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PURE MATHEMATICS (PAPER I)

MARKING SCHEME

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Solutions	Marks	Remarks
1. (a) Since α is a root of $P(x) - x = 0$, $P(\alpha) - \alpha = 0$		
i.e. $P(\alpha) = \alpha$	1	
$P(P(\alpha)) - \alpha = P(\alpha) - \alpha$		
$= 0$	1	
$\therefore \alpha$ is also a root of $P(P(x)) - x = 0$.	2	
(b) (i) $P(P(x)) - x = (x^2 + ax + b)^2 + a(x^2 + ax + b) + b - x$		
$= x^4 + 2ax^3 + (a^2+2b+a)x^2 + (2ab+a^2-1)x + b^2 + ab + b$	1	
By (a), $x^2 + (a - 1)x + b$ must be a factor of $P(P(x)) - x$	2	
By inspection, $P(P(x)) - x = (x^2 + (a-1)x + b)(x^2 + (a+1)x + a + b + 1)$	1	
(ii) $P(P(x)) - x = 0$ has four real roots		
iff $(a - 1)^2 \geq 4b$	1	
and $(a + 1)^2 \geq 4(a + b + 1)$.	1	
The second inequality can be written as $(a - 1)^2 \geq 4b + 4$		
$> 4b$	1	
$\therefore$ the required condition is $(a - 1)^2 \geq 4b + 4$.	1	
	5	
(c) Set $P(x) = x^2 - 3x + 1$. Two of the roots of $P(x) - x = 0$,	1	
and hence of $P(P(x)) - x = 0$, are $\frac{4 \pm \sqrt{16 - 4}}{2}$ i.e. $2 \pm \sqrt{3}$	1	
The other two roots are given by $x^2 + (-3 + 1)x + (-3 + 1 + 1) = 0$	1	
$x = 1 \pm \sqrt{2}$	1	
	4	

Alternative Solution

(c) $(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + (1 - x)$		
$= x^4 - 6x^3 + 8x^2 + 2x - 1$	1	
$= (x^2 - 4x + 1)(x^2 - 2x - 1)$	1	
The solutions are $x = \frac{4 \pm \sqrt{16 - 4}}{2}$		
or $x = \frac{2 \pm \sqrt{4 + 4}}{2}$		
i.e. $x = 2 \pm \sqrt{3}$ or $1 \pm \sqrt{2}$.	1+1	
	4	

Solutions	Marks	Remarks
2 (a) $M\underline{x} = \lambda\underline{x}$ has a non-zero solution		
iff $(M - \lambda I)\underline{x}$ has a non-zero solution		
iff $\lvert M - \lambda I\rvert = 0$	1	
Since $\lvert M - \lambda I\rvert = 0$ is a cubic equation in λ, it has at most 3 solutions. i.e. M has at most 3 eigenvalues.	1	
	3	
(b) Suppose λ is an eigenvalue of M. $M\underline{x} = \lambda\underline{x}$ for some non-zero $\underline{x}$.		
Then $a_0 I\underline{x} = a_0\underline{x}$	1	
$a_1 M\underline{x} = a_1\lambda\underline{x}$		
$a_2 M^2\underline{x} = (a_2 M)\lambda\underline{x} = a_2\lambda^2\underline{x}$	1	
etc.		
$a_n M^n\underline{x} = a_n\lambda^n\underline{x}$		
$(a_0 I + a_1 M + \ldots + a_n M^n)\underline{x} = a_0 I\underline{x} + a_1 M\underline{x} + \ldots + a_n M^n\underline{x}$	1	
$= a_0\underline{x} + a_1\lambda\underline{x} + \ldots + a_n\lambda^n\underline{x}$		
$= (a_0 + a_1\lambda + \ldots + a_n\lambda^n)\underline{x}$	1	
	4	
(c) (i) $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\lvert A - \lambda I\rvert = 0$		
iff $\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$	1	
iff $\lambda = 0$ or $\pm\sqrt{2}$	1	
By (a), these are all the eigenvalues of A.		
(ii) $A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$		
Consider $a_0 I + a_1 A + a_2 A^2$	1	
$= \begin{pmatrix} a_2 + a_0 & a_1 & a_2 \\ a_1 & 2a_2 + a_0 & a_1 \\ a_2 & a_1 & a_2 + a_0 \end{pmatrix}$	1	
$= B$		
By (b), if λ is an eigenvalue of A, then $a_0 + a_1\lambda + a_2\lambda^2$ is an eigenvalue of B.	1	
Putting $\lambda = 0, \pm\sqrt{2}$, three eigenvalues of B are a_0, $a_0 + a_1\sqrt{2} + 2a_2$, $a_0 - a_1\sqrt{2} + 2a_2$	1	
As a_0, a_1, a_2 are non-zero integers, these eigenvalues are distinct and the answer follows from (a).	1	
	7	

