Solutions | Marks

1. $\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} a^2-c^2 & b^2-c^2 & c^2 \\ a-c & b-c & c \\ 0 & 0 & 1 \end{vmatrix}$ 1

$= (a-c)(b-c)\begin{vmatrix} a^2+ac+c^2 & b^2+bc+c^2 & c^2 \\ 1 & 1 & c \\ 0 & 0 & 1 \end{vmatrix}$ 1

$= (a-c)(b-c)\begin{vmatrix} (a^2-b^2)+c(a-b) & b^2+bc+c^2 \\ 0 & 1 \end{vmatrix}$ 1

$= (a-c)(b-c)(a-b)(a+b+c)$ 1A

(N.B. Candidates may use direct expansion and factorize.)

4

Solutions | Marks

$f(x) = \frac{1}{(1-x)(2-x)}$

$= \frac{1}{x-1} + \frac{1}{2-x}$

When $|x| < 1$,

$f(x) = \frac{1}{x-1} + \frac{1}{2-x}$ 1M

$= (-1)\frac{1}{1-x} + \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)$ 1M

$= (-1)\sum_{k=0}^{\infty} x^k + \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k$

$= \sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}} - 1\right)x^k$

$a_k = \frac{1}{2^{k+1}} - 1$ 1A

When $|x| > 2$,

$f(x) = \frac{1}{x-1} + \frac{1}{2-x}$

$= \frac{1}{x}\left(\frac{1}{1-\frac{1}{x}}\right) - \frac{1}{x}\left(\frac{1}{1-\frac{2}{x}}\right)$ 1M

$= \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{1}{x}\right)^k - \frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{2}{x}\right)^k$

$= \sum_{k=0}^{\infty}\frac{1}{x^{k+1}} - \sum_{k=0}^{\infty}\frac{2^k}{x^{k+1}}$

$= \sum_{k=0}^{\infty}(1-2^k)\frac{1}{x^{k+1}}$

$= \sum_{k=1}^{\infty}(1-2^{k-1})\frac{1}{x^k}$ 1A

$= \sum_{k=0}^{\infty} b_k\left(\frac{1}{x^k}\right)$

where $b_k = \begin{cases} 0 & k = 0 \\ 1 - 2^{k-1} & k = 1, 2, \ldots \end{cases}$ 1A

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$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & -1 & q^2 & q \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & q^2 & q \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & q^2-1 & q-1 \end{pmatrix}$$

(a) No solution

⇒ the 3rd row is a contradiction — 1M

⇒ $q = -1$ — 1A — 2

(b) Infinitely many solutions

⇒ the 3rd row is always true — 1M

⇒ $q = 1$ — 1A — 2

Solutions	Marks

(a) (i) For $i = 1, 2, \ldots, n$,

$$P(a_i) = \frac{a_i(a_i - a_1)\ldots(a_i - a_{i-1})(a_i - a_{i+1})\ldots(a_i - a_n)}{(a_i - a_1)\ldots(a_i - a_{i-1})(a_i - a_{i+1})\ldots(a_i - a_n)}$$

$= a_i$ — 1A

(ii) By (a)(i), $a_1, a_2, \ldots, a_n$ are n distinct roots of — 1M

$P(x) - x = 0$.

(iii) Since $\deg(P(x) - x) \le n - 1$ and $P(x) - x = 0$ has n distinct roots, — 1M

$\therefore P(x) - x \equiv 0$.

(b) By (a)(iii), $P(0) = 0$ — 1A

$$\Rightarrow (a_1 a_2 \ldots a_n)\left\{\frac{1}{(a_1 - a_2)\ldots(a_1 - a_n)} + \frac{1}{(a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n)} + \ldots + \frac{1}{(a_n - a_1)\ldots(a_n - a_{n-1})}\right\} = 0$$ — 1M — 2

$$\Rightarrow \left\{\frac{1}{(a_1 - a_2)\ldots(a_1 - a_n)} + \frac{1}{(a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n)} + \ldots + \frac{1}{(a_n - a_1)\ldots(a_n - a_{n-1})}\right\} = 0 \quad (\because a_i \ne 0\ \forall i)$$

Solutions	Marks
5. (a) (⇒)	
⇒ [illegible]	
⇒ $u\bar{v} + \bar{u}v = 0$	1A
⇒ $2\text{Re}(u\bar{v}) = 0$	1A
⇒ $u\bar{v}$ is purely imaginary	
⇒ $u\bar{v} = ih$ for some $h \in \mathbb{R}$	
⇒ $\frac{uv\bar{v}}{v} = ih$ for some $h \in \mathbb{R}$	1M
⇒ $\frac{u}{v} = ik$ for some $k \in \mathbb{R}$	
(⇐)	
If $\frac{u}{v} = ik$, then $u = ikv$.	
So, $u\bar{v} + \bar{u}v = ikv\bar{v} + \overline{ikv}v$	1M
$= ikv\bar{v} - ik\bar{v}v$	1M
$= 0$	
(b) $\arg u - \arg v = \pm\frac{\pi}{2}$	1A

Solutions	Marks
6. (a) (i) $a^2 + b^2 + c^2 - (ab + bc + ca)$	
$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \geq 0$	1M
(ii) $(a^3 + b^3 + c^3) - 3abc$	
$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$	1M
≥ 0 (since $a + b + c > 0$ and use (i))	
[alternatively,	
multiply the inequality in (i) by $(a + b + c)$]	
(b) (i) Since $-\ln 2 < x < \ln 2$,	
we have $\frac{1}{2} < e^x$ and $e^x < 2$	1A
hence, $(e^{-x})^{\frac{1}{3}} + (2 - e^x)^{\frac{1}{3}} + (e^x - e^{-x} + 1)^{\frac{1}{3}}$	
$> (\frac{1}{2})^{\frac{1}{3}} + (2-2)^{\frac{1}{3}} + (\frac{1}{2} - 2 + 1)^{\frac{1}{3}}$	1M
$= (\frac{1}{2})^{\frac{1}{3}} + 0 + (-\frac{1}{2})^{\frac{1}{3}}$	1M
$= 0$	
[N.B. A.M. ≥ G.M. cannot be used, because the values may be negative.]	
(ii) Let $a = (e^{-x})^{\frac{1}{3}}$	
$b = (2 - e^x)^{\frac{1}{3}}$	1M
$c = (e^x - e^{-x} + 1)^{\frac{1}{3}}$	
by (b)(i), $a + b + c > 0$.	
Hence using (a)(ii), we have	
$e^{-x}(2 - e^x)(e^x - e^{-x} + 1)$	
$= (abc)^3$	
$\leq \left[\frac{a^3 + b^3 + c^3}{3}\right]^3$	1M
$= \left[\frac{e^{-x} + (2 - e^x) + (e^x - e^{-x} + 1)}{3}\right]^3$	
$= 1$	

Solutions | Marks

(a) (i) For $n = 1$,

$b_2 = \frac{1}{2}(a_1 + \frac{1}{2}a_1) = (\frac{3}{4})a_1 > \frac{1}{2}a_1 = b_1$ — 1A

$c_2 = \sqrt{a_1(\frac{1}{2}a_1)} = \sqrt{\frac{1}{2}}a_1 > \frac{1}{2}a_1 = c_1$

Assume $b_{k+1} > b_k$ and $c_{k+1} > c_k$.

Then,

$b_{k+2} = \frac{1}{2}(a_{k+1} + c_{k+1}) > \frac{1}{2}(a_k + c_k) = b_{k+1}$ — 1M

$c_{k+2} = \sqrt{a_{k+1}b_{k+1}} > \sqrt{a_k b_k} = c_{k+1}$ — 2

(ii) For $n = 1$,

$b_1 = c_1 = \frac{1}{2}a_1 < a_1$ $a_1 > 0$ — 1A

Assume $b_k < a_k$ and $c_k < a_k$.

Then,

$b_{k+1} = \frac{1}{2}(a_k + c_k) < \frac{1}{2}(a_k + a_k) = a_k < a_{k+1}$ — 1M

$c_{k+1} = \sqrt{a_k b_k} < \sqrt{a_k a_k} = a_k < a_{k+1}$ — 1M — 3

(b) Since (b_n) and (c_n) are increasing and bounded above by L, they are convergent.

Let $b_n \to p$ and $c_n \to q$ as $n \to \infty$.

Then $p = \frac{1}{2}(L + q)$ and $q = \sqrt{Lp}$ — 1M

$\Rightarrow q^2 = \frac{1}{2}L(L + q)$ — 2

$\Rightarrow (L - q)(L + 2q) = 0$

$\Rightarrow q = L$ or $q = -\frac{1}{2}L$ (rejected because $q \ge 0$) — 1A

Hence $\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L$.

