89 I

Solutions	Marks	Remarks
1. (a) $AB^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -1 \end{pmatrix}$		
$= \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$	1A	
$B^TA = \begin{pmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$		
$= \begin{pmatrix} 1 & -2 & 1 \\ 0 & 3 & 0 \\ 2 & -1 & 2 \end{pmatrix}$	1A	
(b) $\lvert AB^T \rvert = 9 \neq 0$. AB^T is invertible and	1M	
$(AB^T)^{-1} = \frac{1}{9}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$	1A	
$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$		
As $\lvert B^TA \rvert = 0$, B^TA is not invertible.	1M 5	
2. $[\prod_{k=1}^{n}(a^k+b^k)]^2 = [\prod_{k=1}^{n}(a^k+b^k)][\prod_{k=1}^{n}(a^{n+1-k}+b^{n+1-k})]$	2	
$= \prod_{k=1}^{n}(a^{n+1}+b^{n+1}+a^kb^{n+1-k}+a^{n+1-k}b^k)$	1A	
$> \prod_{k=1}^{n}(a^{n+1}+b^{n+1})$ as $a, b > 0$	1A	
$= (a^{n+1}+b^{n+1})^n$	1A 5	

Solutions	Marks	Remarks
1. (a) $\lim_{x\to\infty} x\left[\sqrt{1+\frac{1}{x}} - \sqrt{1-\frac{1}{x}}\right] = \lim_{x\to\infty} x\left[\frac{(1+\frac{1}{x})-(1-\frac{1}{x})}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}\right]$	1M	May use L'Hospital's rule
$= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}}$		
$= 1$ (as $\lim_{x\to\infty}\frac{1}{x} = 0$)	1A	
(b) $\lim_{n\to\infty} \frac{n}{1+nh+\frac{n(n-1)}{2}h^2} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}+h+\frac{n-1}{2}h^2}$		
$= 0$	1A	
Now $0 < \frac{n}{(1+h)^n} = \frac{n}{1+nh+\frac{n(n-1)}{2}h^2+\ldots \text{ positive terms}}$		
$\leq \frac{n}{1+nh+\frac{n(n-1)}{2}h^2}$ for $n \geq 2$	1M	Accept L'Hospital's Rule
As $\lim_{n\to\infty} \frac{n}{1+nh+\frac{n(n-1)}{2}h^2} = 0$, by the sandwich theorem,		
$\lim_{n\to\infty} \frac{n}{(1+h)^n} = 0$	1A	
	5	
4. The system has infinitely many solutions only if the determinant of its coeff. matrix is zero.	1M	
$\begin{vmatrix} 1 & 1 & 3 \\ 4 & h & -1 \\ 6 & 7 & 5 \end{vmatrix} = -13h + 65$	1A	
$= 0$		
iff $h = 5$	1A	
Now $6x + 7y + 5z = 2(x + y + 3z) + 4x + 5y - z$	1M	
For the system to have infinitely many solutions,		
$2 = 2k + 1$	1M	
$\therefore k = \frac{1}{2}$	1A	
	6	

Solutions	Marks	Remarks
5. The number of 4-digit numbers formed $= P_4^7 = 840$.	1A	
For a number to be divisible by 3, the sum of its digits must be divisible by 3.	1	
Now $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$		
For the sum to be 21, we have $21 = 28 - 7 = 2 - 4$	1M	For first given combination
There are $P_4^4 = 24$ numbers.	1A	
Similarly,		
$18 = 28 - 1 - 2 - 7 = 28 - 1 - 3 - 6 = 28 - 1 - 4 - 5$		
$= 28 - 2 - 3 - 5$		
There are $4 \times P_4^4 = 96$ numbers.	1A	
$15 = 28 - 1 - 5 - 7 = 28 - 2 - 4 - 7 = 28 - 2 - 5 - 6$		
$= 28 - 3 - 4 - 6$		
There are 96 numbers.		
$12 = 28 - 3 - 6 - 7 = 28 - 4 - 5 - 7$		
There are 48 numbers.		
Altogether there are 264 numbers.	1A	
	6	
Alternatively:		
Possible combinations are:		
$\{1, 2, 3, 6\}\{1, 2, 4, 5\}\{1, 2, 5, 7\}\{1, 3, 4, 7\}$	1A	← For any combination
$\{1, 3, 5, 6\}\{1, 4, 6, 7\}\{2, 3, 4, 6\}\{2, 3, 6, 7\}$	+	
$\{2, 4, 5, 7\}\{3, 4, 5, 6\}\{3, 5, 6, 7\}$	1A	
6. $\frac{1}{2}(z + z^{-1}) = \frac{1}{2}[(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)]$		OR $\frac{1}{2}(z + \bar{z})$
$= \cos\theta$	1A	
$\therefore \cos^n\theta = \frac{1}{2^n}(z + z^{-1})^n$	1A	
$= \frac{1}{2^n}\sum_{r=0}^{n} C_r^n z^{n-r} z^{-r}$	1A	
$= \frac{1}{2^n}\sum_{r=0}^{n} C_r^n z^{n-2r}$	1A	
$= \frac{1}{2^n}\sum_{r=0}^{n} C_r^n [\cos(n-2r)\theta + i\sin(n-2r)\theta]$	1A	
$= \frac{1}{2^n}\sum_{r=0}^{n} C_r^n \cos(n-2r)\theta$ as $\cos^n\theta$ is real.	1A	
	6	

Solutions	Marks	Remarks

7. (a) (i) For any z, z' and z'' in $\mathbb{C}$,

(1) $z\ S\ z$ as $\mathrm{Re}(z) \le \mathrm{Re}(z)$ 1A

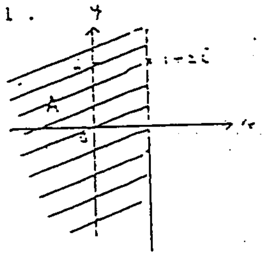
(2) if $z\ S\ z'$ and $z'\ S\ z''$,

then $\mathrm{Re}(z) \le \mathrm{Re}(z') \le \mathrm{Re}(z'')$

$\Rightarrow z\ S\ z''$. 1A

Hence S is both reflexive and transitive.

(ii) $z\ S\ (1+2i)$ iff $\mathrm{Re}(z) \le 1$. 1A



(b) For any z, z', z'' in $\mathbb{C}$,

(1) $z \sim z$ as $z\ S\ z$ and $z\ S\ z$ by (a)

(2) if $z \sim z'$, then $z\ S\ z'$ and $z'\ S\ z$

$\therefore$ $z'\ S\ z$ and $z\ S\ z'$ and hence $z' \sim z$ 1A

(3) if $z \sim z'$ and $z' \sim z''$, then ($z\ S\ z'$ and $z'\ S\ z$)

and ($z'\ S\ z''$ and $z''\ S\ z'$)

i.e. ($z\ S\ z'$ and $z'\ S\ z''$) and ($z''\ S\ z'$ and $z'\ S\ z$)

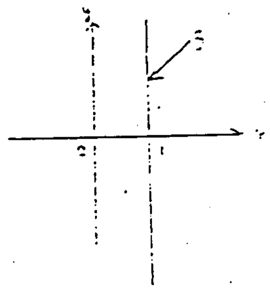
$\therefore$ $z\ S\ z''$ and $z''\ S\ z$ as S is transitive.