Solutions	Marks	Remarks
3 (a) We shall prove by induction.		
For $n = 1$, $p_2 q_1 - p_1 q_2 = (a_2 a_1 + 1) - a_1 a_2 = 1$		
$= (-1)^2$	1	
Assuming that $p_{k+1}q_k - p_k q_{k+1} = (-1)^{k+1}$ for some $k \geq 1$,		
then $p_{k+2}q_{k+1} - p_{k+1}q_{k+2} = (a_{k+2}p_{k+1} + p_k)q_{k+1} - p_{k+1}(a_{k+2}q_{k+1} + q_k)$		
$= p_k q_{k+1} - p_{k+1}q_k$		
$= (-1)^{k+2}$ by the induction hypothesis.	2	
Hence the equality holds for all positive n.		
	3	
(b) (i) For any $n \geq 1$, $b_{n+2} - b_n = \dfrac{p_{n+2}}{q_{n+2}} - \dfrac{p_n}{q_n}$	1	
$= \dfrac{(a_{n+2}p_{n+1} + p_n)q_n - (a_{n+2}q_{n+1} + q_n)p_n}{q_{n+2}q_n} = \dfrac{a_{n+2}(p_{n+1}q_n - p_n q_{n+1})}{q_{n+2}q_n}$		
$= \dfrac{(-1)^{n+1}a_{n+2}}{q_{n+2}q_n}$	1	
As a_{n+2}, q_{n+2}, q_n are positive, $b_{n+2} > b_n$ if n is odd	1	
$b_{n+2} < b_n$ if n is even	1	
Hence $\{b_{2n-1}\}$ is strictly increasing and $\{b_{2n}\}$ strictly decreasing.		
(ii) For any $n \geq 1$, $b_{2n} - b_{2n-1} = \dfrac{p_{2n}}{q_{2n}} - \dfrac{p_{2n-1}}{q_{2n-1}}$		
$= \dfrac{p_{2n}q_{2n-1} - p_{2n-1}q_{2n}}{q_{2n}q_{2n-1}}$		
$= \dfrac{(-1)^{2n}}{q_{2n}q_{2n-1}} > 0$	1	
If n is odd, $b_{2n} > b_{2n-1} \geq b_1$ by (i)	1+1	) 2 for 1st
If n is even, $b_{2n-1} < b_{2n} \leq b_2$	1	) given) part
(iii) As $\{b_{2n-1}\}$ is strictly increasing and bounded above by b_2, $\{b_{2n}\}$ is strictly decreasing and bounded below by b_1, both sequences converge.	1	
$\lim\limits_{n\to\infty}(b_{2n} - b_{2n-1}) = \lim\limits_{n\to\infty}\dfrac{(-1)^{2n}}{q_{2n}q_{2n-1}}$	1	
$= 0$		
as $\{q_n\}$ is a sequence of strictly increasing integers.		
Thus $\lim\limits_{n\to\infty} b_{2n} = \lim\limits_{n\to\infty} b_{2n-1}$.	1	
	11	

Solutions — 88 I — Marks — Remarks

4. (a) (i) For any $(a, b), (a', b') \in \mathbb{N}^2$,

$$f((a, b)) = f((a', b')) \Rightarrow (a, b+1) = (a', b'+1)$$
$$\Rightarrow a = a',\ b = b'.$$

Hence f is injective. 1

$(1, 0) \in \mathbb{N}^2$ but $(1, 0) \notin f[\mathbb{N}^2]$.

$\therefore$ f is not surjective. 1

(ii) First, obviously $(a, b) = (k+2, 0)$ is the only element in S_{k+2} with $b = 0$. 1

Next, for $b > 0$, $(a, b) \in f[S_k] \Leftrightarrow (a, b-1) \in S_k$ 1

$$\Leftrightarrow a + 2(b-1) = k$$
$$\Leftrightarrow (a, b) \in S_{k+2}$$

$\therefore S_{k+2} = f[S_k] \cup \{(k+2, 0)\}$, 1

and $n(S_{k+2}) = n(S_k) + 1$ as f is injective and $(k+2, 0) \notin f[S_k]$. 1

(iii) $S_0 = \{(0, 0)\}$ and $S_1 = \{(1, 0)\}$.

$\Rightarrow n(S_0) = 1 = \frac{0 + 2}{2}$

and $n(S_1) = 1 = \frac{1 + 1}{2}$ 1

Assume that for some $j \geq 0$, $n(S_j) = \begin{cases} \frac{j + 2}{2} & \text{if } j \text{ is even,} \\ \frac{j + 1}{2} & \text{if } j \text{ is odd.} \end{cases}$

By (ii) $n(S_{j+2}) = n(S_j) + 1$ 1

$= \begin{cases} \frac{(j + 2) + 2}{2} & \text{if } j + 2 \text{ is even} \\ \frac{(j + 2) + 1}{2} & \text{if } j + 2 \text{ is odd} \end{cases}$. Hence the result. $\frac{1}{9}$