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Solutions | Marks

8. (a) (i) Consider $f(t) = \sum_{k=1}^{n}(a_k + tb_k)^2 = \sum_{k=1}^{n} a_k^2 + 2t\sum_{k=1}^{n} a_k b_k + t^2\sum_{k=1}^{n} b_k^2$ — 1M

$\because f(t) \ge 0 \ \forall t$ — 1M

$\therefore \Delta \le 0$

$\Rightarrow \left(2\sum_{k=1}^{n} a_k b_k\right)^2 - 4\left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} a_k^2\right) \le 0$

$\Rightarrow \left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$ — 1M

(ii) $p \le \frac{b_k}{a_k} \le q$, $k = 1, \ldots, n$

$\Rightarrow a_k^2(p - \frac{b_k}{a_k})(q - \frac{b_k}{a_k}) \le 0$, $k = 1, \ldots, n$ — 1M

$\Rightarrow pqa_k^2 - (p+q)a_k b_k + b_k^2 \le 0$

$\Rightarrow \sum_{k=1}^{n}(pqa_k^2 - (p+q)a_k b_k + b_k^2) \le 0$ — 1M

$\Rightarrow pq\sum_{k=1}^{n} a_k^2 - (p+q)\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} b_k^2 \le 0$ — 1M

$\Rightarrow (p+q)\sum_{k=1}^{n} a_k b_k \ge \sum_{k=1}^{n} b_k^2 + pq\sum_{k=1}^{n} a_k^2$

(iii) $\frac{m}{M} \le \frac{b_k}{a_k} \le \frac{M}{m}$ — 1M

by (a)(ii), $\left(\frac{m}{M} + \frac{M}{m}\right)\sum_{k=1}^{n} a_k b_k$

$\ge \sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} a_k^2$ — 1M

$\ge 2\sqrt{\sum_{k=1}^{n} b_k^2 \sum_{k=1}^{n} a_k^2}$ ($\because$ A.M. $\ge$ G.M.) — 1M

Hence, $\frac{1}{4}\left(\frac{m}{M} + \frac{M}{m}\right)^2\left(\sum_{k=1}^{n} a_k b_k\right)^2 \ge \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$ — 1M

10

(b) Choose $a_k = 1 + \frac{1}{3^k}$

$b_k = 1 - \frac{1}{3^{k+1}}$ — 1A

then $1 - \frac{1}{3^2} \le a_k, b_k \le 1 + \frac{1}{3^1}$

$\Rightarrow \frac{8}{9} \le a_k, b_k \le \frac{4}{3}$

$\therefore$ take $m = \frac{8}{9}$, $M = \frac{4}{3}$ — 1A

Solutions | Marks

8. (b) by (a)(iii),

$$\left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}\right)^2\right\}\left\{\sum_{k=1}^{n}\left(1-\frac{1}{3^{k+1}}\right)^2\right\}$$

$$\le \frac{1}{4}\left(\frac{\frac{4}{3}}{\frac{8}{9}}+\frac{\frac{8}{9}}{\frac{4}{3}}\right)^2\left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}\right)\left(1-\frac{1}{3^{k+1}}\right)\right\}^2$$

$$= \frac{169}{144}\left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}-\frac{1}{3^{k+1}}-\frac{1}{3^{2k+1}}\right)\right\}^2$$

$$= \frac{169}{144}\left\{\sum_{k=1}^{n}1+\sum_{k=1}^{n}\frac{1}{3^k}-\frac{1}{3}\sum_{k=1}^{n}\frac{1}{3^k}-\frac{1}{3}\sum_{k=1}^{n}\frac{1}{9^k}\right\}^2$$

$$= \frac{169}{144}\left\{n+\frac{2}{3}\sum_{k=1}^{n}\frac{1}{3^k}-\frac{1}{3}\sum_{k=1}^{n}\frac{1}{9^k}\right\}^2$$ 1M

$$< \frac{169}{144}\left\{n+\frac{2}{3}\sum_{k=1}^{n}\frac{1}{3^k}\right\}^2$$

$$< \frac{169}{144}\left\{n+\frac{2}{3}\cdot\frac{\frac{1}{3}}{1-\left(\frac{1}{3}\right)}\right\}^2$$

$$= \frac{169}{144}\left(n+\frac{1}{3}\right)^2$$

by (a)(i),

$$\left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}\right)^2\right\}\left\{\sum_{k=1}^{n}\left(1-\frac{1}{3^{k+1}}\right)^2\right\}$$

$$\ge \left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}\right)\left(1-\frac{1}{3^{k+1}}\right)\right\}^2$$ 1M

$$= \left\{n+\frac{2}{3}\sum_{k=1}^{n}\frac{1}{3^k}-\frac{1}{3}\sum_{k=1}^{n}\frac{1}{9^k}\right\}^2$$

$$> \left\{n+\frac{2}{3}\sum_{k=1}^{n}\frac{1}{3^k}-\frac{1}{3}\sum_{k=1}^{n}\frac{1}{3^k}\right\}^2$$ 1M

$$= \left\{n+\frac{1}{3}\sum_{k=1}^{n}\frac{1}{3^k}\right\}^2$$

$$\ge \left(n+\frac{1}{3}\cdot\frac{1}{3}\right)^2$$

$$= \left(n+\frac{1}{9}\right)^2$$

$$\therefore \left(n+\frac{1}{9}\right)^2 < \left\{\sum_{k=1}^{n}\left(1+\frac{1}{3^k}\right)^2\right\}\left\{\sum_{k=1}^{n}\left(1-\frac{1}{3^{k+1}}\right)^2\right\} < \frac{169}{144}\left(n+\frac{1}{3}\right)^2$$

5

Solutions | Marks

9. (a) (i) $\phi(\alpha z) = \phi(\alpha z + 0z) = \alpha\phi(z) + 0\phi(z) = \alpha\phi(z) + 0 = \alpha\phi(z)$ 2M

(ii) $\phi(0) = \phi(0+0) = \phi(0)+\phi(0)$ 1M

$\Rightarrow \phi(0) = 0$

3

(b) $\forall z \in C$,

Let $z = x + yi$, $x, y \in R$.

Then $\phi(z) = \phi(x+yi)$ 1M

$= x\phi(1) + y\phi(i)$

$= x\psi(1) + y\psi(i)$ 1M

$= \psi(x+yi)$

$= \psi(z)$

$\therefore \phi = \psi$

2

(c) (i) $\phi(1) = \phi(1\times 1) = \phi(1)\phi(1)$ 1M

$\Rightarrow \phi(1) - \phi(1)\phi(1) = 0$

$\Rightarrow \phi(1)(1-\phi(1)) = 0$

$\Rightarrow \phi(1) = 0$ or $\phi(1) = 1$ 1A

but if $\phi(1) = 0$, then $\phi(z) = \phi(1\times z)$

$= \phi(1)\phi(z)$

$= 0\times\phi(z)$ 1M

$= 0 \; \forall z \in C$

implying $\phi \ne 0$!!!

$\therefore \phi(1) = 1$ 1A

Hence $\forall x \in R$,

$\phi(x) = \phi(x\times 1)$ 1M

$= x\phi(1)$

$= x\times 1$

$= x$

Solutions	Marks
9. (c) (ii) We first show $\phi(i) = i$ or $-i$.	
$\phi(-1) = \phi(i \times i)$	
$= \phi(i)\phi(i)$	1M
and $\phi(-1) = -\phi(1)$	
$= -1$	
$\therefore\ -1 = \phi(i)\phi(i)$	1M
$\Rightarrow\ \phi(i) = i$ or $-i$	1A
<u>Case 1</u> $\phi(i) = i$	
$\forall z \in C$, let $z = x + yi$, $x, y \in R$	
We have $\phi(z) = \phi(x + yi)$	
$= x\phi(1) + y\phi(i) = x + yi$	1M
$= z$	
<u>Case 2</u> $\phi(i) = -i$	
$\forall z \in C$, let $z = x + yi$, $x, y \in R$	
We have $\phi(z) = \phi(x + yi)$	
$= x\phi(1) + y\phi(i)$	1M
$= x - yi$	
$= \bar{z}$	
	10

Solutions	Marks
10. (a) (i) $u \otimes (\alpha x + \beta y)$	
$= \begin{pmatrix} u_2(\alpha x_3 + \beta y_3) - (\alpha x_2 + \beta y_2)u_3 \\ u_3(\alpha x_1 + \beta y_1) - (\alpha x_3 + \beta y_3)u_1 \\ u_1(\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1)u_2 \end{pmatrix}$	1A
$= \begin{pmatrix} \alpha(u_2x_3 - x_2u_3) + \beta(u_2y_3 - y_2u_3) \\ \alpha(u_3x_1 - x_3u_1) + \beta(u_3y_1 - y_3u_1) \\ \alpha(u_1x_2 - x_1u_2) + \beta(u_1y_2 - y_1u_2) \end{pmatrix}$	
$= \alpha\begin{pmatrix} u_2x_3 - x_2u_3 \\ u_3x_1 - x_3u_1 \\ u_1x_2 - x_1u_2 \end{pmatrix} + \beta\begin{pmatrix} u_2y_3 - y_2u_3 \\ u_3y_1 - y_3u_1 \\ u_1y_2 - y_1u_2 \end{pmatrix}$	1M
$= \alpha(u \otimes x) + \beta(u \otimes y)$	
(ii) $u \otimes x$	
$= \begin{pmatrix} u_2x_3 - x_2u_3 \\ u_3x_1 - x_3u_1 \\ u_1x_2 - x_1u_2 \end{pmatrix}$	
$= -\begin{pmatrix} x_2u_3 - u_2x_3 \\ x_3u_1 - u_3x_1 \\ x_1u_2 - u_1x_2 \end{pmatrix}$	1M
$= -x \otimes u$	
	3
(b) $0 = u \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ u_3 \\ -u_2 \end{pmatrix} \Rightarrow u_2 = u_3 = 0$	1+1
$0 = u \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \\ 0 \end{pmatrix} \Rightarrow u_1 = 0$	1
$\therefore\ u = 0$	
$u \otimes x = v \otimes x\ \forall x$	
$\Rightarrow\ u \otimes x - v \otimes x = 0\ \forall x$	1M
$\Rightarrow\ -(x \otimes u - x \otimes v) = 0\ \forall x$ (by (a)(ii))	
$\Rightarrow\ -x \otimes (u - v) = 0\ \forall x$ (by (a)(i))	1M
$\Rightarrow\ (u - v) \otimes x = 0\ \forall x$ (by (a)(ii))	1M
$\Rightarrow\ u - v = 0$	
$\Rightarrow\ u = v$	
	6

