13. (a) Let $G(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $G'(x) < a + bG(x)$ for every $x > 0$, where a and b are constants and $b \neq 0$.

(i) Show that $\frac{d}{dx}[G(x)e^{-bx}] < ae^{-bx}$ for every $x > 0$.

(ii) Deduce that for $x > 0$, $G(x) < G(0)e^{bx} + \frac{a}{b}(e^{bx} - 1)$.

(5 marks)

(b) Let $f(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $|f'(x)| < M|f(x)|$ for every $x > 0$, where M is a positive constant.

(i) Show that

$$|f(x)| < |f(0)| + M\int_0^x |f(t)|\,dt$$

for every $x > 0$.

(ii) By putting $G(x) = \int_0^x |f(t)|\,dt$ in (a), or otherwise, show that

$$|f(x)| < |f(0)|e^{Mx}$$

for every $x > 0$.

(6 marks)

(c) Let $h(x)$ be a function continuously differentiable on $\mathbb{R}$ such that $h'(x) = \sin(h(x))$ for every $x > 0$ and $h(0) = 0$. Using (b), or otherwise, show that $h(x) = 0$ for every $x > 0$.

(4 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1990

PURE MATHEMATICS PAPER I

9.00 am-12.00 noon (3 hours)

This paper must be answered in English

This paper consists of two sections BOTH of which are to be answered.

INSTRUCTIONS FOR SECTION A

1. Answer ALL questions. Write your answers in the light yellow AL(C1) answer book.

2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

INSTRUCTIONS FOR SECTION B

1. Answer any FOUR questions. Write your answers in the separate orange AL(C2) answer book.

2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

SECTION A (40 marks)
Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. Consider the following system of linear equations:

$$(*)\begin{cases} 3x - y + z = 1 \\ 2x - 4y - 5z = 1 \\ 4x + 2y + 7z = c\ , \end{cases}$$

where $c \in \mathbf{R}$.

Suppose $(*)$ is consistent. Find c and solve $(*)$. (4 marks)

2. (a) Resolve $\dfrac{1}{x(x+1)(x+2)}$ into partial fractions.

(b) Evaluate $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)(k+2)}$. (6 marks)

3. (a) If α , β and γ are the roots of $x^3 + Ax^2 + Bx + C = 0$, express $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$ in terms of A , B and C .

(b) Find a cubic equation whose roots are the squares of the roots of $x^3 - 3x + 1 = 0$. (5 marks)

4. Let k and n be non-negative integers.

Prove that (a) $C_k^n = \dfrac{k+1}{n+1} C_{k+1}^{n+1}$, where $0 \leq k \leq n$;

(b) $\sum\limits_{k=0}^{n+1} (-1)^k C_k^{n+1} = 0$;

(c) $\sum\limits_{k=0}^{n} \dfrac{(-1)^k}{k+1} C_k^n = \dfrac{1}{n+1}$. (6 marks)

5. (a) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Using (a), or otherwise, solve $16\cos^4\theta - 20\cos^2\theta + 5 = 0$ for values of θ between 0 and 2π . Hence find the value of

$$\cos^2\frac{\pi}{10}\cos^2\frac{3\pi}{10} .$$

(7 marks)

6. Solve the inequality $|x-1| - |x+2| > 2$. (5 marks)

7. A sequence $\{a_0 , a_1 , a_2 , \ldots\}$ of real numbers is defined by

$$a_0 = 0\ ,\quad a_1 = 1 \quad \text{and} \quad a_n = -a_{n-1} + a_{n-2} \quad \text{for all } n = 2, 3, \ldots .$$

Show that for all non-negative integers n , $a_n = \dfrac{1}{\sqrt{5}}(\alpha^n - \beta^n)$, where α , β are roots of $x^2 + x - 1 = 0$ with $\alpha > 0$, $\beta < 0$.

Also prove that $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \beta$. (7 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. (a) Let X and Y be two square matrices such that $XY = YX$.

Prove that

(i) $(X+Y)^2 = X^2 + 2XY + Y^2$,

(ii) $(X+Y)^n = \sum_{r=0}^{n} C_r^n X^{n-r} Y^r$ for $n = 3, 4, 5, \ldots$.