$\therefore$ $z \sim z''$ 1A

Thus $\sim$ is an equivalence relation. 1A

$z \sim (1+2i)$ iff $\mathrm{Re}(z) = 1$ 1A

The set B is the line $x = 1$.



Total: 7

Solutions	Marks	Remarks

8. (a)(i) For any $\theta, \phi \in \mathbb{R}$, $A(\theta)A(\phi)$

$= [I - (\sin\theta)S + (1-\cos\theta)S^2][I - (\sin\phi)S + (1-\cos\phi)S^2]$

$= I^2 - (\sin\theta + \sin\phi)S + [\sin\theta\sin\phi + (1-\cos\theta) + (1-\cos\phi)]S^2$

$- [\sin\theta(1-\cos\phi) + \sin\phi(1-\cos\theta)]S^3 + (1-\cos\theta)(1-\cos\phi)S^4$ 1

$= I - (\sin\theta\cos\phi + \cos\theta\sin\phi)S + (1 + \sin\theta\sin\phi - \cos\theta\cos\phi)S^2$ (as $S^3 = -S$) 1

$= I - \sin(\theta+\phi)S + (1 - \cos(\theta+\phi))S^2$

$= A(\theta+\phi)$ 1

(ii) We shall prove by induction. The case where $n=1$ is trivial.

Assume $[A(\theta)]^k = A(k\theta)$ for some positive integer k.

Then $[A(\theta)]^{k+1} = [A(\theta)]^k A(\theta)$

$= A(k\theta)A(\theta)$ 1

$= A((k+1)\theta)$ by (i) 1

Hence $[A(\theta)]^n = A(n\theta)$ $\forall n \ge 1$

(iii) For any $\theta \in \mathbb{R}$, $A(-\theta) = [A(\theta)]^{-1}$ 1

as $A(-\theta)A(\theta) = A(-\theta+\theta)$

$= A(0)$

$= I$ 1

Total: 7

(b)(i) $T^3 = T^2T$

$$= \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$ 1

$$= \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} = -T \quad \text{i.e. } T^3 + T = 0$$ 1

(ii) As $T^3 + T = 0$, putting $S = T$ and $\theta = \frac{3\pi}{2}$ in (a).

$I + T + T^2 = A(\frac{3\pi}{2})$. 2

(1) By (a)(iii) $(I + T + T^2)^{-1} = A(-\frac{3\pi}{2})$ 1

$= I - T + T^2$ 1

(2) $(I + T + T^2)^{1989} = A(\frac{3\pi}{2} \times 1989)$ 1

$= A(1491 \times 2\pi + \frac{3\pi}{2})$

$= A(\frac{3\pi}{2})$

$= I + T + T^2$ 1

Total: 9

Solutions	Marks	Remarks
(a) Let $f(x) = x^n + x + 1$.		
$f'(x) = nx^{n-1} + 1$		
$> 0 \quad \forall x$ if n is odd.	1	
$\therefore f(x)$ increases monotonically.		
Further $f(-1) < 0$ and $f(1) > 0$, $f(x)=0$ has exactly one real root.	1	
Suppose n is even.		
$f'(x) = 0$ iff $x = -\sqrt[n-1]{\frac{1}{n}}$		
$f''(x) = n(n-1)x^{n-2}$		
> 0 at $x = -\sqrt[n-1]{\frac{1}{n}}$		
As f is continuously differentiable, f attains its absolute minimum at $x = -\sqrt[n-1]{\frac{1}{n}}$	1	
$f(-\sqrt[n-1]{\frac{1}{n}}) = (\sqrt[n-1]{\frac{1}{n}})^n - \sqrt[n-1]{\frac{1}{n}} + 1$		
$> (\sqrt[n-1]{\frac{1}{n}})^n$		
> 0	1	
$\therefore f(x) > 0 \quad \forall x$.		
i.e. $f(x) = 0$ has no real root.	1 / 5	
(b) (i) If α is a root of (*),		
$\alpha^n + \alpha + 1 = 0 \Rightarrow \overline{\alpha^n + \alpha + 1} = 0$	1	
$\Rightarrow \overline{\alpha^n} + \overline{\alpha} + \overline{1} = 0$		
$\Rightarrow (\overline{\alpha})^n + \overline{\alpha} + 1 = 0$	1	
$\therefore \overline{\alpha}$ is also a root of (*).		
Now $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\} \subset \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ as the latter is the set of all roots.	1	
On the other hand, for any r, α_r is a root $\Rightarrow \overline{\alpha}_r$ is also a root, i.e. $\overline{\alpha}_r \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.		
$\therefore \alpha_r = \overline{\overline{\alpha}}_r \in \{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\}$		
Thus $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\}$	1	

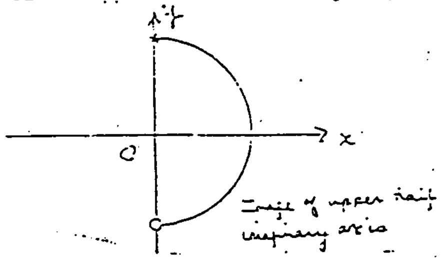
Solutions	Marks	Remarks
(b) (ii) For any integer k,		
$\overline{(\sum_{r=1}^{n} \alpha_r^k)} = (\sum_{r=1}^{n} (\overline{\alpha}_r)^k) = \sum_{r=1}^{n} \alpha_r^k$ by (i) ($\alpha_r \neq 0$)	1	
$\therefore \sum_{r=1}^{n} \alpha_r^k$ is real	1	
(iii) $\alpha_r \neq 0$		OR put $y = \frac{1}{x}$ in (*)
(1) $\sum_{r=1}^{n} \frac{1}{\alpha_r} = \frac{\sum_{r=1}^{n} (\prod_{j \neq r} \alpha_j)}{\prod_{r=1}^{n} \alpha_r}$		
$= \frac{(-1)^{n-1}}{(-1)^n}$	1	
$= -1$	1	
(2) From (*), $\alpha_r^{n-1} + 1 + \frac{1}{\alpha_r} = 0$.		
$\sum_{r=1}^{n} \alpha_r^{n-1} = -\sum_{r=1}^{n} (1 + \frac{1}{\alpha_r})$	1	
$= -n - (-1)$		
$= -n + 1$	1 / 10	