Solutions	Marks
10. (c) $\forall x \in H$, let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	
Then $Mx = M\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$	
$= M\left(x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$	1M
$= x_1 M\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2 M\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3 M\begin{pmatrix}0\\0\\1\end{pmatrix}$	
$= x_1 Me_1 + x_2 Me_2 + x_3 Me_3$	1M
$= x_1(u \otimes e_1) + x_2(u \otimes e_2) + x_3(u \otimes e_3)$	1M
$= u \otimes (x_1 e_1 + x_2 e_2 + x_3 e_3)$	1M
$= u \otimes x$	4
(d) Let $M = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$	
Put $Me_k = u \otimes e_k$, $k = 1, 2, 3$	1M
we have $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\r\\-q\end{pmatrix}$	
$\Rightarrow \begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\r\\-q\end{pmatrix}$	
$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-r\\0\\p\end{pmatrix}$	
$\Rightarrow \begin{pmatrix}d\\e\\f\end{pmatrix} = \begin{pmatrix}-r\\0\\p\end{pmatrix}$	
$\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}q\\-p\\0\end{pmatrix}$	
$\Rightarrow \begin{pmatrix}g\\h\\i\end{pmatrix} = \begin{pmatrix}q\\-p\\0\end{pmatrix}$	
$\therefore M = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$	1A
	2

Solutions	Marks
11. (a) (Existence)	
$(\sqrt{3} + \sqrt{2})^{2n}$ <Bin. Expan.>	
$= \sum_{k=0}^{2n} \binom{2n}{k} (\sqrt{3})^k (\sqrt{2})^{2n-k}$	
$= \sum_{l=0}^{n} \binom{2n}{2l} (\sqrt{3})^{2l} (\sqrt{2})^{2(n-l)} + \sum_{l=1}^{n} \binom{2n}{2l-1} (\sqrt{3})^{2l-1} (\sqrt{2})^{2n-2l+1}$	1M
$= \left\{ \sum_{l=0}^{n} \binom{2n}{2l} 3^l 2^{n-l} \right\} + \sum_{l=1}^{n} \binom{2n}{2l-1} 3^{l-1} 2^{n-l} \sqrt{6}$	
$= \left\{ \sum_{l=0}^{n} \binom{2n}{2l} 3^l 2^{n-l} \right\} + \left\{ \sum_{l=1}^{n} \binom{2n}{2l-1} 3^{l-1} 2^{n-l} \right\} \sqrt{6}$	1M
We see that $\left\{ \sum_{l=0}^{n} \binom{2n}{2l} 3^l 2^{n-l} \right\}$ and $\left\{ \sum_{l=1}^{n} \binom{2n}{2l-1} 3^{l-1} 2^{n-l} \right\}$ are positive integers.	
(Alternatively, use mathematical induction)	
(Uniqueness)	
Suppose $(\sqrt{3} + \sqrt{2})^{2n} = r_n + s_n\sqrt{6}$ where r_n, s_n are positive integers.	
Then $p_n + q_n\sqrt{6} = r_n + s_n\sqrt{6}$	
$\Rightarrow p_n - r_n = (s_n - q_n)\sqrt{6}$	1M
$\Rightarrow p_n - r_n = s_n - q_n = 0$	
$\Rightarrow p_n = r_n$ and $s_n = q_n$	
$(\sqrt{3} - \sqrt{2})^{2n}$	
$= \sum_{k=0}^{2n} \binom{2n}{k} (\sqrt{3})^k (-\sqrt{2})^{2n-k}$	
$= \sum_{l=0}^{n} \binom{2n}{2l} (\sqrt{3})^{2l} (-\sqrt{2})^{2(n-l)} + \sum_{l=1}^{n} \binom{2n}{2l-1} (\sqrt{3})^{2l-1} (-\sqrt{2})^{2n-2l+1}$	
$= \left\{ \sum_{l=0}^{n} \binom{2n}{2l} 3^l 2^{n-l} \right\} - \left\{ \sum_{l=1}^{n} \binom{2n}{2l-1} 3^{l-1} 2^{n-l} \right\} \sqrt{6}$	1M
$= p_n - q_n\sqrt{6}$	
(Alternatively, use mathematical induction)	
$0 < \sqrt{3} - \sqrt{2} < 1$	
$\Rightarrow 0 < (\sqrt{3} - \sqrt{2})^{2n} < 1$	
$\Rightarrow 0 < p_n - q_n\sqrt{6} < 1$	
$\Rightarrow 0 < 2p_n - (p_n + q_n\sqrt{6}) < 1$	1M
$\Rightarrow 0 < 2p_n - (\sqrt{3} + \sqrt{2})^{2n} < 1$	
$\Rightarrow 0 < 2p_n - (\sqrt{3} + \sqrt{2})^{2n}$ and $2p_n - (\sqrt{3} + \sqrt{2})^{2n} < 1$	1M
$\Rightarrow (\sqrt{3} + \sqrt{2})^{2n} < 2p_n$ and $2p_n - 1 < (\sqrt{3} + \sqrt{2})^{2n}$	
$\Rightarrow 2p_n - 1 < (\sqrt{3} + \sqrt{2})^{2n} < 2p_n$	6

Solutions	Marks
11. (b) (i) $2^{3n} - 2^n$	
$= (2^3)^n - 2^n$	
$= (2^3 - 2)[(2^3)^{n-1} + (2^3)^{n-2}2 + \cdots + 2^{n-1}]$	1A
$= 30 \times$ (a positive integer)	
$= 10 \times$ (a positive integer)	
(ii) $3^{4n} - 1$	
$= (3^4)^n - 1^n$	
$= (3^4 - 1)[(3^4)^{n-1} + (3^4)^{n-2} + \cdots + 1]$	1A
$= 80 \times$ (a positive integer)	
$= 10 \times$ (a positive integer)	
(iii) $p_{2n} + q_{2n}\sqrt{6}$	
$= (\sqrt{3} + \sqrt{2})^{4n}$	
$= (5 + 2\sqrt{6})^{2n}$	1M
$= \sum_{k=0}^{2n}\binom{2n}{k}5^k(2\sqrt{6})^{2n-k}$	
$= \sum_{i=0}^{n}\binom{2n}{2i}5^{2i}(2\sqrt{6})^{2n-2i} + \sum_{i=1}^{n}\binom{2n}{2i-1}5^{2i-1}(2\sqrt{6})^{2n-2i+1}$	
$= \left\{\sum_{i=0}^{n}\binom{2n}{2i}5^{2i}2^{2n-2i}6^{n-i}\right\} + \sum_{i=1}^{n}\binom{2n}{2i-1}5^{2i-1}2^{2n-2i+1}6^{n-i}\sqrt{6}$	
$\therefore \quad p_{2n} = \sum_{i=0}^{n}\binom{2n}{2i}5^{2i}2^{2(n-i)}6^{n-i}$	1A
$\Rightarrow \quad 2p_{2n} = \binom{2n}{0}5^0 2^{2n+1}6^n + 10\sum_{i=1}^{n}\binom{2n}{2i}5^{2i-1}2^{2(n-i)}6^{n-i}$	
$\Rightarrow \quad 2p_{2n} = 2^{2n+1}6^n + 10 \times$ (a positive integer)	
$\Rightarrow \quad 2p_{2n} - 2^{2n+1}6^n = 10 \times$ (a positive integer)	1M
$\Rightarrow \quad 2p_{2n} - 2^{2n+1}2^n3^n = 10 \times$ (a positive integer)	
$\Rightarrow \quad 2p_{2n} - 2^{3n+1}3^n = 10 \times$ (a positive integer)	5
(c) By (a)(iii), the integral part of $(\sqrt{3} + \sqrt{2})^{100}$ is	
$2p_{50} - 1$	1A
$= \{10 \times$ (a positive integer) $+ 2^{76}3^{25}\} - 1$ (by (b)(iii))	1M
$= \{10 \times$ (a positive integer) $+ 2 \times 2^{3\times 25} \times 3 \times 3^{4\times 6}\} - 1$	
$= \{10 \times$ (a positive integer) $+ 6 \times 2^{3+3} \times 1\} - 1$	1M
$= \{10 \times$ (a positive integer) $+ 6 \times 2^3 \times 1\} - 1$	
$= \{10 \times$ (a positive integer) $+ 8\} - 1$	
$= 10 \times$ (a positive integer) $+ 7$	
$\therefore$ The unit digit is 7.	1A 4