(b) The number of solutions $= \sum_{k=0}^{p} n(S_k)$ 1

If p is even, $\sum_{k=0}^{p} n(S_k) = (1 + 1) + (2 + 2) + \ldots + (\frac{p}{2} + \frac{p}{2}) + \frac{p + 2}{2}$ 1

$= 2 \times \frac{p}{4}(1 + \frac{p}{2}) + \frac{p + 2}{2} = \frac{(p + 2)^2}{4}$ 1

If p is odd, $\sum_{k=0}^{p} n(S_k) = (1 + 1) + (2 + 2) + \ldots + (\frac{p + 1}{2} + \frac{p + 1}{2})$ 1

$= 2 \times \frac{p + 1}{4}(1 + \frac{p + 1}{2}) = \frac{(p + 1)(p + 3)}{4}$ $\frac{1}{5}$

Solutions — 88 I — Marks — Remarks

5. (a) (i) Consider the function $f(x) = a^x + a^{-x}$ for $x > 0$.

$f'(x) = (a^x - a^{-x}) \ln a = \left(\frac{a^{2x} - 1}{a^x}\right) \ln a$ 1

for $x>0$,

For $0 < a < 1$, $a^{2x} < 1$ and $\ln a < 0$, $\therefore f'(x) > 0$.

For $1 < a$, $a^{2x} > 1$ and $\ln a > 0$, $\therefore f'(x) > 0$. .... 1 — for either

Thus $a^x + a^{-x}$ is strictly increasing for $x > 0$. 1

(ii) By (i), $a^{\frac{1}{2}(p-q)} + a^{-\frac{1}{2}(p-q)} < a^{\frac{1}{2}(r-s)} + a^{-\frac{1}{2}(r-s)}$ 1

$\frac{a^{\frac{1}{2}p}}{a^{\frac{1}{2}q}} + \frac{a^{\frac{1}{2}q}}{a^{\frac{1}{2}p}}$

$\frac{1}{a^{\frac{1}{2}(p+q)}}(a^p + a^q) < \frac{1}{a^{\frac{1}{2}(r+s)}}(a^r + a^s)$ 1

$\frac{a^p + a^q}{a^{\frac{1}{2}(p+q)}}$

As $a^{\frac{1}{2}(p+q)} = a^{\frac{1}{2}(r+s)} > 0$, the result follows. $\frac{1}{6}$

(b) (i) As $\frac{u}{v}$ is positive and not equal to 1, putting $a = \frac{u}{v}$ in (a)(ii). 1

$(\frac{u}{v})^p + (\frac{u}{v})^q < (\frac{u}{v})^r + (\frac{u}{v})^s$

$\frac{u^p v^q + u^q v^p}{v^{p+q}} < \frac{u^r v^s + u^s v^r}{v^{r+s}}$ 1

The result follows as $v^{p+q} = v^{r+s} > 0$ ① (Given)

$u^{p+q} = u^{r+s} > 0$ ② 1

①+② Since $u^{p+q} + v^{p+q} = u^{r+s} + v^{r+s}$, 1

$(u^p v^q + u^q v^p) + (u^{p+q} + v^{p+q}) < (u^r v^s + u^s v^r) + (u^{r+s} + v^{r+s})$

$(u^p + v^p)(u^q + v^q) < (u^r + v^r)(u^s + v^s)$ ....