Solutions	Marks	Remarks
10. (a) $f(x) = e^{x-1} - x$		
$f'(x) = e^{x-1} - 1$	1	
$f''(x) = e^{x-1}$		
$f'(x) = 0$ iff $x = 1$ at which	1	
$f''(1) = e^0 > 0$		
As $f(x)$ is continuously differentiable in $\mathbb{R}$, $f(x) \geq f(1)$		
$\Rightarrow e^{x-1} - x \geq e^{1-1} - 1 = 0$, i.e. $e^{x-1} \geq x \quad \forall x \in \mathbb{R}$	1	
	3	
(b) As $b_i \neq 0$, put $x_i = \frac{a_i}{b_i}$ $(i = 1, 2, \ldots, n)$ in (a).		
$e^{\left(\frac{a_i}{b_i} - 1\right)} \geq \frac{a_i}{b_i}$	1	
$\prod_{i=1}^{n} e^{\left(\frac{a_i}{b_i} - 1\right)} \geq \prod_{i=1}^{n} \frac{a_i}{b_i}$ as $\frac{a_i}{b_i} > 0$	1	
$\therefore e^{\left\{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) - n\right\}} \geq \prod_{i=1}^{n} \frac{a_i}{b_i}$	1	
If $\sum_{i=1}^{n} \frac{a_i}{b_i} \leq n$, $1 \geq e^{\left\{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right) - n\right\}} \geq \frac{\prod_{i=1}^{n} a_i}{\prod_{i=1}^{n} b_i}$		
$\Rightarrow \prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$ as $a_i, b_i > 0$.	1	
	4	
(c) (i) For $i = 1, 2, \ldots, n$, put $b_i = \frac{1}{n} \sum_{r=1}^{n} a_r = m \ (> 0)$ in (b).	1	
Then $\sum_{i=1}^{n} \frac{a_i}{b_i} = \frac{1}{m} \sum_{i=1}^{n} a_i = n \leq n$	1	
By (b), $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} m = m^n$	1	
$= \left[\frac{1}{n} \sum_{i=1}^{n} a_i\right]^n$		
$\Rightarrow \left[\prod_{i=1}^{n} a_i\right]^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$	1	

Solutions	Marks	Remarks
10. (c) (ii) Consider the positive numbers $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$.	1	
By (i), $\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_i} \geq \left[\prod_{i=1}^{n} \frac{1}{a_i}\right]^{\frac{1}{n}}$	1	
$= \frac{1}{\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}}$		
$\geq \frac{1}{\frac{1}{n} \sum_{i=1}^{n} a_i}$	1	
$= \frac{1}{m}$		
$\therefore \sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{n}{m}$		
$\Rightarrow \sum_{i=1}^{n} \left(\frac{1}{a_i} - \frac{1}{m}\right) \geq 0$.	1	
	3	

Solutions	Marks	Remarks
11. (a) We first prove the uniqueness. Suppose ∃ integers a, b, c, d such that $a\sqrt{2}+b=c\sqrt{2}+d$.	1	
Then $(a-c)\sqrt{2}=d-b$.		
As $\sqrt{2}$ is irrational, $(a-c)$, $(d-b)$ are integers only if $a-c=d-b=0$, i.e. $a=c$ and $b=d$.	1	
Hence the uniqueness.		
Next observe that the given statement is true for $n=1$ with $a_1=b_1=1$.	1	
Assume that for some $k\ge 1$, $(\sqrt{2}+1)^k=a_k\sqrt{2}+b_k$, where a_k and b_k are positive integers with b_k odd and $b_k\ge a_k\ge 2^{k-1}$.		
$(\sqrt{2}+1)^{k+1}=(a_k\sqrt{2}+b_k)(\sqrt{2}+1)$		
$=(a_k+b_k)\sqrt{2}+(2a_k+b_k)=a_{k+1}\sqrt{2}+b_{k+1}$, say	1	
Now (a_k+b_k) and $(2a_k+b_k)$ are positive integers and		
$2a_k+b_k$ is odd as b_k is odd.	1	
Further $2a_k+b_k\ge a_k+b_k\ge 2a_k\ge 2^k$.	1	
Thus the statement is true ∀ positive n.		
To prove that a_n is odd for even n, first $a_1=1$ is odd.		
Assume that a_k is odd for some odd k,		
$(\sqrt{2}+1)^{k+2}=(a_k\sqrt{2}+b_k)(2\sqrt{2}+3)$	1	
$=(3a_k+2b_k)\sqrt{2}+(4a_k+3b_k)$		
As a_k is odd, $3a_k+2b_k$ is odd.	1	
The answer follows.	6	
(b) For $n=1$, $(\sqrt{2}-1)^1=(-1)^2(1\times\sqrt{2}-1)$		
Suppose $(\sqrt{2}-1)^k=(-1)^{k+1}(a_k\sqrt{2}-b_k)$, $k\ge 1$.		
$(\sqrt{2}-1)^{k+1}=(-1)^{k+1}(a_k\sqrt{2}-b_k)(\sqrt{2}-1)$	2	
$=(-1)^{k+1}[-(a_k+b_k)\sqrt{2}+(2a_k+b_k)]$		
$=(-1)^{k+2}(a_{k+1}\sqrt{2}-b_{k+1})$ by (a).	1	
Thus $(\sqrt{2}-1)^n=(-1)^{n+1}(a_n\sqrt{2}-b_n)$ ∀ $n\ge 1$.		

Solutions	Marks	Remarks
11. (b) Now $0<(\sqrt{2}-1)<\frac{1}{2}$	1	
$\Rightarrow 0<(\sqrt{2}-1)^n<\frac{1}{2^n}$	1	
$\Rightarrow \lvert a_n\sqrt{2}-b_n\rvert<\frac{1}{2^n}$	1	
$\Rightarrow \left\lvert\sqrt{2}-\frac{b_n}{a_n}\right\rvert<\frac{1}{a_n 2^n}$		
$<\frac{1}{2^{2n-1}}$ by (a)	1	
	7	