Solutions	Marks
12. (a) $\forall u, v \in A \cap B$, $\alpha, \beta \in R$ such that $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$,	
$\alpha u + \beta v \in A$ ($\because A$ convex and $u, v \in A$)	1M
$\alpha u + \beta v \in B$ ($\because B$ convex and $u, v \in B$)	
$\therefore \quad \alpha u + \beta v \in A \cap B$	1M
$A = \{(0,0,0)\}$; $B = \{(1,1,1)\}$ are convex.	1M
But $A \cup B = \{(0,0,0), (1,1,1)\}$ is not convex because	
$\frac{1}{2}(0,0,0) + \frac{1}{2}(1,1,1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	1M
$\notin A \cup B$	4
(b) $\forall \alpha, \beta \in R$ such that $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$,	
$\forall w_1, w_2 \in A + B$,	
we have $w_1 = u_1 + v_1$ for some $u_1 \in A$, $v_1 \in B$	
$w_2 = u_2 + v_2$ for some $u_2 \in A$, $v_2 \in B$	
then $\alpha w_1 + \beta w_2$	
$= \alpha(u_1 + v_1) + \beta(u_2 + v_2)$	1M
$= (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2)$	
$= u' + v'$ where $u' = \alpha u_1 + \beta u_2 \in A$ ($\because A$ convex)	
and $v' = \alpha v_1 + \beta v_2 \in B$ ($\because B$ convex)	1M
$\in A + B$	2
(c) $\forall \alpha, \beta \geq 0$ and $\alpha + \beta = 1$,	
$\forall u, v \in \gamma A$,	
$u = \gamma u_1$, $v = \gamma v_1$ for some $u_1, v_1 \in A$.	
Hence, $\alpha u + \beta v$	
$= \alpha(\gamma u_1) + \beta(\gamma v_1)$	1M
$= \gamma(\alpha u_1 + \beta v_1)$	
$= \gamma u'$ where $u' = \alpha u_1 + \beta v_1 \in A$ ($\because A$ convex)	1M
$\in \gamma A$	2

Solutions	Marks

12. (d) (i) $\forall \alpha, \beta \geq 0$ and $\alpha + \beta = 1$,

$\forall u, v \in \text{cov}(a_1, \ldots, a_n)$,

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$u = \alpha_1 a_1 + \ldots + \alpha_n a_n$, $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$

$v = \beta_1 a_1 + \ldots + \beta_n a_n$, $\beta_i \geq 0$ and $\sum_{i=1}^{n} \beta_i = 1$

then $\alpha u + \beta v = \alpha(\alpha_1 a_1 + \ldots + \alpha_n a_n) + \beta(\beta_1 a_1 + \ldots + \beta_n a_n)$

$= (\alpha\alpha_1 + \beta\beta_1)a_1 + \ldots + (\alpha\alpha_n + \beta\beta_n)a_n$ — 1A

It remains to show that $\alpha\alpha_1 + \beta\beta_1, \ldots, \alpha\alpha_n + \beta\beta_n \geq 0$

and $(\alpha\alpha_1 + \beta\beta_1) + \ldots + (\alpha\alpha_n + \beta\beta_n) = 1$.

Since $\alpha, \beta, \alpha_i, \beta_i \geq 0$

$\alpha\alpha_1 + \beta\beta_1, \ldots, \alpha\alpha_n + \beta\beta_n \geq 0$ — 1M

Also,

$(\alpha\alpha_1 + \beta\beta_1) + \ldots + (\alpha\alpha_n + \beta\beta_n) = \alpha(\alpha_1 + \ldots + \alpha_n) + \beta(\beta_1 + \ldots + \beta_n)$

$= \alpha \cdot 1 + \beta \cdot 1$

$= \alpha + \beta$ — 1M

$= 1$

(ii) It is equivalent to proving that $\alpha_1 a_1 + \ldots + \alpha_n a_n \in S$

for all $\alpha_i \geq 0$, $\sum_{i=1}^{n} \alpha_i = 1$.

We shall use induction on n.

For $n = 1$, $a_1 \in S$

$\Rightarrow \quad 1 \cdot a_1 \in S$. — 1M

Assume that it is true for $n = k$

i.e. $\alpha_1, \ldots, \alpha_k \geq 0$ and $\alpha_1 + \ldots + \alpha_k = 1 \Rightarrow$ — 1M

$\Rightarrow \sum_{i=1}^{k} \alpha_i a_i \in S$

then for $n = k + 1$,

If $\sum_{i=1}^{k+1} \alpha_i = 1$, $\alpha_i \geq 0 \ \forall i$ then

$\sum_{i=1}^{k+1} \alpha_i a_i = \sum_{i=1}^{k} \alpha_i a_i + \alpha_n a_n$

$= \lambda\left(\sum_{i=1}^{k} \frac{\alpha_i}{\lambda} a_i\right) + \alpha_n a_n$ where $\lambda = \sum_{i=1}^{k} \alpha_i$

$= \lambda w + \alpha_n a_n$ where $w = \sum_{i=1}^{k} \frac{\alpha_i}{\lambda} a_i \in S$ $\left(\because \sum_{i=1}^{k} \frac{\alpha_i}{\lambda} = \frac{1}{\lambda}\sum_{i=1}^{k} \alpha_i = 1\right)$ — 1M

$\in S$ ($\because \lambda + \alpha_n = 1$ and $w, a_n \in S$ and S is convex) — 1M

Total: 7

Solutions	Marks

13. (a) (reflexive)

$v - v = 0 = 0u \ \forall v \in R^3$ — 1M

(symmetric)

$v \sim w \Rightarrow v - w = ku$

$\Rightarrow w - v = (-k)u$ — 1M

(transitive)

$v \sim w$ and $w \sim x$

$\Rightarrow v - w = ku$ and $w - x = k'u$

$\Rightarrow v - x = (v - w) + (w - x)$

$= ku + k'u$ — 1M

$= (k + k')u$

$\Rightarrow v \sim x$

Total: 3

(b) (i) $\forall v \in R^3$, $v - (v \cdot u)u$ exists $\Rightarrow f([v])$ exists. — 1

Now, if $[v] = [w]$, — 1M

then $v \sim w$

$\Rightarrow \quad v - w = ku$ for some $k \in R$

$\Rightarrow \quad v = w + ku$ — 1M

Hence, $f([v]) = v - (v \cdot u)u$

$= (w + ku) - ((w + ku) \cdot u)u$

$= w + ku - (w \cdot u + ku \cdot u)u$

$= w + ku - (w \cdot u + k)u$ — 1M

$= w + ku - (w \cdot u)u - ku$

$= w - (w \cdot u)u$

$= f([w])$

(ii) If $f([v]) = f([w])$ — 1M

then $v - (v \cdot u)u = w - (w \cdot u)u$

$\Rightarrow \quad v - w = ((v \cdot u) - (w \cdot u))u$

$\Rightarrow \quad v \sim w$ — 1M

$\Rightarrow \quad [v] = [w]$

(iii) ($\Rightarrow$)

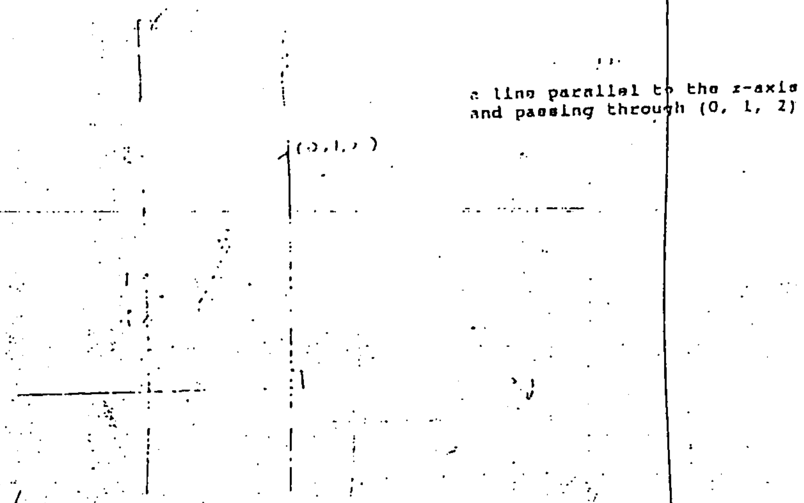
If $w \perp u$, then $f([w]) = w - (w \cdot u)u$

$= w - 0$ — 1M

$= w$

$\Rightarrow w \in f(R^3/\sim)$

Solutions	Marks
13. (b) (iii) (⇐) $\Rightarrow w - v - (v \cdot u)u \quad \exists\, v \in R^3$	
If $w \in f(R^3/\sim)$ then $w = v - (v \cdot u)u$ for some $v \in R^3$	1M
$\Rightarrow w \cdot u = (v - (v \cdot u)u) \cdot u$	
$= v \cdot u - (v \cdot u)u \cdot u$	1M
$= v \cdot u - v \cdot u$	
$= 0$	
$\therefore w \perp u$	
$\because u \neq 0$ and $u \cdot u \neq 0$	1M
$\therefore u \notin f(R^3/\sim)$ i.e. f is not surjective	10
(c)	



2

Solutions	Marks
1. Substituting y [illegible] into [illegible], we have	
$2x^2 + 2x(x + c) + (x + c)^2 = 1$	
$5x^2 + 4cx + c^2 - 1 = 0$	
Let (x_1, y_1) and (x_2, y_2) be the intersection points.	
Then, $x_1 + x_2$ = sum of roots	
$= -\frac{4c}{5}$	1A
$\Rightarrow y_1 + y_2 = x_1 + c + x_2 + c$	
$= -\frac{4c}{5} + 2c$	
$= \frac{6c}{5}$	
Let $M(x, y)$ be the mid-point.	
We have $x = \frac{1}{2}(x_1 + x_2) = -\frac{2c}{5}$	
and $y = \frac{1}{2}(y_1 + y_2) = \frac{3c}{5}$	1A
Eliminating c, we obtain $3x + 2y = 0$, a straight line.	1A

Solution | Marks

$(1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta)\sin(\frac{1}{2}\theta)$

$= \sin(\frac{1}{2}\theta) + \cos\theta\sin(\frac{1}{2}\theta) + \cos 2\theta\sin(\frac{1}{2}\theta) + \cdots + \cos n\theta\sin(\frac{1}{2}\theta)$

$= \sin(\frac{1}{2}\theta) + \frac{1}{2}[\sin(\frac{3\theta}{2}) - \sin(\frac{\theta}{2})] + \frac{1}{2}[\sin(\frac{5\theta}{2}) - \sin(\frac{3\theta}{2})] + \cdots + \frac{1}{2}[\sin(\frac{(2n+1)\theta}{2}) - \sin(\frac{(2n-1)\theta}{2})]$ — 1M, 1A

$= \frac{1}{2}[\sin(\frac{1}{2}\theta) + \sin(\frac{(2n+1)\theta}{2})]$ — 1A

$= \sin(\frac{1}{2}(n+1)\theta)\cos(\frac{1}{2}n\theta)$

To solve $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = 0$, $0 \le \theta < 2\pi$:

Firstly, $\theta = 0$ is obviously not a solution. — 1M

Then, for $\theta \in (0, 2\pi)$,

$\sin\frac{1}{2}\theta \ne 0$,

hence the equation becomes $\sin(\frac{1}{2}(n+1)\theta)\cos(\frac{1}{2}n\theta) = 0$.

