{11} + a_{22}$.

(a) Show that for any $A, B, C \in M$ and $\alpha, \beta \in \mathbf{R}$,

(i) $\operatorname{tr}(\alpha A + \beta B) = \alpha\operatorname{tr}(A) + \beta\operatorname{tr}(B)$,

(ii) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$,

(iii) the equality "$\operatorname{tr}(ABC) = \operatorname{tr}(BAC)$" is *not* necessarily true.

(5 marks)

(b) Let $A \in M$.

(i) Show that $A^2 - \operatorname{tr}(A)A = -(\det A)I$, where I is the 2×2 identity matrix.

(ii) If $\operatorname{tr}(A^2) = 0$ and $\operatorname{tr}(A) = 0$, use (a) and (b)(i) to show that A is singular and $A^2 = 0$.

(5 marks)

(c) Let $S, T \in M$ such that $(ST - TS)S = S(ST - TS)$.

Using (a) and (b) or otherwise, show that

$$(ST - TS)^2 = 0.$$

(5 marks)

END OF PAPER

92-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1992

PURE MATHEMATICS PAPER II

2.00 pm-5.00 pm (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B. Answer BOTH sections.
2. Section A: Answer ALL questions, using AL(C1) answer book.
3. Section B: Answer any FOUR questions, using AL(C2) answer book.

Section A (40 marks)

Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. (a) Evaluate $\lim_{x\to 0}\dfrac{\tan x - x}{x^2 \sin x}$.

   (b) Prove that $\left|x\sin\dfrac{1}{x}\right| \le |x|$ for all $x \ne 0$.

   Hence evaluate $\lim_{x\to 0}\dfrac{\dfrac{1}{x}+\sin\dfrac{1}{x}}{\dfrac{1}{x}-\sin\dfrac{1}{x}}$.

   (6 marks)

2. Sketch the curve with polar equation $r = a(1 + \cos\theta)$, where $a > 0$ and $\theta \in [0, 2\pi]$.

   Also find the area enclosed by the curve.

   (5 marks)

3. If the lines

$$\frac{x-2}{1} = \frac{y-4}{p} = \frac{z-4}{1}$$

and $$\frac{x}{1} = \frac{y-3}{-1} = \frac{z-2}{q}$$

are coplanar and perpendicular to each other, find p and q .

(6 marks)

4. Evaluate $\int_0^2 xe^{|x-1|}\,dx$.

(4 marks)

5. Using a definite integral, or otherwise, evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2n^2+k^2}{n^3+k^3} .$$

(7 marks)

6. Consider the line $(L) : y = 2a$ and the circle $(C) : x^2 + y^2 = a^2$, where $a > 0$. Let P be a variable point on (L) . If the tangents from P to (C) touch the circle (C) at points Q and R respectively, show that the mid-point of QR lies on a fixed circle, and find the centre and radius of this circle.

(6 marks)

7. Let f be a differentiable function such that

$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \text{for all } x, y \in \mathbf{R} .$$

(a) Show that $f'(0) = \lim_{h\to 0}\frac{f(h)}{h}$.

(b) Hence, or otherwise, show that for all $x \in \mathbf{R}$,

$$f'(x) = f'(0) + 3x^2 ,$$

and deduce that

$$f(x) = f'(0)x + x^3 .$$

(6 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. Let $f(x) = xe^{-x^2}$ for $x \in \mathbf{R}$.

(a) Find $f'(x)$ and $f''(x)$.

(2 marks)

(b) Determine the values of x for each of the following cases:

(i) $f'(x) = 0$,

(ii) $f'(x) > 0$,

(iii) $f'(x) < 0$,

(iv) $f''(x) = 0$,

(v) $f''(x) > 0$,

(vi) $f''(x) < 0$.

(3 marks)

(c) Find all relative extrema and points of inflexion of $f(x)$.

(3 marks)

(d) Find the asymptote of the graph of $f(x)$.

(1 mark)

(e) Sketch the graph of $f(x)$.

(3 marks)

(f) Hence sketch the curve $x + y = (x - y)e^{-\frac{1}{2}(x-y)^2}$.

(3 marks)

9. (a) Let g be a continuously differentiable function and $p \geq 1$.

Prove that $\int_0^x (x-t)^p g'(t)\,dt = -x^p g(0) + p\int_0^x g(t)(x-t)^{p-1}\,dt$

for any $x \in \mathbf{R}$.

(2 marks)

(b) For any $n = 1, 2, \cdots$, and $x \in \mathbf{R}$, prove that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} e^t\,dt.$$

Hence or otherwise, show that

$$\left| \left(e + \frac{1}{e}\right) - 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2n)!}\right) \right| < \frac{3}{(2n)!}.$$

(7 marks)

(c) (i) Let f_0 be a continuous function. For any $n = 1, 2, \cdots$, and $x \in \mathbf{R}$, define

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt.$$

Prove that $f_n(x) = \dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1} f_0(t)\,dt$.