Solutions	Marks	Remarks
12. (a) For any $z_1, z_2 \in \mathbb{C}\setminus\{-1\}$, $f(z_1) = f(z_2)$		
$\Rightarrow \frac{i(1-z_1)}{1+z_1} = \frac{i(1-z_2)}{1+z_2}$	1	
$\Rightarrow 1 - z_1 + z_2 - z_1z_2 = 1 + z_1 - z_2 - z_1z_2$		
$\Rightarrow z_1 = z_2$		
Hence f is injective.	1	
For any $w \in \mathbb{C}\setminus\{-i\}$, consider $w = \frac{i(1-z)}{1+z}$.		
Changing subject, we have $z = \frac{i-w}{i+w}$. (as $w \neq -i$)	1	
As $z \neq -1$ and $f(z) = w$, f is surjective and thus bijective.	1	
	4	
(b) (i) Let $z = ti$, $t \geq 0$, be any point on the upper half of the imaginary axis	1	OR May use $z + \bar{z} = 0$ iff $\frac{i-w}{i+w} + \overline{\left(\frac{i-w}{i+w}\right)} = 0$ iff $w\bar{w} = 1$ etc.
$f(z) = \frac{i(1-ti)}{1+ti} = \frac{2t + (1-t^2)i}{1+t^2}$	1	
$= x + iy$ where $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$		
We see that $x^2 + y^2 = \frac{4t^2}{(1+t^2)^2} + \frac{1-2t^2+t^4}{(1+t^2)^2} = 1$.	1	
As $x = \frac{2t}{1+t^2} \geq 0$, $f(z)$ lies on right half of the unit circle (including the end point i)	1	
For any point $w = x + iy$ ($\neq -i$) on the right half of the unit circle, we have $x^2 + y^2 = 1$, $x \geq 0$.		
By (a), the pre-image of w is given by		
$z = \frac{i-w}{i+w} = \frac{i-(x+iy)}{i+(x+iy)} = \frac{-x+(1-y)i}{x+(1+y)i} = \frac{-x^2-y^2+1-2xi}{x^2+(1+y)^2} = \frac{2xi}{x^2+(1+y)^2}$ $(x \geq 0)$ as $x^2+y^2=1$		
$\therefore$ z lies on the upper half of the imaginary axis.	1	
	1	

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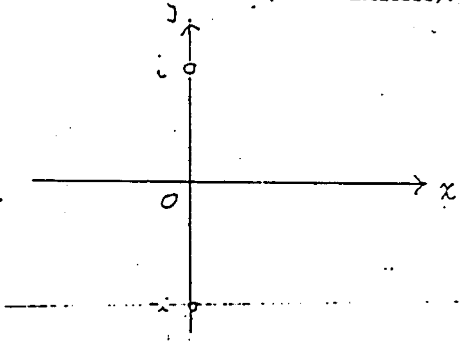
Solutions	Marks	Remarks
12. (b) (ii) Let $z = t$, $t > 0$.	1	OR May show that $z = \bar{z}$ iff $i(w + \bar{w}) = 0$
$f(z) = \frac{i(1-t)}{1+t}$ lies on the imaginary axis.	1	
Further $-1 < \frac{1-t}{1+t} < 1$,		
i.e. $f(z)$ lies between $-i$ and i (end points excluded).	1	
For any $w = yi$, $-1 < y < 1$,		
$f^{-1}(w) = \frac{i-yi}{i+yi} = \frac{1-y}{1+y} > 0$.	1	
The image is exactly the part of the imaginary axis lying between $-i$ and i (end points excluded).		
	1	
	11	

Solutions	Marks	Remarks
13. (a) (i) $T(\underline{0}) = T(\underline{0}+\underline{0})$	1	
$= T(\underline{0}) + T(\underline{0})$		
$\Rightarrow T(\underline{0}) = \underline{0}$	1	
(ii) For any $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$ and $\alpha, \beta, \gamma \in \mathbb{R}$,		
$T(\alpha\underline{x}+\beta\underline{y}+\gamma\underline{z}) = T(\alpha\underline{x}+\beta\underline{y}) + T(\gamma\underline{z})$		
$= \alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z})$	1	
(iii) For any linearly dependent $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$,		
$\exists\, \alpha, \beta, \gamma \in \mathbb{R}$ (not all zero) such that		
$\alpha\underline{x}+\beta\underline{y}+\gamma\underline{z} = \underline{0}$	1	
$T(\alpha\underline{x}+\beta\underline{y}+\gamma\underline{z}) = \underline{0}$		
$\therefore \alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z}) = \underline{0}$		
i.e. $T(\underline{x}), T(\underline{y}), T(\underline{z})$ are linearly dependent.	1	
	5	
(b) To prove (1) $\Rightarrow$ (2), suppose T is injective.		
For any linearly independent $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^3$ and $\alpha, \beta, \gamma \in \mathbb{R}$,		
$\alpha T(\underline{x}) + \beta T(\underline{y}) + \gamma T(\underline{z}) = \underline{0}$	1	
$\Rightarrow T(\alpha\underline{x}+\beta\underline{y}+\gamma\underline{z}) = \underline{0}$	1	
$\Rightarrow \alpha\underline{x}+\beta\underline{y}+\gamma\underline{z} = 0$ by (a) and injectivity of T	1	
$\Rightarrow \alpha = \beta = \gamma = 0$ as $\underline{x}, \underline{y}, \underline{z}$ are linearly independent.	1	
Hence $T(\underline{x}), T(\underline{y}), T(\underline{z})$ are linearly independent.		
To prove (2) $\Rightarrow$ (3), observe that $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are		
linearly independent because if $\exists\, \alpha, \beta, \gamma \in \mathbb{R}$ such that		
$\alpha\underline{e}_1 + \beta\underline{e}_2 + \gamma\underline{e}_3 = \underline{0}$,		
then $(\alpha, \beta, \gamma) = \underline{0}$		
i.e. $\alpha = \beta = \gamma = 0$	1	
$\therefore$ by (2), $T(\underline{e}_1), T(\underline{e}_2), T(\underline{e}_3)$ are linearly independent.	1	
To prove (3) $\Rightarrow$ (1), suppose $T(\underline{x}) = T(\underline{y})$ for some $\underline{x}, \underline{y} \in \mathbb{R}^3$.		
$\exists\, \alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that		
$\underline{x} = \sum_{i=1}^{3} \alpha_i \underline{e}_i, \quad \underline{y} = \sum_{i=1}^{3} \beta_i \underline{e}_i$	1	
Now $T(\underline{x}) = T(\underline{y}) \Rightarrow T(\Sigma \alpha_i \underline{e}_i) = T(\Sigma \beta_i \underline{e}_i)$		
$\Rightarrow \Sigma \alpha_i T(\underline{e}_i) = \Sigma \beta_i T(\underline{e}_i)$	1	
$\Rightarrow \Sigma(\alpha_i - \beta_i) T(\underline{e}_i) = \underline{0}$	1	
$\Rightarrow \alpha_i = \beta_i \;\forall\, i$ as $T(\underline{e}_i)$ are linearly independent by assumption.	1	
	10	

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89 II

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八九年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1989

Pure Mathematics (Paper II)

MARKING SCHEME

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89 II

Solutions	Marks	Remarks
1. f is continuously differentiable for $x > 0$ and		
$f'(x) = \dfrac{x^e \cdot e^x - e \cdot x^{e-1} \cdot e^x}{x^{2e}}$		
$= \dfrac{e^x(x-e)}{x^{e+1}}$	1A	
$f'(x) = 0$ iff $x = e$.	1A	
For $0 < x < e$, $f'(x) < 0 \Rightarrow f$ is strictly decreasing there; For $x > e$, $f'(x) > 0 \Rightarrow f$ is strictly increasing there	2	May consider $f''(x) = \dfrac{e^x}{x^2}\left[\left(1-\dfrac{e}{x}\right)^2 + \dfrac{e}{x^2}\right]$
$\therefore f(x) > f(e) = 1$ if $x \neq e$		
Now $f(\pi) = \dfrac{e^\pi}{\pi^e} > f(e) = 1$.		
$\Rightarrow e^\pi > \pi^e$ (as $\pi^e > 0$)	1A 5	