∴ $\sin(\frac{1}{2}(n+1)\theta) = 0$ or $\cos(\frac{1}{2}n\theta) = 0$.

∴ $\frac{1}{2}(n+1)\theta = k\pi$ or $\frac{1}{2}n\theta = \frac{1}{2}(2k+1)\pi$, $k \in \mathbb{Z}$. — 1A

∴ $\theta = \frac{2k\pi}{(n+1)}$, $k = 1, 2, \ldots, n$ — 1A

or

$\theta = \frac{(2k+1)\pi}{n}$, $k = 0, 1, 2, \ldots, n-1$

Alternative solution for the first part

Use mathematical induction :

When $n = 1$, L.H.S $= (1 + \cos\theta)\sin\frac{\theta}{2}$

$= \sin\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}$

$= \sin\frac{\theta}{2} + \frac{1}{2}(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2})$ — 1A

$= \frac{1}{2}(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2})$

$= \sin\theta\cos\frac{\theta}{2}$

$=$ R.H.S — 1M

Assume $(1 + \cos\theta + \cos 2\theta + \cdots + \cos k\theta)\sin\frac{1}{2}\theta = \sin(\frac{1}{2}(k+1)\theta)\cos(\frac{1}{2}k\theta)$

then $(1 + \cos\theta + \cos 2\theta + \cdots + \cos k\theta + \cos(k+1)\theta)\sin\frac{1}{2}\theta$

$= \sin\frac{1}{2}(k+1)\theta\cos\frac{1}{2}k\theta + \cos(k+1)\theta\sin\frac{1}{2}\theta$ — 1A

$= \frac{1}{2}(\sin(\frac{2k+1}{2})\theta + \sin(\frac{1}{2})\theta + \sin(\frac{2k+3}{2})\theta - \sin(\frac{2k+1}{2})\theta)$

$= \frac{1}{2}(\sin(\frac{2k+3}{2})\theta + \sin\frac{1}{2}\theta) = \sin(\frac{k+2}{2})\theta\cos(\frac{k+1}{2})\theta$

Solution | Marks

(a) Length $= \int_0^{\frac{\pi}{2}}\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$ — 1M

$= \int_0^{\frac{\pi}{2}}\sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t}\,dt$

$= \int_0^{\frac{\pi}{2}} 3\sin t\cos t\,dt$ — 1A

$= [3\frac{\sin^2 t}{2}]_0^{\frac{\pi}{2}}$

$= \frac{3}{2}$ — 1A

(b) Area $= \int_0^1 y\,dx$ — 1M

$= \int_0^{\frac{\pi}{2}}(\cos^3 t)(3\sin^2 t\cos t)\,dt$

$= 3\int_0^{\frac{\pi}{2}}\sin^2 t\cos^4 t\,dt$ — 1A

$= 3\int_0^{\frac{\pi}{2}}(\frac{\sin 2t}{2})^2\cos^2 t\,dt$

$= 3\int_0^{\frac{\pi}{2}}(\frac{\sin 2t}{2})^2(\frac{\cos 2t + 1}{2})\,dt$ — 1M

$= \frac{3}{8}\{\int_0^{\frac{\pi}{2}}\sin^2 2t\cos 2t\,dt + \int_0^{\frac{\pi}{2}}\sin^2 2t\,dt\}$

$= \frac{3}{8}\{\frac{1}{2}[\frac{\sin^3 2t}{3}]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{1 - \cos 4t}{2}\,dt\}$

$= \frac{3}{8}\{0 + \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 - \cos 4t\,dt\}$

$= \frac{3}{16}[t - \frac{\sin 4t}{4}]_0^{\frac{\pi}{2}}$

$= \frac{3\pi}{32}$ — 1A

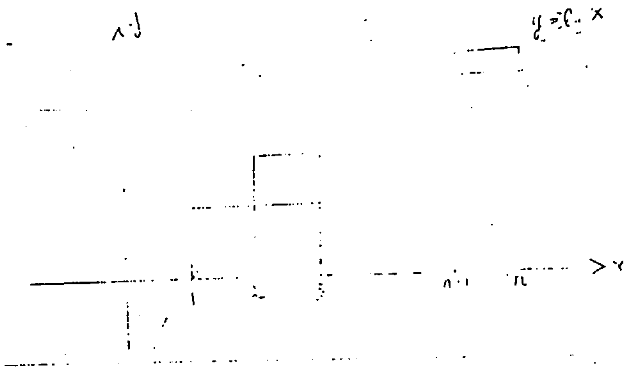
Solutions	Marks
4. Use the substitution $t = \tan\frac{1}{2}\theta$.	1M
Then $\sin\theta = \frac{2t}{(1+t^2)}$ and $d\theta = \frac{2dt}{(1+t^2)}$, at $\theta = 0$, $t = 0$; at $\theta = \frac{1}{2}\pi$, $t = 1$	1A
Therefore, the integral	
$= \int_0^1 \frac{1}{1+2\left(\frac{2t}{(1+t^2)}\right)} \times \frac{2}{1+t^2}\,dt$	
$= \int_0^1 \frac{2}{1+t^2+4t}\,dt$	1A
$= \int_0^1 \frac{2}{(t+2)^2-3}\,dt$	1M
$= \left(\frac{1}{\sqrt{3}}\right)\int_0^1 \frac{1}{t+2-\sqrt{3}} - \frac{1}{t+2+\sqrt{3}}\,dt$	1M
$= \left(\frac{1}{\sqrt{3}}\right)\left[\ln\lvert t+2-\sqrt{3}\rvert - \ln\lvert t+2+\sqrt{3}\rvert\right]_0^1$	
$= \left(\frac{1}{\sqrt{3}}\right)\left\{\ln\left\lvert\frac{3-\sqrt{3}}{3+\sqrt{3}}\right\rvert - \ln\left\lvert\frac{2-\sqrt{3}}{2+\sqrt{3}}\right\rvert\right\}$	1A

Solutions	Marks
5. (a) L'Hospital's rule cannot be used.	
Let $y = \left(\frac{a^x-1}{a-1}\right)^{\frac{1}{x}}$	
then $\lim \ln y = \lim \frac{1}{x}\ln\frac{a^x-1}{a-1}$	
$= \left(\lim \frac{1}{x}\right)\left(\lim \ln\frac{a^x-1}{a-1}\right)$	1M
$= 0 \times \ln\left(\frac{0-1}{a-1}\right)$	
$= 0$	
hence, $\lim y = 1$	1A
(b) —', use L'Hospital's rule:	
$\lim \ln y = \lim \frac{1}{x}\ln\frac{a^x-1}{a-1}$	1M
$= \lim\left[\left(\frac{a-1}{a^x-1}\right)\frac{d}{dx}\left(\frac{a^x-1}{a-1}\right)\right]$	1M
$= \lim\left[\left(\frac{a-1}{a^x-1}\right)\left(\frac{a^x\ln a}{a-1}\right)\right]$	
$= \lim\left[\left(\frac{1}{1-\left(\frac{1}{a^x}\right)}\right)(\ln a)\right]$	
$= \frac{1}{1-0}\ln a$	1A
$= \ln a$	
hence $\lim y = a$	1A

Solutions	Marks
6. (a) $\frac{d}{du}\int_0^u (\sqrt{2})^{t^2} dt = (\sqrt{2})^{u^2}$	1A
(b) $F(x) = \int_0^{\sec x} (\sqrt{2})^{t^2} dt - \int_0^{\tan x} (\sqrt{2})^{t^2} dt$	1M
$F'(x) = \sec x \tan x (\sqrt{2})^{\sec^2 x} - \sec^2 x (\sqrt{2})^{\tan^2 x}$	1A
Since $1 + \tan^2 x = \sec^2 x$ [illegible]	1M
$F'(x) = 0 \Leftrightarrow \sqrt{2}^{\tan^2 x} \sec x(\sqrt{2}\tan x - \sec x) = 0$	1A
$\Leftrightarrow \sin x = \frac{1}{\sqrt{2}}$	
$\Leftrightarrow x = \frac{\pi}{4}$	1A
7. (a) $\lvert f(x) - f(c) \rvert \le \lvert x - c \rvert^2$ for all $x \in R$	
$\Rightarrow -\lvert x - c\rvert^2 \le \lvert f(x) - f(c)\rvert \le \lvert x - c\rvert^2$ for all $x \in R$	
$\Rightarrow -\lvert x - c\rvert \le \left\lvert \frac{f(x) - f(c)}{(x - c)} \right\rvert \le \lvert x - c\rvert$ for all $x \in R\backslash\{c\}$	2
Since $\lim_{x \to c}(x - c) = 0$, by squeezing theorem, $\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)} = 0$.	1
$\Rightarrow f'(c) = 0$	
(b) For all $y \in R$, by (a), $f'(y) = 0$	
since $f'(y) = 0$ for all $y \in R$, f is constant.	