(ii) Putting $p = 1000$, $q = 988$, $r = 1988$, $s = 0$ in (i),

$(u^{1988} + v^{1988})(u^0 + v^0) > (u^{1000} + v^{1000})(u^{988} + v^{988})$

$(u^{1988} + v^{1988}) > \frac{1}{2}(u^{1000} + v^{1000})(u^{988} + v^{988})$

Similarly $u^{988} + v^{988} > \frac{1}{2}(u^{900} + v^{900})(u^{88} + v^{88})$

and $u^{88} + v^{88} > \frac{1}{2}(u^{80} + v^{80})(u^8 + v^8)$

The answer follows. $\frac{1}{[illegible]}$

Solutions	Marks	Remarks
6. (a) (i) For any $a, b \in \mathbb{R}$ and $\underline{x}, \underline{y} \in \mathbb{R}^3$,		
$\phi_{\underline{u}}(a\underline{x}+b\underline{y}) = \underline{u}\cdot(a\underline{x}+b\underline{y})$	1	
$= a(\underline{u}\cdot\underline{x}) + b(\underline{u}\cdot\underline{y})$		
$= a\,\phi_{\underline{u}}(\underline{x}) + b\,\phi_{\underline{u}}(\underline{y})$	1	
$\therefore \phi_{\underline{u}}$ is linear.		
(ii) If $\phi_{\underline{u}}(\underline{x}) = \underline{u}\cdot\underline{x} = 0$ for any $\underline{x} \in \mathbb{R}^3$, then $\underline{u}\cdot\underline{u} = 0$	1	
$\underline{u} = \underline{0}$	1	
(iii) Given $\underline{u}, \underline{v} \in \mathbb{R}^3$, if $\phi_{\underline{u}} = \phi_{\underline{v}}$,		
then $\underline{u}\cdot\underline{x} = \underline{v}\cdot\underline{x} \quad \forall\, \underline{x} \in \mathbb{R}^3$	1	
$\underline{u}\cdot\underline{x} - \underline{v}\cdot\underline{x} = 0$		
$(\underline{u}-\underline{v})\cdot\underline{x} = 0 \quad \forall\, \underline{x} \in \mathbb{R}^3$.	1	
By (ii),		
$\underline{u} - \underline{v} = \underline{0}$ or $\underline{u} = \underline{v}$	1 / 7	
(b) Given $f : \mathbb{R}^3 \to \mathbb{R}$,		
let $\underline{w} = (f(\underline{i}), f(\underline{j}), f(\underline{k}))$	2	
Then for any $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$		
$\underline{w}\cdot\underline{x} = f(\underline{i})x_1 + f(\underline{j})x_2 + f(\underline{k})x_3$	1	
$= f(x_1\underline{i} + x_2\underline{j} + x_3\underline{k})$ as f is linear	1	
$= f(\underline{x})$	1	
If $\exists\, \underline{w}' \in \mathbb{R}^3$ such that $f(\underline{x}) = \underline{w}'\cdot\underline{x} \quad \forall\, \underline{x} \in \mathbb{R}^3$,		
by (a)(iii), $\underline{w}' = \underline{w}$.		
Hence the uniqueness	2 / 7	

Solutions	Marks	Remarks
7. (a) (i) $p(1) = \frac{1}{6}$	1	
$p(2) = \frac{1}{6} + \frac{1}{6}\times\frac{1}{6} \quad (= \frac{7}{36})$		
$p(3) = \frac{1}{6} + \frac{1}{6}\times\frac{1}{6}\times 2 + \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} \quad (= \frac{49}{216})$	1	
(ii) $p(4) = \frac{1}{6} + \frac{1}{6}(p(1) + p(2) + p(3))$	1	
(iii) For $n > 6$, $p(n) = \frac{1}{6}(p(n-6) + p(n-5) + \dots + p(n-1))$	2 / 5	
(b) (i) For $k < 0$, $p(k) = 0$; $p(0) = 1$.		May use induction
For $k > 0$, $p(k) = \frac{1}{6}[p(k-6) + p(k-5) + \dots + p(k-1)]$	1	
$\therefore \sum_{k=1}^{n} p(k) = \sum_{k=1}^{n} \frac{1}{6}[p(k-6) + p(k-5) + \dots + p(k-1)]$	2	
$= \frac{1}{6}\left[\sum_{k=1}^{n} p(k-6) + \sum_{k=1}^{n} p(k-5) + \dots + \sum_{k=1}^{n} p(k-1)\right]$		
$= \frac{1}{6}\left[\sum_{k=0}^{n-6} p(k) + \sum_{k=0}^{n-5} p(k) + \dots + \sum_{k=0}^{n-1} p(k)\right]$	2	
$= \frac{1}{6}\left[6\times\sum_{k=0}^{n} p(k) - 6p(n) - 5p(n-1) - 4p(n-2) - 3p(n-3) - 2p(n-4) - p(n-5)\right]$	1	
$\therefore p(n) + \frac{5}{6}p(n-1) + \frac{4}{6}p(n-2) + \frac{3}{6}p(n-3) + \frac{2}{6}p(n-4) + \frac{1}{6}p(n-5)$		
$= \sum_{k=0}^{n} p(k) - \sum_{k=0}^{n} p(k) = 1$	1	
(ii) Since $\lim_{n\to\infty} p(n)$ exists, $[1 + \frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6}]\lim_{n\to\infty} p(n) = 1$	1	
$\therefore \lim_{n\to\infty} p(n) = \frac{2}{7}$	1 / 9	

8. (a) (i) As $\alpha_j \bar{\alpha}_j = 1$, ... 1

May be awarded below when used.

$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \bar{b}_4$

$= -\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \overline{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)}$

$= -(\alpha_2\alpha_3\alpha_4\alpha_5 + \alpha_1\alpha_3\alpha_4\alpha_5 + \alpha_1\alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_2\alpha_3\alpha_4)$

$= -b_1$ 1

$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \bar{b}_2$

$= -\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \overline{(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_5 + \alpha_2\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_5)}$

$= -(\alpha_4\alpha_5 + \alpha_3\alpha_5 + \alpha_3\alpha_4 + \alpha_2\alpha_5 + \alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_1\alpha_5 + \alpha_1\alpha_4 + \alpha_1\alpha_3 + \alpha_1\alpha_2)$

$= -b_3$ 1

(ii) $\sum_{j=1}^{5} \alpha_j = 0 \Rightarrow b_4 = 0$

$\Rightarrow b_1 = 0$ by (i) as $\alpha_j \neq 0$. 1

Next $0 = (\sum_{j=1}^{5} \alpha_j)^2$ 1

$= \sum_{j=1}^{5} \alpha_j^2 + 2b_3$

$\sum_{j=1}^{5} \alpha_j^2 = 0 \Rightarrow b_3 = 0$ 1

$\Rightarrow b_2 = 0$

As $b_5 = 1$ and $b_0 = -\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$, the result follows. 1 / 7

8 (b) Consider the fifth roots of the complex number $\alpha = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$, which form a regular pentagon. ($|\alpha| = 1$).