(Note: For any square matrix A, define $A^0 = I$.)

(3 marks)

(b) By using (a)(ii) and considering $\begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$, or otherwise, find

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{100}$.

(4 marks)

(c) If X and Y are square matrices,

(i) prove that $(X+Y)^2 = X^2 + 2XY + Y^2$ implies $XY = YX$;

(ii) prove that $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$ does NOT imply $XY = YX$.

(Hint: Consider a particular X and Y,

e.g. $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$).

(8 marks)

9. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two vectors in $\mathbf{R}^3$ satisfying

$$\mathbf{w}_1 \cdot \mathbf{w}_1 = \mathbf{w}_2 \cdot \mathbf{w}_2 = 1 \text{ and } \mathbf{w}_1 \cdot \mathbf{w}_2 = 0 .$$

Let $S = \{\alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 \in \mathbf{R}^3 : \alpha_1, \alpha_2 \in \mathbf{R}\}$.

(a) Show that $\mathbf{u} = (\mathbf{u} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{u} \cdot \mathbf{w}_2)\mathbf{w}_2$ for all $\mathbf{u} \in S$.

(3 marks)

(b) For any $\mathbf{v} \in \mathbf{R}^3$, let $\mathbf{w} = (\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v} \cdot \mathbf{w}_2)\mathbf{w}_2$.

Show that

(i) $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in S$;

(ii) $\mathbf{w} \cdot \mathbf{w} \leq \mathbf{v} \cdot \mathbf{v}$, where equality holds if and only if $\mathbf{v} \in S$.

Draw a figure to show the geometrical relationship between $\mathbf{v}$, $\mathbf{w}$ and S.

(12 marks)

10. Let $f : \mathbf{C} \setminus \{0\} \to \mathbf{C}$ be defined by $f(z) = z + \frac{1}{z}$.

(a) If $z = r(\cos\theta + i\sin\theta)$ and $f(z) = u + iv$ where $r > 0$ and θ, u, $v \in \mathbf{R}$, express u and v in terms of r and θ.

(2 marks)

(b) Find and sketch the image of each of the following circles under f:

(i) $|z| = 1$;

(ii) $|z| = a$, $0 < a < 1$.

(4 marks)

(c) Show that f is surjective but not injective.

(4 marks)

(d) Let $E = \{z \in \mathbf{C} \setminus \{0\} : |z| < 1\}$ and $f_E : E \to \mathbf{C}$ be defined by $f_E(z) = f(z)$ for all $z \in E$.
Show that f_E is injective but not surjective.

(5 marks)

11. Let u_0 and v_0 be real numbers such that $0 < v_0 < u_0$.

For $n = 1, 2, \ldots$, define

$$u_n = \frac{u_{n-1} + v_{n-1}}{2} \quad , \quad v_n = \frac{2u_{n-1} v_{n-1}}{u_{n-1} + v_{n-1}} \quad .$$

(a) (i) Show that $u_n \geqslant v_n$ for $n = 0, 1, 2, \ldots$.

(ii) Deduce that $\{u_n\}$ is monotonic decreasing and $\{v_n\}$ is monotonic increasing.

(iii) Show that both $\lim_{n\to\infty} u_n$ and $\lim_{n\to\infty} v_n$ exist.

(5 marks)

(b) (i) Prove that $u_n - v_n \leqslant \frac{1}{2^n}(u_0 - v_0)$ for $n = 0, 1, 2, \ldots$.

(ii) Prove that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

(iii) Evaluate $\lim_{n\to\infty} (u_n v_n)$ and $\lim_{n\to\infty} u_n$.

(10 marks)

12. (a) Let $p > 1$ and $f(x) = x^p - px$ for all $x > 0$.

(i) Find the absolute minimum of $f(x)$ on the interval $(0, \infty)$.

(ii) Deduce that $x^p - 1 \geqslant p(x - 1)$ for all $x > 0$.

(4 marks)

(b) (i) Let α , β , γ and δ be positive numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 \text{ and } \gamma + \delta = 1 .$$

By taking $x = \alpha\gamma$ and $\beta\delta$ respectively, prove that, for $p > 1$,

$$\alpha^{p-1} \gamma^p + \beta^{p-1} \delta^p \geqslant 1 ,$$

where the equality holds if and only if $\alpha\gamma = \beta\delta = 1$.