Solutions	Marks
8. (a) $f'(x) = 1 - \frac{1}{1 + x} = \frac{x}{1 + x}$	1A
$f'(x) < 0$ on $(-1, 0)$	1M
and $f'(x) > 0$ on $(0, \infty)$	
now $f(0) = 0$, the result follows.	1A
	$\overline{3}$
(b) $a_{n+1} - a_n$	
$= \frac{1}{n + 1} - \ln(n + 2) + \ln(n + 1)$	
$= \frac{1}{n + 1} - \ln\left(1 + \frac{1}{n + 1}\right)$	1M
> 0 (by (a))	
$b_n - b_{n+1}$	
$= -\ln n - \frac{1}{n + 1} + \ln(n + 1)$	
$= -\frac{1}{n + 1} - \ln\left(1 - \frac{1}{n + 1}\right)$	1M
> 0 ($\because -1 < \frac{1}{n + 1} < 0$ and use (a))	
$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln(n + 1)$	
$< 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$	
$= b_n \le b_1$	1M
$\therefore$ $\{a_n\}$ is bounded above and increasing.	
$\Rightarrow$ $\lim_{n \to \infty} a_n$ exists.	
$b_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$	
$> 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln(n + 1)$	
$= a_n \ge a_1$	1M
$\therefore$ $\{b_n\}$ is bounded below and decreasing.	
$\Rightarrow$ $\lim_{n \to \infty} b_n$ exists	
$\lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n = \lim_{n \to \infty}(b_n - a_n)$	
$= \lim_{n \to \infty}(\ln(n + 1) - \ln n)$	
$= \lim_{n \to \infty}\ln\left(1 + \frac{1}{n}\right)$	1M
$= 0$	
$\therefore$ $\lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n$	$\overline{5}$

Solutions	Marks
8. (c) (i) $\frac{1}{kn+1}+\frac{1}{kn+2}+\cdots+\frac{1}{kn+n}$	
$=\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{(k+1)n}\right]-\left[1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{kn}\right]$	1M
$=[b_{(k+1)n}+\ln(k+1)n]-[b_{kn}+\ln(kn)]$	1M
$=b_{(k+1)n}-b_{kn}+\ln\left(\frac{k+1}{k}\right)$	
$\to 0+\ln\left(\frac{k+1}{k}\right)$ as $n\to\infty$	1M
$=\ln\left(\frac{k+1}{k}\right)$	1A
(ii) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$	
$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)$	1M
$=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$	
$=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$	1M
$=\ln\left(\frac{1+1}{1}\right)$ (by (c)(i), put $k=1$)	
$=\ln 2$	1A
	7

Solutions	Marks
9. (a) Let $y=\sqrt[n]{n}$	
$\ln y=\frac{1}{n}\ln n$	
$\lim_{n\to\infty}\ln y=\lim_{n\to\infty}\frac{\ln n}{n}$	1M
$=0$ ($\because \lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$)	1M
$\therefore \lim_{n\to\infty} y=1$	1A
	3
(b) 	1M
$\because$ Σ area of small rectangle $\le$ area under the curve $\le$ Σ area of big rectangle	1M
$\therefore$ $\ln 1+\ln 2+\cdots+\ln(n-1)\le\int_1^n \ln x\,dx\le\ln 2+\ln 3+\cdots+\ln n$	
$\ln((n-1)!)\le\int_1^n \ln x\,dx\le\ln(n!)$	1A
Now $\int_1^n \ln x\,dx$	
$=[x\ln x]_1^n-\int_1^n x\,d(\ln x)$	1M
$=n\ln n-\int_1^n 1\,dx$	
$=n\ln n-(n-1)$	1A
$\therefore$ $\ln(n-1)!\le n\ln n-(n-1)\le\ln(n!)$	1A
$\Rightarrow$ $(n-1)!\le e^{n\ln n}\cdot e^{-n+1}\le n!$	1M
$\Rightarrow$ $(n-1)!\le n^n e^{-n+1}\le n!$	
	7

Solutions	Marks

9. (c) By (b),

$(n-1)! \le n^n e^{-(n-1)} \le n!$

$\Rightarrow \quad \frac{1}{n} \le \frac{n^n}{n!} e^{-(n-1)} \le 1$

$\Rightarrow \quad \frac{e^{n-1}}{n} \le \frac{n^n}{n!} \le e^{n-1}$ — 1A

$\Rightarrow \quad e^{1-n} \le \frac{n!}{n^n} \le n \cdot e^{1-n}$

$\Rightarrow \quad e^{\frac{1-n}{n}} \le \frac{(n!)^{\frac{1}{n}}}{n} \le \sqrt[n]{n}\, e^{\frac{1-n}{n}}$

Now $\lim_{n\to\infty} e^{\frac{1}{n}-1} = e^{-1}$ — 1A

and $\lim_{n\to\infty} \sqrt[n]{n}\, e^{\frac{1}{n}-1} = \lim_{n\to\infty} \sqrt[n]{n} \lim_{n\to\infty} e^{\frac{1}{n}-1}$ — 1M

$= 1 \cdot e^{-1}$

$= e^{-1}$ — 1A

By squeezing theorem,

$\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$ — 1A

Total: 5

Solutions	Marks

10. (a) $f(x) = 0$

$\Leftrightarrow \quad x^3 - x^2 - x + 1 = 0$

$\Leftrightarrow \quad x(x^2 - 1) - (x^2 - 1) = 0$

$\Leftrightarrow \quad (x^2 - 1)(x - 1) = 0$

$\Leftrightarrow \quad (x + 1)(x - 1)^2 = 0$

$\Leftrightarrow \quad x = -1 \text{ or } x = 1$ — 1A

Total: 1

(b) $f(x) = [(x+1)(x-1)^2]^{\frac{1}{3}}$

for $x \ne \pm 1$, $f'(x) = \frac{1}{3}[(x+1)(x-1)^2]^{-\frac{2}{3}} \frac{d}{dx}(x^3 - x^2 - x + 1)$

$= \frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{4}{3}}(3x^2 - 2x - 1)$

$= \frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{4}{3}}(3x+1)(x-1)$

$= \frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{1}{3}}(3x+1)$ — 1A

$\frac{f(x) - f(1)}{x - 1}$

$= \frac{\sqrt[3]{(x+1)(x-1)^2} - 0}{x - 1}$

$= \frac{\sqrt[3]{x+1}}{\sqrt[3]{x-1}}$ — 1M

$\to \infty$ as $x \to 1$

$\therefore$ $f'(1)$ does not exist.

$\frac{f(x) - f(-1)}{x - (-1)}$

$= \frac{\sqrt[3]{(x+1)(x-1)^2} - 0}{x + 1}$ — 1M

$= \sqrt[3]{\frac{(x-1)^2}{(x+1)^2}}$

$\to \infty$ as $x \to -1$

$\therefore$ $f'(-1)$ does not exist.

Total: 3

(c) (i) $f'(x) = 0$

$\Leftrightarrow \quad \frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{1}{3}}(3x+1) = 0$

$\Leftrightarrow \quad x = -\frac{1}{3}$ — 1A

Solutions	Marks

(c) (ii) $f'(x) > 0$

$\Leftrightarrow \frac{1}{3}\left((x+1)^{-\frac{1}{3}}\right)^2 \cdot \frac{1}{\sqrt[3]{x-1}} \cdot (3x+1) > 0$

$\Leftrightarrow \frac{1}{\sqrt[3]{x-1}} \cdot (3x+1) > 0 \quad (x \neq -1)$

$\Leftrightarrow x > 1$ or $x < -\frac{1}{3}$ — 1A

(iii) $f'(x) < 0$

$\Leftrightarrow \frac{1}{\sqrt[3]{x-1}} \cdot (3x+1) < 0$

$\Leftrightarrow -\frac{1}{3} < x < 1$ — 1A

3

(d) From (c), relative max. is

$\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$

$= \left(-\frac{1}{3}, \sqrt[3]{\left(-\frac{1}{3}+1\right)\left(-\frac{1}{3}-1\right)^2}\right)$

$= \left(-\frac{1}{3}, \sqrt[3]{\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)^2}\right)$

$= \left(-\frac{1}{3}, \frac{1}{3}\sqrt[3]{32}\right)$

$= \left(-\frac{1}{3}, \frac{2}{3}\sqrt[3]{4}\right)$ — 1A

relative min. is $(1, f(1))$

$= (1, 0)$ — 1A

$f''(x) = \frac{d}{dx}\left\{\frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{1}{3}}(3x+1)\right\}$

$= \frac{1}{3}\left\{-\frac{2}{3}(x+1)^{-\frac{5}{3}}(x-1)^{-\frac{1}{3}}(3x+1) - \frac{1}{3}(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{4}{3}}(3x+1)\right.$

$\left. + 3(x+1)^{-\frac{2}{3}}(x-1)^{-\frac{1}{3}}\right\}$

$= \frac{1}{9}(x+1)^{-\frac{5}{3}}(x-1)^{-\frac{4}{3}}[-2(x-1)(3x+1)$

$- (x+1)(3x+1) + 9(x+1)(x-1)]$

$= \frac{1}{9}\left(\frac{1}{(x+1)^{\frac{5}{3}}}\right)\left(\frac{1}{(x-1)^{\frac{4}{3}}}\right)((3x+1)(-3x+1) + 9(x^2-1))$

$= \frac{1}{9}\left(\frac{1}{(x+1)^{\frac{5}{3}}}\right)\left(\frac{1}{(x-1)^{\frac{4}{3}}}\right)(-9x^2 + 1 + 9x^2 - 9)$

$= -\frac{8}{9}\left(\frac{1}{(x+1)^{\frac{5}{3}}}\right)\left(\frac{1}{(x-1)^{\frac{4}{3}}}\right)$ — 1A

Solutions	Marks

10. (d) $\therefore f''(x) > 0$ on $(-\infty, -1)$

and $f''(x) < 0$ on $(-1, \infty)$

Hence the point of inflexion is at $x = -1$.