Solutions	Marks	Remarks
2. $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} = \dfrac{1}{3}\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)$	1M+1A	1M for attempt to solve by partial fractions
$\therefore \dfrac{1}{3}\displaystyle\int\left(\dfrac{1}{x+1} - \dfrac{x-2}{x^2-x+1}\right)dx$		
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{3}\displaystyle\int \dfrac{x-2}{x^2-x+1}dx$	1A	For $\displaystyle\int\dfrac{1}{x+1}dx = \ln\lvert x+1\rvert$
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{3}\displaystyle\int\left(\dfrac{x-\frac{1}{2}}{x^2-x+1} - \dfrac{\frac{3}{2}}{x^2-x+1}\right)dx$	1M	
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{6}\ln\lvert x^2-x+1\rvert + \dfrac{1}{2}\displaystyle\int\dfrac{1}{x^2-x+1}dx$		
$= \dfrac{1}{3}\ln\lvert x+1\rvert - \dfrac{1}{6}\ln\lvert x^2-x+1\rvert + \dfrac{1}{2}\displaystyle\int\dfrac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx$		
$= \dfrac{1}{6}\ln\left\lvert\dfrac{(x+1)^2}{x^2-x+1}\right\rvert + \dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right) + C$	1A 5	

Solutions	Marks	Remarks
(a) Consider x fixed and put $u = xt$. Then $du = x\,dt$. $t = \frac{1}{x} \rightarrow u = 1$; $t = x \rightarrow u = x^2$	1A	
$\therefore f(x) = \displaystyle\int_1^{x^2} \sin\sqrt{u}\,\frac{du}{x}$		
$= \dfrac{1}{x}\displaystyle\int_1^{x^2}\sin\sqrt{u}\,du$	1A	
(b) $\dfrac{df}{dx} = -\dfrac{1}{x^2}\displaystyle\int_1^{x^2}\sin\sqrt{u}\,du + \dfrac{1}{x}\cdot 2x\cdot\sin\sqrt{x^2}$	1A+1A	
$= 2\sin 1$ at $x = 1$ (as $\displaystyle\int_1^1 \sin\sqrt{u}\,du = 0$)	1A	Withheld if answer given as 0.0349
$= (1.683)$	5	

Solutions	Marks	Remarks
4. (a) The two curves intersect at $x = 0$ and $x = 1$.		
Area bounded by the curves is $\displaystyle\int_0^1(\sqrt{x} - x^2)dx$	1A	
$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$	1A	
(b) $y = \ln\cos x \Rightarrow \dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} = -\tan x$	1A	
Arc length $= \displaystyle\int_0^{\frac{\pi}{4}}\sqrt{1+(-\tan x)^2}\,dx$	1M	
$= \displaystyle\int_0^{\frac{\pi}{4}}\sec x\,dx$		
$= [\ln\lvert\sec x + \tan x\rvert]_0^{\frac{\pi}{4}}$	1A	
$= \ln(\sqrt{2}+1)$ units (≈ 0.881)	1A 6	

Solutions	Marks	Remarks

Differentiating $y(1+x^2) = 1$ with respect to x, by Leibnitz rule, $\sum_{r=0}^{n} C_r^n y^{(n-r)} (1+x^2)^{(r)} = 0$. — 1A — May use induct:

As $(1+x^2)' = 2x$, $(1+x^2)^{(2)} = 2$, $(1+x^2)^{(r)} = 0$ for $r \geq 3$,

$(1+x^2)y^{(n)} + n\cdot 2x\cdot y^{(n-1)} + \frac{n(n-1)}{2}\cdot 2y^{(n-2)} = 0$ for $n \geq 2$. — 1A

Now $y^{(n)}(0) = -n(n-1)y^{(n-2)}(0)$ for $n \geq 2$ — 1A

$y(0) = 1$

$y'(0) = 0$

$\therefore y^{(n)}(0) = 0$ if n is odd. .... — 1A

If n is even, $y^{(n)}(0) = -n(n-1)y^{(n-2)}(0)$

$= (-1)^2(n)(n-1)(n-2)(n-3)y^{(n-4)}(0)$ — 1A

$=$ etc.

$= (-1)^{\frac{n}{2}} n!$ $(y^{(0)}(0) = 1)$ .... — 1A — 6

6. (a) Let (r, θ) be the polar coordinates of a point on Γ.

Then $x = r\cos\theta$, $y = r\sin\theta$.

Substituting in Γ, $r^2\sin^2\theta = 1 + 2r\cos\theta$ — 1A

$r^2\sin^2\theta - 2r\cos\theta - 1 = 0$

$r = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta + 4\sin^2\theta}}{2\sin^2\theta}$

$= \frac{\cos\theta + 1}{\sin^2\theta}$ or $\frac{\cos\theta - 1}{\sin^2\theta}$ — 1A — For either

i.e. $\frac{1}{1-\cos\theta}$ or $\frac{-1}{1+\cos\theta}$

Either $r = \frac{1}{1-\cos\theta}$ or $r = \frac{-1}{1+\cos\theta}$ could be the required equation, depending on the restrictions on r.

(b) Let $r = \frac{1}{1-\cos\theta}$ be the polar equation of Γ.

Since PQ passes through O, let $P = (r_1, \theta)$, $Q = (r_2, \theta + \pi)$. — 1

We have $r_1 = \frac{1}{1-\cos\theta}$, $r_2 = \frac{1}{1-\cos(\theta+\pi)} = \frac{1}{1+\cos\theta}$ — 1A

$\frac{8}{3} = r_1 + r_2$ .... — 1M

$= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$

$= \frac{2}{\sin^2\theta}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$.

$\therefore P = (2, \frac{\pi}{3})$, $Q = (\frac{2}{3}, \frac{4\pi}{3})$ — 1A — 6

Solutions	Marks	Remarks

(a) Let $y = [\ln(e+h)]^{\frac{1}{h}}$

$\ln y = \frac{1}{h}\ln(\ln(e+h))$ — 1

$\lim_{h\to 0}\ln y = \lim_{h\to 0}\frac{1}{h}\ln(\ln(e+h))$

$= \lim_{h\to 0}\frac{\frac{1}{(e+h)\ln(e+h)}}{1}$ (By L'Hospital's Rule)

$= \frac{1}{e}$ .... — 1A

$\ln \lim_{h\to 0} y = \frac{1}{e}$

$\therefore \lim_{h\to 0} y = e^{\frac{1}{e}}$ .... — 1A

(b) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{1}{1+(\frac{k}{n})^2}\right)$