These are the roots of the equation $z^5 - \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = 0$. 1

If $\sum_{j=1}^{5} \alpha_j = \sum_{j=1}^{5} \alpha_j^2 = 0$, by (a)(ii),

$(z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) = z^5 - \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = 0$

iff $z = \alpha_1, \alpha_2, \alpha_3, \alpha_4$ or α_5 1

$\therefore$ the fifth roots are exactly the α_j's.

On the other hand, if the α_j's form a regular pentagon,

without loss of generality, let $\alpha_1 = \cos\theta + i\sin\theta$,

$\alpha_2 = \cos(\theta + \frac{2\pi}{5}) + i\sin(\theta + \frac{2\pi}{5}) = \alpha_1 w$, ... 1

$\alpha_3 = \alpha_1 w^2$, $\alpha_4 = \alpha_1 w^3$ and $\alpha_5 = \alpha_1 w^4$, where $w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$.

$\alpha_1 + \alpha_2 + \dots + \alpha_5 = \alpha_1(1 + w + w^2 + w^3 + w^4)$ 1

Now $w^5 = 1 \Rightarrow w^5 - 1 = 0$

$\Rightarrow (w-1)(1 + w + w^2 + w^3 + w^4) = 0$

$\Rightarrow 1 + w + w^2 + w^3 + w^4 = 0$ as $w \neq 1$.

$\therefore \sum_{j=1}^{5} \alpha_j = 0$ 1

Similarly, writing $\alpha_j^2 = \alpha_1^2 w^{2(j-1)}$, $j = 1, 2, \dots, 5$. 1

$\sum_{j=1}^{5} \alpha_j^2 = 0$, noting that $1 + w^2 + w^4 + w^6 + w^8 = 0$. 1 / 7

<u>Alternative solution to the 'only if' part</u>

If the α_j's form a regular pentagon, they are the fifth roots of some complex number $\alpha = a + bi$, $a, b \in \mathbb{R}$, i.e. the roots of the equation $z^5 - \alpha = 0$.

Let $z = \cos\theta + i\sin\theta$ (as $|\alpha| = 1$).

$z^5 - \alpha = 0$ iff $(\cos\theta + i\sin\theta)^5 - (a + bi) = 0$

iff $(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta - a)$

$+ (5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta - b)i = 0$ 1

iff $(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta - a) + (16\sin^5\theta - 20\sin^3\theta + 5\sin\theta - b)i = 0$

As the coefficient of the term $\cos^4\theta$ in the equation $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta - a = 0$ is zero, the sum of the roots $\cos\theta_1 + \cos\theta_2 + \dots + \cos\theta_5 = 0$. Similar consideration 1

of the imaginary part gives $\sin\theta_1 + \sin\theta_2 + \dots + \sin\theta_5 = 0$.

$\therefore$ the sum of the roots of the equation $z^5 - \alpha = 0$ is zero. 1

Next if the α_j's form a regular pentagon, the α_j^2's also form a regular pentagon 1