(ii) Deduce that, if a , b , c and d are positive and $p > 1$, then $\left(\frac{a+b}{a}\right)^{p-1} c^p + \left(\frac{a+b}{b}\right)^{p-1} d^p \geqslant (c + d)^p$.

(4 marks)

(c) Suppose $\{a_i\}_{i=1,2,\ldots}$ and $\{b_i\}_{i=1,2,\ldots}$ are two sequences of positive numbers and $p > 1$.

By considering $a = \left(\sum_{j=1}^{n} a_j{}^p\right)^{\frac{1}{p}}$ and $b = \left(\sum_{j=1}^{n} b_j{}^p\right)^{\frac{1}{p}}$,

prove that $\left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}} \geqslant \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}$,

where the equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \frac{a}{b}$.

(7 marks)

13. Let $M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\theta \in \mathbf{R}$.

(a) Show that

(i) $M_\theta M_\phi = M_{\theta+\phi}$ for all θ, $\phi \in \mathbf{R}$;

(ii) $(M_\theta)^{-1} = M_{(-\theta)}$ for all $\theta \in \mathbf{R}$.

(2 marks)

(b) A relation $\sim$ is defined in $\mathbf{R}^2$ as follows:

For all (x_1, y_1) , $(x_2, y_2) \in \mathbf{R}^2$,

$(x_1, y_1) \sim (x_2, y_2)$ iff $M_\theta \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ for some $\theta \in \mathbf{R}$.

(i) Show that $\sim$ is an equivalence relation.

(ii) Sketch the set $S = \{(x, y) \in \mathbf{R}^2 : (x, y) \sim (1, 0)\}$ in $\mathbf{R}^2$.

(4 marks)

(c) Let $\mathbf{R}^2/\sim$ be the quotient set defined by $\sim$, and let $[x, y]$ denote the equivalence class containing $(x, y) \in \mathbf{R}^2$.

Let $\mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$.

A function $f : \mathbf{R}^2/\sim \to \mathbf{R}_+$ is defined by $f([x, y]) = \sqrt{x^2 + y^2}$ for all $[x, y] \in \mathbf{R}^2/\sim$.

(i) Show that f is well-defined.

(ii) Show that f is bijective.

(iii) Sketch the set $T = \{(x, y) \in \mathbf{R}^2 : (x, y) \in f^{-1}(2)\}$.

(9 marks)

END OF PAPER

90-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1990

PURE MATHEMATICS PAPER II

2.00 pm-5.00 pm (3 hours)

This paper must be answered in English

This paper consists of two sections BOTH of which are to be answered.

INSTRUCTIONS FOR SECTION A

1. Answer ALL questions. Write your answers in the light yellow AL(C1) answer book.
2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

INSTRUCTIONS FOR SECTION B

1. Answer any FOUR questions. Write your answers in the separate orange AL(C2) answer book.
2. Write your Candidate Number, Centre Number and Seat Number in the spaces provided on the cover of the answer book.

SECTION A (40 marks)

Answer ALL questions in this section. Write your answers in the light yellow AL(C1) answer book.

1. By differentiating the function $\frac{\ln x}{x}$, or otherwise, prove that if $e < a < b$, then $a^b > b^a$.

(5 marks)

2. Let n be a positive integer and $x \in \left(0, \frac{\pi}{n+1}\right)$.

Show that

$$\cot kx - \cot (k+1)x = \frac{\sin x}{\sin kx \sin (k+1)x}$$

for all $k = 1, 2, 3, \ldots, n$.

Deduce that

$$\frac{1}{\sin x \sin 2x} + \frac{1}{\sin 2x \sin 3x} + \ldots + \frac{1}{\sin nx \sin (n+1)x} = \frac{\sin nx}{\sin^2 x \sin (n+1)x}.$$

(5 marks)

3. Suppose $f(x)$ and $g(x)$ are real-valued continuous functions on $[0, a]$ satisfying the conditions that $f(x) = f(a-x)$ and $g(x) + g(a-x) = K$ where K is a constant.