$= \int_0^1 \frac{1}{1+x^2}\,dx$ — 2A

$= [\tan^{-1}x]_0^1$ — 1A

$= \frac{\pi}{4}$ — 1A — 7

Solutions	Marks	Remarks
(a) $I_0 = \int_0^1 e^{ax}dx = \frac{1}{a}e^{ax}\Big\vert_0^1 = \frac{1}{a}(e^a - 1)$	1	
For $n \geq 1$, $I_n = \int_0^1 x^n e^{ax}dx$		
$= \frac{1}{a}\int_0^1 x^n de^{ax}$	1	
$= \frac{1}{a}x^n e^{ax}\Big\vert_0^1 - \frac{n}{a}\int_0^1 x^{n-1}e^{ax}dx$	1	
$= \frac{e^a}{a} - \frac{n}{a}I_{n-1}$	1 / 4	
(b) We shall prove inductively.		
First $I_1 = \frac{e^a}{a} - \frac{1}{a}I_0$		
$= \frac{1}{a^2} + e^a[\frac{1}{a} - \frac{1}{a^2}]$	1	
Hence the statement is true for $n = 1$.		
Assume that for some $k \geq 1$,		
$I_k = \frac{(-1)^{k+1}k!}{a^{k+1}} + e^a\left[\frac{1}{a} + \sum_{r=1}^{k}\frac{(-1)^r k(k-1)\ldots(k-r+1)}{a^{r+1}}\right]$		

Solutions	Marks	Remarks
(b) then $I_{k+1} = \frac{e^a}{a} - \frac{k+1}{a}I_k$	2	
$= \frac{e^a}{a} - \left\{\frac{k+1}{a} \times \frac{(-1)^{k+1}k!}{a^{k+1}} + e^a\left[\frac{k+1}{a} \times \frac{1}{a} + \sum_{r=1}^{k}\frac{(-1)^r(k+1)(k)(k-1)\ldots(k-r+1)}{a^{r+2}}\right]\right.$		
$= \frac{(-1)^{k+2}(k+1)!}{a^{k+2}} + e^a\left[\frac{1}{a} - \frac{k+1}{a^2} - \sum_{r=2}^{k+1}\frac{(-1)^{r-1}(k+1)(k)(k-1)\ldots(k+1-r+1)}{a^{r+1}}\right]$	1	
$= \frac{(-1)^{k+2}(k+1)!}{a^{k+2}} + e^a\left\{\frac{1}{a} + \sum_{r=1}^{k+1}\frac{(-1)^r(k+1)(k)\ldots(k+1-r+1)}{a^{r+1}}\right]$	1	
Thus the statement is true for $n = k+1$ and hence $\forall n \geq 1$.	1 / 6	
(c) Put $x = \log\sqrt{u}$;	1	
Then $u = e^{2x}$, $du = 2e^{2x}dx$. When $u = 1$, $x = 0$; when $u = e^2$, $x = 1$.	1	
$\int_1^{e^2}\left(\frac{\log u}{u}\right)^3 du = 16\int_0^1 x^3 e^{-4x}dx$	1	
$= 16I_3$ with $a = -4$	1	
$= 16\left[\frac{(-1)^4 \cdot 3 \cdot 2}{(-4)^4} + e^{-4}\left[\frac{1}{-4} + \frac{-3}{(-4)^2} + \frac{3 \cdot 2}{(-4)^3} - \frac{3 \cdot 2 \cdot 1}{(-4)^4}\right]\right]$		
$= \frac{3}{8} - \frac{71}{8}e^{-4}$ (= 0.2124)	1 / 5	

Solutions	Marks	Remarks

9. (a) Slope of the chord $= \dfrac{\dfrac{t_2^2}{1+t_2^3} - \dfrac{t_1^2}{1+t_1^3}}{\dfrac{t_2}{1+t_2^3} - \dfrac{t_1}{1+t_1^3}}$ [1]

$$= \frac{t_1^2 t_2^2 - t_1 - t_2}{t_1 t_2 (t_1 + t_2) - 1} \quad (\text{for } t_1 \neq t_2)$$ [1]

Equation of the chord is

$$y - \frac{t_1^2}{1+t_1^3} = \frac{t_1^2 t_2^2 - t_1 - t_2}{t_1 t_2 (t_1 + t_2) - 1}\left(x - \frac{t_1}{1+t_1^3}\right)$$ [1]

i.e. $(t_1^2 t_2^2 - t_1 - t_2)x + [1 - t_1 t_2 (t_1 + t_2)]y + t_1 t_2 = 0$

Letting $t_1, t_2 \to t$, the equation of the tangent at t

is $(t^4 - 2t)x + (1 - 2t^3)y + t^2 = 0$ [1]

[4]

(b)

By (a), putting $x = \dfrac{t_3}{1+t_3^3}$, $y = \dfrac{t_3^2}{1+t_3^3}$, a necessary and

sufficient condition for the three points to be collinear

is $(t_1^2 t_2^2 - t_1 - t_2)\dfrac{t_3}{1+t_3^3} + [1 - t_1 t_2 (t_1 + t_2)]\dfrac{t_3^2}{1+t_3^3} + t_1 t_2 = 0$ [1]

$\Leftrightarrow t_1^2 t_2^2 t_3 - t_1 t_3 - t_2 t_3 + t_3^2 - t_1^2 t_2 t_3^2 - t_1 t_2^2 t_3^2 + t_1 t_2 + t_1 t_2 t_3^2 = 0$

$\Leftrightarrow t_1 t_2 t_3 (t_1 t_2 - t_1 t_3 - t_2 t_3 + t_3^2) + (t_1 t_2 - t_1 t_3 - t_2 t_3 + t_3^2) = 0$ [1]

$\Leftrightarrow (t_1 t_2 t_3 + 1)[t_1 (t_2 - t_3) - t_3 (t_2 - t_3)] = 0$

$\Leftrightarrow (t_1 t_2 t_3 + 1)(t_1 - t_3)(t_2 - t_3) = 0$ [1]

$\Leftrightarrow t_1 t_2 t_3 = -1$ as t_1, t_2, t_3 are distinct. [1]

[4]

Solutions	Marks	Remarks

9. (c) Equation of tangent at t is $(t^4 - 2t)x + (1 - 2t^3)y + t^2 = 0$

Putting $x = \dfrac{T}{1+T^3}$, $y = \dfrac{T^2}{1+T^3}$, the tangent intersects the curve at $P(T)$ [1]

iff $t^2 T^3 + (1 - 2t^3)T^2 + (t^4 - 2t)T + t^2 = 0$

iff $(T - t)(t^2 T^2 - (t^3 - 1)T - t) = 0$ [1]

iff $(T - t)(T - t)(t^2 T + 1) = 0$

iff $T = t$ or $-\dfrac{1}{t^2}$ [1]

$T = t$ is the point of contact.