and $\sum_{j=1}^{5} \alpha_j^2 = 0$ 1

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Solutions	Marks	Remarks
9 (a) (i) Since the 4 vectors $\underline{v}_i - \underline{v}_5$, $i = 1, 2, 3, 4$, are linearly dependent, there exist real numbers t_1, t_2, t_3, t_4, not all zero, such that		
$t_1(\underline{v}_1 - \underline{v}_5) + t_2(\underline{v}_2 - \underline{v}_5) + t_3(\underline{v}_3 - \underline{v}_5) + t_4(\underline{v}_4 - \underline{v}_5) = \underline{0}$	1	
or $t_1\underline{v}_1 + t_2\underline{v}_2 + t_3\underline{v}_3 + t_4\underline{v}_4 - (t_1 + t_2 + t_3 + t_4)\underline{v}_5 = \underline{0}$		
Putting $t_5 = -(t_1 + t_2 + t_3 + t_4)$, we have $\sum_{i=1}^{5} t_i = 0$ and $\sum_{i=1}^{5} t_i\underline{v}_i = \underline{0}$	1	
(ii) (1) As $\left\lvert\frac{t_r}{\lambda_r}\right\rvert \geq \left\lvert\frac{t_i}{\lambda_i}\right\rvert \geq 0$, $t_r = 0 \Rightarrow t_i = 0 \ \forall i$, which contradicts the definition of t_i above. Hence $t_r \neq 0$.	1	
(2) $\sum_{i=1}^{5} \mu_i = \sum_{i=1}^{5} \lambda_i - \frac{\lambda_r}{t_r}\sum_{i=1}^{5} t_i$	1	
$= \sum_{i=1}^{5} \lambda_i = 1$	1	
$\left\lvert\frac{t_r}{\lambda_r}\right\rvert \geq \left\lvert\frac{t_i}{\lambda_i}\right\rvert \Rightarrow \lvert\lambda_i\rvert \geq \left\lvert\frac{\lambda_r}{t_r}\right\rvert \lvert t_i\rvert$	1	
Also $\lambda_i > 0$, $\lambda_i = \lvert\lambda_i\rvert$		
$\geq \left\lvert\frac{\lambda_r}{t_r}\right\rvert \lvert t_i\rvert \geq \frac{\lambda_r}{t_r} t_i$	1	
$\therefore \mu_i \geq 0$.		
Further $\mu_r = \lambda_r - \frac{\lambda_r}{t_r} t_r = 0$	1	
(3) Now $\sum_{i=1}^{5} \mu_i\underline{v}_i = \sum_{i=1}^{5} \lambda_i\underline{v}_i - \frac{\lambda_r}{t_r}\sum_{i=1}^{5} t_i\underline{v}_i$		
$= \sum_{i=1}^{5} \lambda_i\underline{v}_i$	1	
	9	
(b) If $\alpha_i = 0$ for some i, there is nothing to be proved.	1	
Assume that $\alpha_i > 0 \ \forall i$. Let r be such that $\left\lvert\frac{t_r}{\alpha_r}\right\rvert \geq \left\lvert\frac{t_i}{\alpha_i}\right\rvert \ \forall i$.	1	
Define $k_i = \alpha_i - \frac{\alpha_r}{t_r} t_i$, $i = 1, 2, 3, 4, 5$,		
then by (a), $k_i \geq 0$, $\sum_{i=1}^{5} k_i = 1$	1	
$\underline{v} = \sum_{i=1}^{5}\alpha_i\underline{v}_i = \sum_{i=1}^{5} k_i\underline{v}_i$ and	1	
finally $k_r = 0$.	1	
	5	

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香港考試局

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一九八八年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1988

PURE MATHEMATICS (PAPER II)

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Abstammungsurkunde

E 2

(Standesamt) Neustadt in Holstein -/-   Nr. 26/1988

Henning Ming Chin Ho, ----------------

männlichen Geschlechts, -/-

ist am 31. Januar 1988 -/-

in Neustadt in Holstein -/- geboren.

Eltern: Hak Lon Ho und Yeat Fong Ho geb. Pong, beide

wohnhaft in Timmendorfer Strand, Krs. -----

Ostholstein. -/-

Änderungen des Geburtseintrags: -/-

Neustadt in Holstein, den 02. Februar 1988

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(Siegel)

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Der Standesbeamte

(Siegel)

Timmendorfer Platz 23

Solutions | '88 II | Marks | Remarks

1\. (a) (i) Put $t = x - \sqrt{x^2 - 1}$ 1

$$(x - t)^2 = x^2 - 1$$

$$x = \frac{t^2 + 1}{2t} = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

$$dx = \frac{1}{2}\left(1 - \frac{1}{t^2}\right)dt$$ 1

$$\int \frac{dx}{x - \sqrt{x^2 - 1}} = \frac{1}{2}\int \frac{1}{t}\left(1 - \frac{1}{t^2}\right)dt = \frac{1}{2}\left[\int \frac{1}{t}\,dt - \int \frac{1}{t^3}\,dt\right]$$ 1

$$= \ln\sqrt{t} + \frac{1}{4t^2} + c = \ln\sqrt{x - \sqrt{x^2-1}} + \frac{1}{4(2x^2-1-2x\sqrt{x^2-1})} + c$$ 1

(ii) Put $t = \tan\frac{x}{2}$ 1

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$dx = \frac{2dt}{1 + t^2}$$ 1

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x} = \int_0^1 \frac{2dt}{3 + t^2}$$ 1

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}}\Big|_0^1 = \frac{\pi}{3\sqrt{3}}$$ 1

5

(b) $$\int_0^k \frac{dx}{(x^2+1)^{n+1}} = \frac{x}{(x^2+1)^{n+1}}\Big|_0^k + \int_0^k \frac{x(n+1)(2x)}{(x^2+1)^{n+2}}\,dx$$ 1

$$= \frac{k}{(k^2+1)^{n+1}} + 2(n+1)\int_0^k \frac{x^2}{(x^2+1)^{n+2}}\,dx$$

$$= \frac{k}{(k^2+1)^{n+1}} + 2(n+1)\left[\int_0^k \frac{(x^2+1)\,dx}{(x^2+1)^{n+2}} - \int_0^k \frac{dx}{(x^2+1)^{n+2}}\right]$$ 1

$$2(n+1)\int_0^k \frac{dx}{(x^2+1)^{n+2}} = \frac{\cancel{k}^{\,0}}{(k^2+1)^{n+1}} + (2n+1)\int_0^k \frac{dx}{(x^2+1)^{n+1}}$$ 1