$\Rightarrow (-1, 0)$ is the point of inflextion. — 1A

4

(e) Clearly there is no vertical asymptote.

Let $y = mx + b$ be an oblique asymptote,

then $\lim_{x\to\infty}(f(x) - (mx+b)) = 0$

$\Rightarrow \lim_{x\to\infty} \frac{f(x) - (mx+b)}{x} = 0$

$\Rightarrow \lim_{x\to\infty} \frac{\sqrt[3]{x^3 - x^2 - x + 1} - (mx+b)}{x} = 0$

$\Rightarrow \lim_{x\to\infty} \sqrt[3]{1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}} - m - \frac{b}{x} = 0$

$\Rightarrow m = 1$ — 1A

Then $\lim_{x\to\infty}(f(x) - (x+b)) = 0$

$\Rightarrow \lim_{x\to\infty}(f(x) - x) - \lim_{x\to\infty} b = 0$

$\Rightarrow b = \lim_{x\to\infty}(f(x) - x)$

$= \lim_{x\to\infty}\left(\sqrt[3]{x^3 - x^2 - x + 1} - x\right)$

$= \lim_{x\to\infty} \frac{x^3 - x^2 - x + 1 - x^3}{(x^3 - x^2 - x + 1)^{\frac{2}{3}} + (x^3 - x^2 - x + 1)^{\frac{1}{3}}x + x^2}$

$= \lim_{x\to\infty} \frac{-x^2 - x + 1}{(x^3 - x^2 - x + 1)^{\frac{2}{3}} + (x^3 - x^2 - x + 1)^{\frac{1}{3}}x + x^2}$

$= \lim_{x\to\infty} \frac{-1 - \frac{1}{x} + \frac{1}{x^2}}{\left(\frac{x^3 - x^2 - x + 1}{x^3}\right)^{\frac{2}{3}} + \left(\frac{x^3 - x^2 - x + 1}{x^3}\right)^{\frac{1}{3}} + 1}$

$= \lim_{x\to\infty} \frac{-1 - \frac{1}{x} + \frac{1}{x^2}}{\left(1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)^{\frac{2}{3}} + \left(1 - \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\right)^{\frac{1}{3}} + 1}$

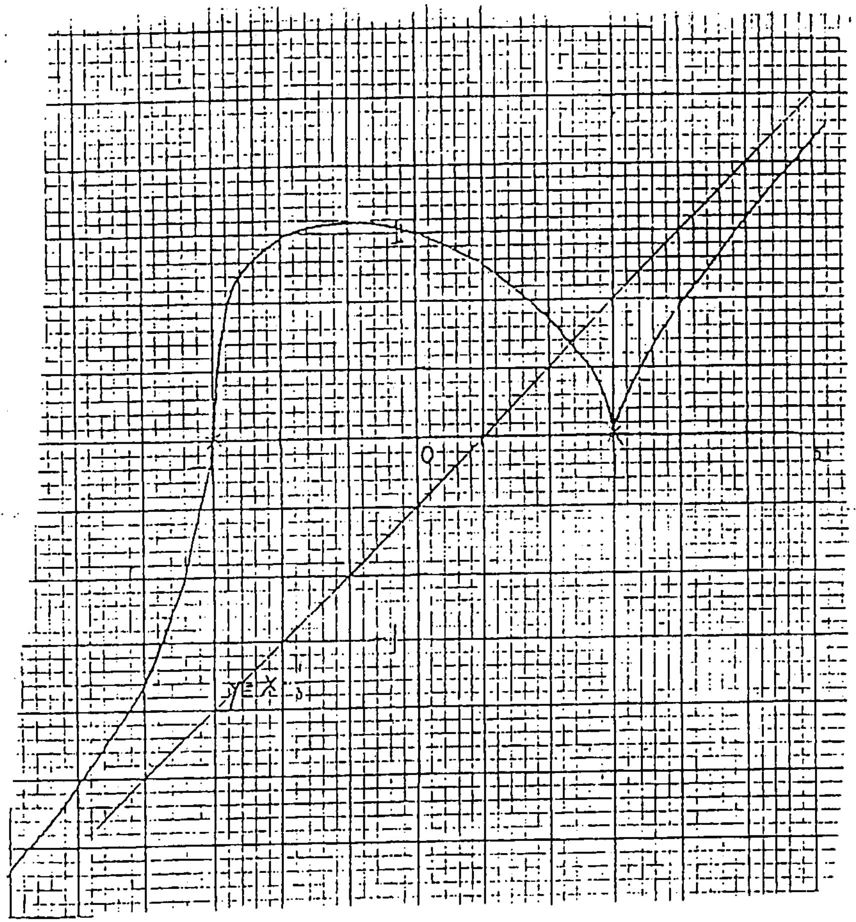
$= -\frac{1}{3}$ — 1A

2

$\therefore$ the asymptote is $y = x - \frac{1}{3}$.

Solutions	Marks

10. (f)



$\frac{2}{2}$

Solutions	Marks

11. (a) (non-parallel)

(1, 2, 3) and (2, 3, 5) are not in proportion — 1M

$\therefore$ L_1, L_2 are non-parallel.

(non-intersecting)

If $(\alpha, \beta, \gamma) \in L_1 \cap L_2$,

then $\frac{\alpha - 2}{1} = \frac{\beta - 3}{2} = \frac{\gamma - 3}{3}$ — 1M

$\frac{\alpha - 4}{2} = \frac{\beta - 6}{3} = \frac{\gamma - 11}{5}$

$$\Rightarrow \begin{cases} \alpha - 2 = \frac{1}{2}(\beta - 3) = \frac{1}{3}(\gamma - 3) \\ \alpha - 2 = \frac{1}{3}(\beta - 6) + 1 = \frac{1}{5}(\gamma - 11) + 1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}(\beta - 3) = \frac{1}{3}(\beta - 6) + 1 \\ \frac{1}{3}(\gamma - 3) = \frac{1}{5}(\gamma - 11) + 1 \end{cases}$$

$$\Rightarrow \begin{cases} 3(\beta - 3) = 2(\beta - 6) + 6 \\ 5(\gamma - 3) = 3(\gamma - 11) + 15 \end{cases}$$

$$\Rightarrow \begin{cases} \beta = -12 + 6 + 9 = 3 \\ \gamma = \frac{1}{2}(-33 + 15 + 15) = -\frac{3}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\beta - 3}{2} = \frac{3 - 3}{2} = 0 \\ \frac{\gamma - 3}{3} = \frac{-\frac{3}{2}}{3} \end{cases}$$

$\Rightarrow$ $\frac{\beta - 3}{2} \neq \frac{\gamma - 3}{3}$, a contradiction. — $\frac{1M}{3}$

(b) (i) Let $P(x, y, z) \in \pi$.

Now $A = (2, 3, 3) \in L_1$

$\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k} \parallel L_1$ — 1A

$\vec{v} = 2\hat{i} + 3\hat{j} + 5\hat{k} \parallel L_2$ — 1M

$\therefore$ $\vec{AP} \times \vec{u} \cdot \vec{v} = 0$ — 1A

$$\begin{vmatrix} x - 2 & y - 3 & z - 3 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

$1 \cdot (x - 2) - (-1)(y - 3) + (-1)(z - 3) = 0$

$\Rightarrow$ $(x - 2) + (y - 3) - (z - 3) = 0$ — 1A

$x + y - z - 2 = 0$

Solutions	Marks
11. (b) (ii) Let $P(x,y,z) \in \pi'$.	
Now $B = (4,6,11) \in L_2$	1A
$\vec{v} = 2\vec{i} + 3\vec{j} + 5\vec{k} // L_2$	
$\vec{w} = \vec{i} + \vec{j} - \vec{k} \perp \pi$	1M
$\therefore \ \vec{BP} \times \vec{v} \cdot \vec{w} = 0$	
$\begin{vmatrix} x-4 & y-6 & z-11 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{vmatrix} = 0$	
$(-8)(x-4) - (-7)(y-6) + (-1)(z-11) = 0$	
$(-8x+32) + (7y-42) - (z-11) = 0$	
$-8x + 7y - z + 1 = 0$	1A
$8x - 7y + z - 1 = 0$	7
(c) (i) $L_1 : x - 2 = \frac{y-3}{2} = \frac{z-3}{3} = \lambda$	
$\pi' : 8x - 7y + z - 1 = 0$	
Substitute L_1 to π' :	
$8(\lambda+2) - 7(2\lambda+3) + (3\lambda+3) - 1 = 0$	
$(8 - 14 + 3)\lambda + (16 - 21 + 3 - 1) = 0$	
$-3\lambda - 3 = 0$	1A
$\lambda = -1$	
$\therefore \ x = -1 + 2 = 1$	
$y = 2(-1) + 3 = 1$	
$z = 3(-1) + 3 = 0$	1A
$\therefore \ S = (1,1,0)$	
(ii) Direction of the line	1M
$= (\vec{i} + 2\vec{j} + 3\vec{k}) \times (2\vec{i} + 3\vec{j} + 5\vec{k})$	1A
$= \vec{i} + \vec{j} - \vec{k}$	
$\therefore$ Equations of the line is	1A
$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-0}{-1}$	5