As $t \neq 0$ or ± 1, $-\dfrac{1}{t^2} \neq t$ or -1

$\therefore$ the tangent meets the curve again at another point

T, where $T = -\dfrac{1}{t^2}$. [1]

Let $P(t_1)$, $P(t_2)$, $P(t_3)$ be three distinct points on the curve and let the tangents at these points meet the curve again at $P(T_1)$, $P(T_2)$, $P(T_3)$ respectively, where

$$T_1 = -\frac{1}{t_1^2},\ T_2 = -\frac{1}{t_2^2},\ T_3 = -\frac{1}{t_3^2}.$$ [1]

By (b), $t_1 t_2 t_3 = -1$.

$\therefore T_1 T_2 T_3 = -\dfrac{1}{t_1^2 t_2^2 t_3^2} = -1$ [1]

By (b) again, $P(T_1)$, $P(T_2)$, $P(T_3)$ are collinear. [1]

[7]

Remarks:

OR

From (b), $t_1 t_2 t_3 = -1$.

Letting $t_1, t_2 \to$ etc.

Solutions	Marks	Remarks

10. (a) (i) As $x \to \pm\infty$, $f(x) \to \pm\infty$ respectively.

$\therefore$ the graph of $f(x)$ does not have any horizontal asymptote. On the other hand, $x^2 + 1$ does not vanish for any real x, there is no vertical asymptote. — 1, 1

Now $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} (1 + \frac{8}{x^2+1}) = 1$

and $\lim_{x\to\pm\infty} (f(x) - x) = \lim_{x\to\pm\infty} \frac{8x}{x^2+1} = 0$ — 1

$\therefore$ $y = x$ is an asymptote and is also the only one of the graph of $f(x)$.

(ii) $f'(x) = \frac{(x^2+1)(3x^2+9) - x(x^2+9)(2x)}{(x^2+1)^2} = \frac{(x^2-3)^2}{(x^2+1)^2}$ — 1

$f'(x) = 0$ iff $x = \sqrt{3}$ or $-\sqrt{3}$ — 1

$f''(x) = \frac{(x^2+1)^2(2)(x^2-3)(2x) - (x^2-3)^2(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{16x(x^2-3)}{(x^2+1)^3}$ — 1

$f''(x) = 0$ iff $x = 0$ or $\sqrt{3}$ or $-\sqrt{3}$ — 1

Consider the following table:

	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
$f'(x)$	+	0	+	+	+	0	+
$f''(x)$	−	0	+	0	−	0	+
$f(x)$	↗ ⌢	pt. of inflexion	↗ ⌣	pt. of inflexion	↗ ⌢	pt. of inflexion	↗ ⌣

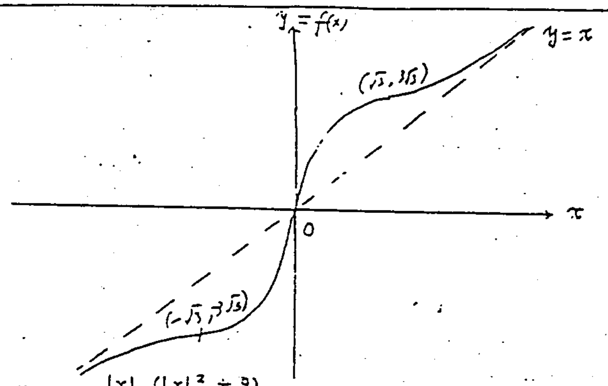
$\therefore$ the graph of $f(x)$ has inflexion points $(-\sqrt{3}, -3\sqrt{3})$, $(0, 0)$ and $(\sqrt{3}, 3\sqrt{3})$ — 1+1

Since f is continuously differentiable, the only possible extreme values occur at x where $f'(x) = 0$. Thus f has no extreme point. — 1

Total: 10

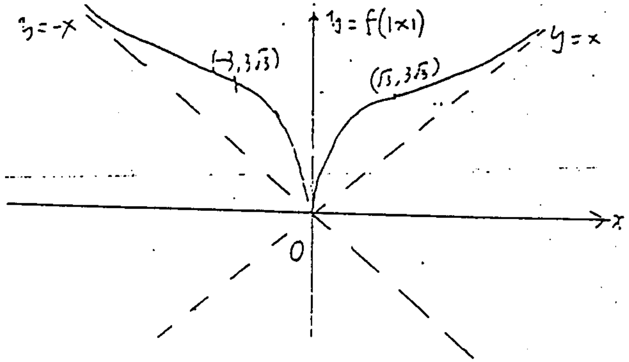
Solutions	Marks	Remarks

10. (b) (i) — 2



(ii) $f(|x|) = \frac{|x|(|x|^2 + 9)}{|x|^2 + 1}$

$= \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$ — 1



Total: 2/5

Solutions	Marks	Remarks
11. (a) $\int_a^b (x-a)f'(x)dx = (x-a)f(x)\Big\vert_a^b - \int_a^b f(x)dx$	1	
$= (b-a)f(b) - \int_a^b f(x)dx$		
$= \int_a^b f(b)dx - \int_a^b f(x)dx$		
$= \int_a^b [f(b) - f(x)]dx$	1	
	2	
(b) $\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\frac{k}{n}) - f(x)]dx = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\frac{k}{n})dx - \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)dx$		
$= \sum_{k=1}^{n} \frac{1}{n} f(\frac{k}{n}) - \int_0^1 f(x)dx$	1	
$= E_n$		
If $\vert f'(x)\vert \leq M \quad \forall x \in [0, 1]$,		
$\vert E_n\vert = \left\vert \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\frac{k}{n}) - f(x)]dx \right\vert$		
$= \left\vert \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \frac{k-1}{n}) f'(x)dx \right\vert$ by (a)	1	
$\leq \sum_{k=1}^{n} \left\vert \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \frac{k-1}{n}) f'(x)dx \right\vert$		
$\leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \vert f'(x)\vert \, \vert x - \frac{k-1}{n}\vert dx$	1	
$\leq \sum_{k=1}^{n} M \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \frac{k-1}{n})dx$	1	
$= \sum_{k=1}^{n} M \left[\frac{1}{2} (x - \frac{k-1}{n})^2 \right]_{\frac{k-1}{n}}^{\frac{k}{n}}$		
$= \frac{M}{2} \cdot \frac{1}{n}$	1	
	5	

Solutions	Marks	Remarks
11. (c) For $1 \leq k \leq n$,		
$\int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\frac{k}{n}) - f(x)]dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}} f'(x)(x - \frac{k-1}{n})dx$ by (a)	1	
$= f'(\xi_k) \int_{\frac{k-1}{n}}^{\frac{k}{n}} (x - \frac{k-1}{n})dx$		
for some $\xi_k \in [\frac{k-1}{n}, \frac{k}{n}]$ by Hint with $h(x) = x - \frac{k-1}{n} \geq 0$		
on $[\frac{k-1}{n}, \frac{k}{n}]$ and $f'(x)$, $h(x)$ are continuous.	1	
$= f'(\xi_k) \left[\frac{1}{2} (x - \frac{k-1}{n})^2 \right]_{\frac{k-1}{n}}^{\frac{k}{n}}$		
$= \frac{f'(\xi_k)}{2n^2}$	1	
$E_n = \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} [f(\frac{k}{n}) - f(x)]dx$		
$= \sum_{k=1}^{n} f'(\xi_k) \frac{1}{2n^2}$ where $\xi_k \in [\frac{k-1}{n}, \frac{k}{n}]$	1	
$\therefore \lim_{n\to\infty} n E_n = \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} f'(\xi_k)\frac{1}{n}$		
$= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} f'(\xi_k)(\frac{k}{n} - \frac{k-1}{n})$	1	
$= \frac{1}{2} \int_0^1 f'(x)dx$ by definition of definite integral	2	
$= \frac{1}{2} [f(1) - f(0)]$		
	8	