$K \to \infty$

Let $k \to \infty$

$$I_{n+1} = \frac{2n + 1}{2n + 2} I_n \quad \text{or} \quad I_n = \frac{2n - 1}{2n} I_{n-1}$$ 1

Hence $$I_n = \frac{(2n-1)(2n-3)\ldots 1}{(2n)(2n-2)\ldots 2} I_0$$ 1

Now $I_0 = \lim\limits_{k\to\infty}\int_0^k \frac{dx}{x^2 + 1} = \lim\limits_{k\to\infty} \tan^{-1} k = \frac{\pi}{2}$ and the answer follows. 1

6

2. (a) (i) $I_n(0) = \int_{\frac{1}{n}}^{n} \frac{dx}{2(1 + x^2)}$ 1

$= \frac{1}{2}[\tan^{-1}x]_{\frac{1}{n}}^{n} = \frac{1}{2}(\tan^{-1} n - \tan^{-1}\frac{1}{n})$ 1

(ii) $I_n(1) = \int_{\frac{1}{n}}^{1} \frac{dx}{(1 + x)(1 + x^2)}$

$= \frac{1}{2}\int_{\frac{1}{n}}^{n} (\frac{1}{1 + x} + \frac{1 - x}{1 + x^2})dx$ 1

$= \frac{1}{2}\int_{\frac{1}{n}}^{n} (\frac{1}{1 + x} + \frac{1}{1 + x^2} - \frac{x}{1 + x^2})dx$

$= \frac{1}{2}[\ln(1 + x) + \tan^{-1}x - \frac{1}{2}\ln(1 + x^2)]_{\frac{1}{n}}^{n}$ 1

$= \frac{1}{2}[\ln\frac{1 + n}{\sqrt{1 + n^2}} + \tan^{-1}n - \ln\frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n^2}}} - \tan^{-1}\frac{1}{n}]$

$= \frac{1}{2}(\tan^{-1}n - \tan^{-1}\frac{1}{n})$ 1

$I_n(-1) = \int_{\frac{1}{n}}^{n} \frac{dx}{(1 + \frac{1}{x})(1 + x^2)}$

$= \frac{1}{2}\int_{\frac{1}{n}}^{n} (-\frac{1}{1 + x} + \frac{1}{1 + x^2} + \frac{x}{1 + x^2})dx$ 1

$= \frac{1}{2}[-\ln(1 + x) + \tan^{-1}x + \frac{1}{2}\ln(1 + x^2)]_{\frac{1}{n}}^{n}$

$= \frac{1}{2}(\tan^{-1}n - \tan^{-1}\frac{1}{n})$ .... $\frac{1}{7}$

(b) (i) Putting $x = \frac{1}{u}$,

$I_n(a) = \int_{n}^{\frac{1}{n}} \frac{-\frac{1}{u^2}}{(1 + \frac{1}{u^a})(1 + \frac{1}{u^2})} du$ 1

$= \int_{\frac{1}{n}}^{n} \frac{u^a}{(1 + u^a)(1 + u^2)} du$ 1

(ii) $2I_n(a) = \int_{\frac{1}{n}}^{n} \frac{1}{(1 + x^a)(1 + x^2)} dx + \int_{\frac{1}{n}}^{n} \frac{x^a}{(1 + x^a)(1 + x^2)} dx$ 2

$= \int_{\frac{1}{n}}^{n} \frac{1 + x^a}{(1 + x^a)(1 + x^2)} dx$ 1

$= \int_{\frac{1}{n}}^{n} \frac{1}{1 + x^2} dx$, which is independent of a 1

(iii) $\lim_{n\to\infty} I_n(a) = \lim_{n\to\infty} \frac{1}{2}(\tan^{-1}n - \tan^{-1}\frac{1}{n}) = \frac{\pi}{4}$ .... $\frac{1}{7}$

3. (a) $2yy' = 4a$

$y' = \frac{2a}{y} = \frac{1}{t}$ (for $t \neq 0$) 1

The equation of the tangent at $(at^2, 2at)$ is

$y - 2at = \frac{1}{t}(x - at^2)$ or $x - ty + at^2 = 0$ (which also holds for $t = 0$) .... $\frac{1}{2}$

(b) The tangents at P and Q are

$x - t_1y + at_1^2 = 0$

$x - t_2y + at_2^2 = 0$

$t_2x - t_1t_2y + at_1^2t_2 = 0$
$t_1x - t_1t_2y + at_2^2t_1 = 0$
$(t_2 - t_1)x = at_1t_2(t_2 - t_1)$

Solving these equations, the point of intersection is given by

$x = at_1t_2$, $y = a(t_1 + t_2)$ .... $\frac{1+1}{2}$

(c) (i) The slopes of PO and QO are respectively $\frac{2}{t_1}$ and $\frac{2}{t_2}$.