Show that $\int_0^a f(x)\,g(x)\,dx = \frac{1}{2}K\int_0^a f(x)\,dx$.

Hence, or otherwise, evaluate $\int_0^\pi x \sin x \cos^4 x\,dx$.

(5 marks)

4. (a) Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\tan x}\right)$.

(b) Evaluate $\int \frac{dx}{\sqrt{x^2 + 4x + 2}}$.

(6 marks)

5. (a) Evaluate $\frac{d}{dx}\int_0^{x^n} f(t)\,dt$, where f is continuous and n is a positive integer.

(b) If $F(x) = \int_{x^3}^{x^2} e^{-t^2}\,dt$, find $F'(1)$.

(6 marks)

6. Find the equation of the plane containing the line $(L): \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-2}{2}$ and the point $A(1, 1, 3)$.

(5 marks)

7. (a) Evaluate $\int \ln (1 + x^2)\,dx$.

(b) Let $u_n = \frac{1}{n^4}\prod_{k=1}^{2n}(n^2 + k^2)^{\frac{1}{n}}$. Prove that $\ln u_n = \sum_{k=1}^{2n}\frac{1}{n}\ln\left(1 + \frac{k^2}{n^2}\right)$.

Hence, or otherwise, find the value of $\lim_{n \to \infty} u_n$.

(8 marks)

SECTION B (60 marks)

Answer any FOUR questions from this section. Write your answers in the separate orange AL(C2) answer book.
Each question carries 15 marks.

8. (a) Let $I_n = \int_0^1 \frac{x^{n+1}}{(1+x)^2}\,dx$ for $n = 0, 1, 2, \ldots$.

(i) Find I_0.

(ii) Prove that $\lim_{n \to \infty} I_n = 0$.

(5 marks)

(b) (i) Prove that

$$\int_0^1 x\left(\frac{1-(-x)^m}{1+x}\right)\left(\frac{1-(-x)^n}{1+x}\right)dx = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}$$

for any positive integers m and n.

(ii) Hence evaluate

$$\lim_{n \to \infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(-1)^{i+j}}{i+j}.$$

(10 marks)

9. Consider the hyperbola $(H): xy = c^2$, $c > 0$.

Let $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ be points on (H) where $t_1^2 \neq t_2^2$, $t_1 \neq 0$ and $t_2 \neq 0$.

(a) Find the equation of the straight line joining the points P and Q, and hence, or otherwise, obtain the equations of the tangents to (H) at P and Q respectively.

(3 marks)

(b) Suppose R is the point of intersection of the tangents at P and Q

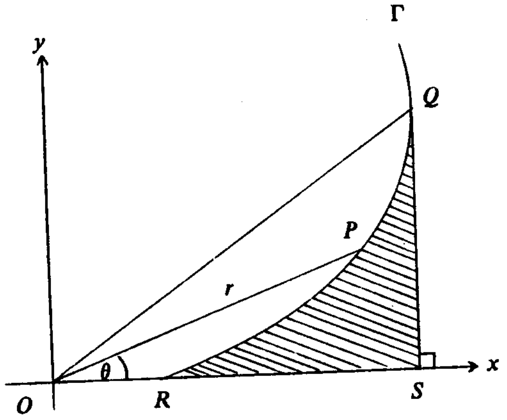
(i) Find the coordinates of R.

(ii) Show that if P and Q are moving in such a way that $t_1 t_2$ is constant, then R lies on a straight line passing through the mid-point of PQ.

(iii) If P and Q are moving in such a way that PQ always touches the ellipse $4x^2 + y^2 = c^2$, show that R lies on an ellipse with centre at the origin. Also find equation of this ellipse.

(12 marks)

10.



In a polar coordinate system in which the origin is the pole and the positive x-axis is the initial line, a curve Γ is given by the polar equation $r = e^{\theta}$, $0 \leq \theta \leq \pi$.