(ii) Evaluate $\dfrac{d^{100}}{dx^{100}}\displaystyle\int_0^x (x-t)^{99}\,|\sin(t^2)|\,dt$.

(6 marks)

10. Let Γ be a Cartesian coordinate system on a plane and Γ' be another Cartesian coordinate system with the same origin, obtained from Γ by an anti-clockwise rotation through an angle θ.

Suppose (x, y) and (x', y') are the coordinates of an arbitrary point P with respect to Γ and Γ' respectively.

(a) Let $V = \begin{pmatrix} x \\ y \end{pmatrix}$, $V' = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

(i) Show that

$$V = MV', \text{ where } M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(ii) If the equation of a conic section in the coordinate system Γ is given by

$V^t A V = C$, where $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$, $C = (c)$, $a, b, h, c \in \mathbf{R}$,

show that this conic section is represented in the coordinate system Γ' by

$$V'^t A' V' = C,$$

where A' is a 2×2 matrix such that $\det A = \det A'$.

Furthermore, show that θ can be chosen such that A' is a diagonal matrix.

(10 marks)

(b) The equation of a conic section (H) in Γ is given by $7x^2 + 2hxy + 13y^2 = 16$. Find h if (H) is

(i) an ellipse,
(ii) a hyperbola,
(iii) a pair of straight lines,
(iv) given by $x'^2 + 4y'^2 = 4$ in Γ'.

(5 marks)

11. (a) Let $f(x)$ be a polynomial and n a positive integer such that $\deg f(x) \geq n$.

Prove that for any $a \in \mathbf{R}$,

if $f(a) = f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$,

then $f(x)$ is divisible by $(x - a)^n$.

(8 marks)

(b) Let $p(x)$, $q(x)$, $r(x)$ and $s(x)$ be polynomials and

$$F(x) = \left(\int_1^x p(t)r(t)\,dt\right)\left(\int_1^x q(t)s(t)\,dt\right) - \left(\int_1^x p(t)q(t)\,dt\right)\left(\int_1^x r(t)s(t)\,dt\right).$$

Prove that if $\deg F(x) \geq 4$, then $F(x)$ is divisible by $(x - 1)^4$.

(7 marks)

12. (a) For any $x > 0$, by considering the integral $\int_1^{1+x} \frac{1}{t}\,dt$ or otherwise, prove that

$$\frac{x}{1+x} < \ln(1+x) < x ,$$

and deduce that

$$\frac{1}{1+x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x} .$$

(3 marks)

(b) For any $x > 0$, define $f(x) = \left(1 + \frac{1}{x}\right)^x$. Using (a) or otherwise, prove that f is strictly increasing and $1 < f(x) < e$.

(4 marks)

(c) For $x > 0$ and $n = 2, 3, \cdots,$ define

$$F_n(x) = f(x) - f(n-1) - \int_x^n \frac{1}{t^2 f(t)}\,dt ,$$

where $f(x) = \left(1 + \frac{1}{x}\right)^x$.

(i) For each fixed n , prove that there exists a unique $\alpha_n \in \mathbf{R}$ such that $F_n(\alpha_n) = 0$.

Does $\lim_{n\to\infty} \alpha_n$ exist? Explain.

(ii) For each fixed x , prove that $\lim_{n\to\infty} F_n(x)$ exists.

(8 marks)

13. Suppose $\{a_k\}$ is a sequence of positive numbers such that $a_0 = a_1 = 1$, and $a_k = a_{k-1} + a_{k-2}$ for $k = 2, 3, \cdots$.

Let $-\frac{1}{3} < x < \frac{1}{3}$ and $S_n(x) = \sum_{k=0}^{n} a_k x^k$.

(a) For $k = 0, 1, 2, \cdots$, prove that

$$a_{k+1} \le 2a_k ,$$

and deduce that $a_k \le 2^k$.

Hence prove that $S_n(x) < 3$ for $n = 0, 1, 2, \cdots$.

(6 marks)

(b) Prove that $\lim_{n\to\infty} S_n(x)$ exists and equals $\frac{1}{1 - x - x^2}$.

[Hint: Put $y = -x$ for the case when $x < 0$.]

(5 marks)

(c) Evaluate:

(i) $\sum_{k=0}^{\infty} a_k \left(\frac{1}{5}\right)^k$,

(ii) $\sum_{k=0}^{\infty} (-1)^k a_k \left(\frac{1}{5}\right)^k$,

(iii) $\sum_{k=0}^{\infty} a_{2k} \left(\frac{1}{25}\right)^k$.

(4 marks)

END OF PAPER

93-AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1993

PURE MATHEMATICS PAPER I

9.00 am-12.00 noon (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Section A: Answer ALL questions in the AL(C1) answer book.
3. Section B: Answer any FOUR questions in the AL(C2) answer book.