Solutions	Marks
12. (a) (i) $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx = 1$	1A
When $n \ge 1$,	
$\int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, d(\sin x)$	
$= [\cos^{2n} x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, d(\cos^{2n} x)$	
$= \int_0^{\frac{\pi}{2}} 2n \cos^{2n-1} x \sin^2 x \, dx$	1A
$= \int_0^{\frac{\pi}{2}} 2n \cos^{2n-1} x (1 - \cos^2 x) \, dx$	
$= 2n \int_0^{\frac{\pi}{2}} \cos^{2n-1} x \, dx - 2n \int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx$	
Hence $I_n = \frac{2n}{2n+1} I_{n-1}$	1A
(ii) When $n = 0$, the result is proved in (a)(i).	1M
Assume the result holds for $n = k \ge 0$,	
i.e. $I_k = \frac{(k!)^2 2^{2k}}{(2k+1)!}$	
Then $I_{k+1} = \frac{2(k+1)}{2(k+1)+1} I_k$	
$= \frac{2(k+1)}{2k+3} \left(\frac{(k!)^2 2^{2k}}{(2k+1)!} \right)$	1M
$= \frac{[(k+1)!]^2 \, 2^{2(k+1)}}{[2(k+1)+1]!}$	
	5
(b) (i) $S_m = \sum_{n=0}^{m} \left(\frac{1}{2}\right)^{n-1} \int_0^{\frac{\pi}{2}} \cos^{2n+1} x \, dx$ (by (a))	1M
$= \int_0^{\frac{\pi}{2}} 2\cos x \sum_{n=0}^{m} \left(\frac{1}{2} \cos^2 x \right)^n dx$	1M
$= \int_0^{\frac{\pi}{2}} 2\cos x \frac{1 - (\frac{1}{2}\cos^2 x)^{m+1}}{1 - (\frac{1}{2}\cos^2 x)} dx$	

Solutions	Marks
(b) (ii) Since $\dfrac{2\cos x(\frac{1}{2}\cos^2 x)^{n+1}}{1-(\frac{1}{2}\cos^2 x)} \geq 0$ for $0 \leq x \leq \frac{\pi}{2}$,	1M

we have

$$S_n = \int_0^{\frac{\pi}{2}} 2\cos x \frac{1-(\frac{1}{2}\cos^2 x)^{n+1}}{1-(\frac{1}{2}\cos^2 x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\frac{1}{2}\cos^2 x} - \frac{2\cos x(\frac{1}{2}\cos^2 x)^{n+1}}{1-(\frac{1}{2}\cos^2 x)}\,dx$$

$$\leq \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\frac{1}{2}\cos^2 x} - 0\,dx \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\frac{1}{2}\cos^2 x}\,dx$$

Also $\dfrac{2\cos x(\frac{1}{2}\cos^2 x)^{n+1}}{1-(\frac{1}{2}\cos^2 x)} \leq \dfrac{1}{2^{n-1}} \quad \forall x$ 1M

because $\dfrac{2\cos x\left(\frac{1}{2}\cos^2 x\right)^{n+1}}{1-\left(\frac{1}{2}\cos^2 x\right)}$

$$= \frac{1}{2^n}\left(\frac{\cos x(\cos^2 x)^{n+1}}{1-\frac{1}{2}\cos^2 x}\right)$$

$$\leq \frac{1}{2^n}\left(\frac{1}{1-\frac{1}{2}\cos^2 x}\right)$$

$$= \frac{1}{2^n}\left(\frac{2}{2-\cos^2 x}\right) \qquad \text{1M}$$

$$\leq \frac{1}{2^n}\left(\frac{2}{2-1}\right)$$

$$= \frac{1}{2^n}\cdot\frac{2}{1}$$

$$= \frac{1}{2^{n-1}}$$

Hence $S_n = \displaystyle\int_0^{\frac{\pi}{2}} 2\cos x \frac{1-\left(\frac{1}{2}\cos^2 x\right)^{n+1}}{1-\frac{1}{2}\cos^2 x}\,dx$

$$\geq \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\left(\frac{1}{2}\cos^2 x\right)} - \frac{1}{2^{n-1}}\,dx \qquad \text{1M}$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\left(\frac{1}{2}\cos^2 x\right)}\,dx - \frac{\pi}{2^n}$$

Solutions	Marks
12. (b) (iii) $\because \lim\limits_{n\to\infty} \frac{\pi}{2^n} = 0$	

$\therefore$ By sandwitch property,

$$\lim_{n\to\infty} S_n = \int_0^{\frac{\pi}{2}} \frac{2\cos x}{1-\frac{1}{2}\cos^2 x}\,dx \qquad \text{1A}$$

$$= \int_0^{\frac{\pi}{2}} \frac{4}{1+\sin^2 x}\,d(\sin x)$$

$$= 4\int_0^1 \frac{1}{1+t^2}\,dt \qquad \text{1A}$$

$$= 4[\tan^{-1} t]_0^1$$

$$= \pi \qquad \text{1A}$$

$\overline{10}$

Solutions	Marks
13. (a) (i) Since $(x^2-1)^n = x^{2n}$ + terms of lower degree	
$\frac{d^n}{dx^n}(x^2-1)^n = \frac{(2n)!}{n!}x^n$ + terms of lower degree	1A
The result follows.	
(ii) When $n = 0$, $P(x) = c = cP_0(x)$.	
Assume the result is true for $n \le k$.	1M
Let $P(x)$ be a polynomial of degree $k+1$.	
Then $\exists\, \alpha_{k+1} \in R$ such that degree of $(P(x) - \alpha_{k+1}P_{k+1}(x))$ is	
less than or equal to k.	1M
By induction assumption, $\exists\, \alpha_i \in R$, $i = 0, \ldots, k$ such that	
$P(x) - \alpha_{k+1}P_{k+1}(x) = \sum_{i=0}^{k} \alpha_i P_i(x)$	1M
Hence $P(x) = \sum_{i=0}^{k+1} \alpha_i P_i(x)$ and the result follows.	4
(b) (i) $R_n(x) = (x^2-1)^n$, $R_n^{(1)}(x) = 2nx(x^2-1)^{n-1}$	
$R_n^{(2)}(x) = 4n(n-1)x^2(x^2-1)^{n-2} + 2n(x^2-1)^{n-1}$	1A
Hence $(1-x^2)R_n^{(2)}(x) + 2x(n-1)R_n^{(1)}(x) + 2nR_n(x)$	
$= [-4n(n-1)x^2(x^2-1)^{n-1} - 2n(x^2-1)^n]$	
$+ 4n(n-1)x^2(x^2-1)^{n-1} + 2n(x^2-1)^n$	
$= 0$	
i.e. When $k = 0$, the result holds.	
Assume the result holds for $k = l \ge 0$.	1M
Then $\frac{d}{dx}\Big[(1-x^2)R_n^{(l+2)}(x) + 2x(n-l-1)R_n^{(l+1)}(x)$	
$+ (l+1)(2n-l)R_n^{(l)}(x)\Big] = 0$	1M
$\Rightarrow \big[(1-x^2)R_n^{(l+3)}(x) - 2xR_n^{(l+2)}(x)\big] + \big[2(n-l-1)R_n^{(l+1)}(x)$	
$+ 2x(n-l-1)R_n^{(l+2)}(x)\big] + (l+1)(2n-l)R_n^{(l+1)}(x) = 0$	
$\Rightarrow (1-x^2)R_n^{((l+1)+2)} + 2x(n-(l+1)-1)R_n^{((l+1)+1)}$	
$+ ((l+1)+1)(2n-(l+1))R_n^{(l+1)}(x) = 0$	
By the principle of M.I., the result follows.	
Putting $k = n$, we have	1M
$(1-x^2)P_n^{(2)}(x) - 2xP_n^{(1)}(x) + n(n+1)P_n(x) = 0$.	
So $[(1-x^2)P_n^{(1)}(x)]'$	
$= (1-x)^2 P_n^{(2)}(x) - 2xP_n^{(1)}(x)$	1M
$= -n(n+1)P_n(x)$	5

Solutions	Marks
13. (c) (i) $n(n+1)\int_{-1}^{1} P_n(x)\,P_m(x)\,dx$	
$= \int_{-1}^{1} P_m(x)(-1)[(1-x^2)P_n'(x)]'\,dx$	1M
$= (-1)\Big\{[P_m(x)(1-x^2)P_n'(x)]_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx\Big\}$	1M
$= (-1)\Big\{0 - \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx\Big\}$	
$= \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx$	
(ii) By symmetry, $m(m+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx = \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx$	1M
hence $n(n+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx = m(m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx$	1A
$\Rightarrow [n(n+1) - m(m+1)]\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$	
$\Rightarrow (n^2 - m^2 + n - m)\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$	
$\Rightarrow (n-m)(n+m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$	1A
If $n \ne m$, then $n - m \ne 0$ and $n + m + 1 > 0$ ($\because n, m \ge 0$)	
$\therefore \int_{-1}^{1}P_m(x)P_n(x)\,dx = 0$	1M
	6