(a) If a point $P(r, \theta)$ on the curve Γ has rectangular coordinates (x, y) , find $\frac{dy}{dx}$ in terms of θ . (3 marks)

(b) Let $P(r, \theta)$ be any point on Γ with $0 < \theta < \pi$. Prove that the tangent at P always makes an angle $\frac{\pi}{4}$ with the line OP . (3 marks)

(c) Find the rectangular coordinates and polar coordinates of the point Q on Γ at which the tangent is perpendicular to the x-axis. (2 marks)

(d) (i) Find the area of the shaded region QRS bounded by the tangent at Q , the curve Γ and the positive x-axis.

(ii) Find the length of the arc RQ . (7 marks)

11. Let $\{a_n\}$ be a sequence of real numbers such that $0 < a_1 < 1$ and $a_{n+1} = \sin(a_n)$ for all $n = 1, 2, \ldots$.

(a) Making use of the fact that $\sin x < x$ for $0 < x < 1$, show that $\lim_{n\to\infty} a_n$ exists and find its value. (7 marks)

(b) (i) Evaluate $\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$.

(ii) Hence find $\lim_{n\to\infty} \left(\frac{1}{a_{n+1}^2} - \frac{1}{a_n^2} \right)$. (5 marks)

(c) It is known that if $\lim_{n\to\infty} x_n$ exists and equals L , then $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n}$ also exists and equals L . Use this fact, or otherwise, to show that $\lim_{n\to\infty} (n a_n^2)$ exists and find its value. (3 marks)

12. Let $f(x) = (2x - 1)x^{\frac{2}{3}}$ for $x \in \mathbf{R}$.

(a) Find $f'(x)$ and $f''(x)$ for $x \neq 0$. (2 marks)

(b) Show that $f'(0)$ does not exist. (1 mark)

(c) Determine those values of x such that

(i) $f'(x) = 0$, (ii) $f'(x) > 0$, (iii) $f'(x) < 0$,
(iv) $f''(x) = 0$, (v) $f''(x) > 0$, (vi) $f''(x) < 0$. (3 marks)

(d) Find the relative extrema and the points of inflexion of the function. (4 marks)

(e) Show that the graph of the function has no asymptotes. (2 marks)

(f) Using the results of (a), (b), (c), (d) and (e), sketch the graph of the function. (3 marks)

13. Let $f(x) = \dfrac{1}{\sqrt{1 + x^2}}$ for all $x \in \mathbf{R}$.

Let $f^{(n)}$ denote the n^{th} derivative of f for $n = 1, 2, \ldots$, and $f^{(0)} = f$.

(a) Prove that

$$(1 + x^2) f'(x) + x f(x) = 0.$$

Deduce that

$$(1 + x^2) f^{(n+1)}(x) + (2n + 1)x f^{(n)}(x) + n^2 f^{(n-1)}(x) = 0$$

for $n = 1, 2, \ldots$.

(2 marks)

(b) Define $P_n(x) = (1 + x^2)^{n+\frac{1}{2}} f^{(n)}(x)$ for $n = 0, 1, 2, \ldots$.

(i) For $n = 0, 1, 2, \ldots,$ prove that

$$P_{n+1}(x) = (1 + x^2) P_n'(x) - (2n + 1)x P_n(x) .$$

Deduce that $P_n(x)$ is a polynomial of degree n with leading coefficient $(-1)^n n!$.

(ii) For $n = 1, 2, \ldots,$ show that

$$P_{n+1}(x) + (2n + 1)x P_n(x) + n^2(1 + x^2) P_{n-1}(x) = 0$$

and find $P_n(0)$.

(iii) For $n = 1, 2, \ldots,$ prove that

$$P_n'(x) = -n^2 P_{n-1}(x)$$

and deduce that, for $r = 1, 2, \ldots, n$,

$$P_n^{(r)}(x) = (-1)^r (n(n-1) \ldots (n - r + 1))^2 P_{n-r}(x) .$$

(iv) Hence show that $P_n(x)$ is either an odd function or an even function for $n = 1, 2, \ldots$.

(Note: $\dfrac{P_n^{(r)}(0)}{r!}$ is the coefficient of x^r in $P_n(x)$.)

(13 marks)

END OF PAPER

91-AL
P MATHS
PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1991

PURE MATHEMATICS PAPER I

9.00 am-12.00 noon (3 hours)

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