Solutions	Marks	Remarks
12. (a) Let R be any point on ℓ with position vector $\vec{r} = \vec{r}_0 + t\vec{a}$		
and let R' be the projection of R on π.		
The unit vector normal to π is $\frac{1}{\sqrt{\vec{n}\cdot\vec{n}}}\vec{n}$.	1	Note
The vector $\overrightarrow{R'R}$ is given by $[(\vec{r} - \vec{r}_0)\cdot\frac{1}{\vec{n}\cdot\vec{n}}\vec{n}]\vec{n}$	2	Candidates may use coordinate geometry method
∴ the vector $\overrightarrow{R_0R'}$ is given by		
$(\vec{r} - \vec{r}_0) - [(\vec{r} - \vec{r}_0)\cdot\frac{1}{\vec{n}\cdot\vec{n}}\vec{n}]\vec{n}$	1	
$= t\vec{a} - t\frac{\vec{a}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$	1	
∴ equation of the projection of ℓ on π is		
$\vec{r} = \vec{r}_0 + t(\vec{a} - \frac{\vec{a}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}),\quad t \in \mathbb{R}$	1 / 6	
(b) (i) Putting $x = -1 - 2t$, $y = 3 + 3t$, $z = 1 + t$ in π_1,		
$4(-1 - 2t) + (3 + 3t) - 2(1 + t) - 4 = 0$		
$t = -1$		
∴ $P_1 = (1, 0, 0)$	1	
Similarly, from ℓ_2 and π_1,		
$4(2 - 8t) + 19t - 2(2 + 4t) - 4 = 0$		
$\Rightarrow t = 0$		
∴ $P_2 = (2, 0, 2)$	1	
$\overrightarrow{P_1P_2} = \vec{i} + 2\vec{k}$	1	
The directions of ℓ_1 and ℓ_2 are given by the vectors		
$-2\vec{i} + 3\vec{j} + \vec{k}$ and $-8\vec{i} + 19\vec{j} + 4\vec{k}$ respectively	1	
$(\vec{i} + 2\vec{k})\cdot(-2\vec{i} + 3\vec{j} + \vec{k}) = 0$		
and $(\vec{i} + 2\vec{k})\cdot(-8\vec{i} + 19\vec{j} + 4\vec{k}) = 0$		
∴ the line segment P_1P_2 is perpendicular to ℓ_1 and ℓ_2.	1	
(ii) $\ell_1 : \vec{r} = (-\vec{i} + 3\vec{j} + \vec{k}) + t(-2\vec{i} + 3\vec{j} + \vec{k})$		
$\ell_2 : \vec{r} = (2\vec{i} + 2\vec{k}) + t(-8\vec{i} + 19\vec{j} + 4\vec{k})$		
By (a), set $\vec{r}_0 = \vec{i}$, $\vec{a} = -2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{n} = 4\vec{i} + \vec{j} - 2\vec{k}$	1	
$\ell_1' : \vec{r} = \vec{i} + t[(-2\vec{i} + 3\vec{j} + \vec{k}) + \frac{1}{3}(4\vec{i} + \vec{j} - 2\vec{k})]$		
$= \vec{i} + \frac{1}{3}t(-2\vec{i} + 10\vec{j} + \vec{k})$	1	
Similarly, $\ell_2' : \vec{r} = 2\vec{i} + 2\vec{k} + 2t(-2\vec{i} + 10\vec{j} + \vec{k})$	1	
Hence, $\ell_1' \mathbin{//} \ell_2'$	1 / 9	

Solutions	Marks	Remarks
(a) (i) $\frac{d}{dx}[G(x)e^{-bx}] = G'(x)e^{-bx} - bG(x)e^{-bx}$	1	
$\le (a + bG(x))e^{-bx} - bG(x)e^{-bx}$		
(as $e^{-bx} > 0$)		
$= ae^{-bx} \quad \forall x \ge 0$	1	
(ii) As $G(x)$ is continuously differentiable, for every		
$x \ge 0$, $\int_0^x \frac{d}{dt}[G(t)e^{-bt}]dt \le \int_0^x ae^{-bt}dt$	1	
$G(x)e^{-bx} - G(0) \le -\frac{a}{b}(e^{-bx} - 1)$	1	
∴ $G(x) \le G(0)e^{bx} + \frac{a}{b}(e^{bx} - 1)$ (as $e^{-bx} > 0$)	1 / 5	
(b) (i) As $f(x) = f(0) + \int_0^x f'(t)dt$,	1	
$\lvert f(x)\rvert \le \lvert f(0)\rvert + \left\lvert\int_0^x f'(t)dt\right\rvert$	1	
$\le \lvert f(0)\rvert + \int_0^x \lvert f'(t)\rvert dt$		
$\le \lvert f(0)\rvert + M\int_0^x \lvert f(t)\rvert dt$ for $x \ge 0$	1	
(ii) $\frac{d}{dx}\int_0^x \lvert f(t)\rvert dt = \lvert f(x)\rvert$		
$\le \lvert f(0)\rvert + M\int_0^x \lvert f(t)\rvert dt$	1	
We see that the function $\int_0^x \lvert f(t)\rvert dt$ satisfies		
the conditions for $G(x)$ in (a) with $a = \lvert f(0)\rvert$ and $b = M > 0$.		
∴ $\int_0^x \lvert f(t)\rvert dt \le e^{Mx}\int_0^0 \lvert f(t)\rvert dt + \frac{\lvert f(0)\rvert}{M}(e^{Mx} - 1)$	1	
$M\int_0^x \lvert f(t)\rvert dt + \lvert f(0)\rvert \le \lvert f(0)\rvert e^{Mx}$		
i.e. $\lvert f(x)\rvert \le \lvert f(0)\rvert e^{Mx}$ by (i)	1 / 6	
(c) As $\lvert h'(x)\rvert = \lvert\sin(h(x))\rvert$		
$\le \lvert h(x)\rvert \quad \forall x \ge 0$	1	
Conditions in (b) are satisfied with $M = 1$.	1	
∴ $\lvert h(x)\rvert \le \lvert h(0)\rvert e^{x}$	1	
$= 0 \quad \forall x \ge 0$	1 / 4	