$\angle POQ = 90° \Rightarrow \frac{2}{t_1} \times \frac{2}{t_2} = -1$ or $t_1t_2 = -4$. 1

From (b), the point of intersection is $(at_1t_2, a(t_1+t_2))$

∴ the locus of this point is $x = -4a$. 1

(ii) The mid-point of PQ is given by $x = \frac{a(t_1^2 + t_2^2)}{2}$, $y = a(t_1 + t_2)$ 1

Eliminating t_1, t_2,

$(\frac{y}{a})^2 = (t_1 + t_2)^2 = \frac{2x}{a} + 2t_1t_2$ 1

As $t_1t_2 = -4$, $(\frac{y}{a})^2 = \frac{2x}{a} - 8$

∴ the locus of the mid-point is the parabola

$y^2 = 2a(x - 4a)$ .... $\frac{1}{5}$

(d) T': $(x + y)^2 = 8(x - y)$

T' can be written as $(\frac{x + y}{\sqrt{2}})^2 = 4\sqrt{2}(\frac{x - y}{\sqrt{2}})$

Consider the transformation

$Y = \frac{x + y}{\sqrt{2}}$; $X = \frac{x - y}{\sqrt{2}}$, ... 1

which is a rotation of the axes through -45°. 1

Relative to the X-Y axes, T' can be written as $Y^2 = 4\sqrt{2}X$ 1

By (c)(i), the locus is given by $X = -4\sqrt{2}$ 1

or $\frac{x - y}{\sqrt{2}} = -4\sqrt{2}$

i.e. $x - y + 8 = 0$ .... $\frac{1}{5}$

Solutions	Marks	Remarks

4\. (a) $\angle PBD = \theta$ 1

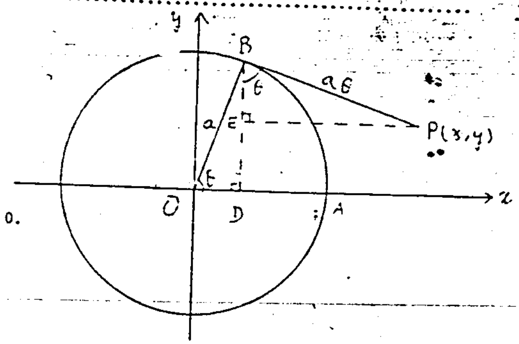
$BP = a\theta$

$x = OD + PE$

$= a(\cos\theta + \theta\sin\theta)$ 1

$y = BD - BE$

$= a(\sin\theta - \theta\cos\theta)$, $\theta \geq 0$.



1/4

(b) $\frac{dy}{d\theta} = a\theta\sin\theta$, $\frac{dx}{d\theta} = a\theta\cos\theta$ 1

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}$

$= \tan\theta$ 1

The slope of the tangent line at any point P on the locus

$= \tan\theta$

$=$ slope of OB.

AS $PB \perp OB$, the thread is normal to the locus

1/3

(c) The area bounded $= \int_0^a x\,dy$ 1

$= \int_0^{\frac{\pi}{2}} a^2(\cos\theta + \theta\sin\theta)\theta\sin\theta d\theta$

$= a^2 \int_0^{\frac{\pi}{2}} (\theta\sin\theta\cos\theta + \theta^2\sin^2\theta)d\theta$ 1

Now $\int_0^{\frac{\pi}{2}} \theta\sin\theta\cos\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta\sin 2\theta d\theta$ 1

$= -\frac{1}{4}\int_0^{\frac{\pi}{2}} \theta d\cos 2\theta$

$= -\frac{1}{4}\left[\theta\cos 2\theta \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos 2\theta d\theta \right]$

$= \frac{\pi}{8}$ 1

$\int_0^{\frac{\pi}{2}} \theta^2\sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} \theta^2 \left(\frac{1 - \cos 2\theta}{2} \right)d\theta$ 1

$= \int_0^{\frac{\pi}{2}} \frac{\theta^2}{2} d\theta - \frac{1}{4}\int_0^{\frac{\pi}{2}} \theta^2 d\sin 2\theta$

$= \frac{\pi^3}{48} - \frac{1}{4}\left[\theta^2\sin 2\theta \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\theta\sin 2\theta d\theta \right]$

$= \frac{\pi^3}{48} + \frac{\pi}{8}$ 1

$\therefore$ the area bounded $= \frac{\pi a^2(12 + \pi^2)}{48}$ $(\approx 1.43a^2)$ 1/7

Solutions	Marks	Remarks

5\. (a) ~~Let $f(x) = ax^2 + bx + c$~~ }

} 1

$f'(x) = 2ax + b$ }

L.S. $= (x - t)(ax + at + b)$.

R.S. $= (x - t)[2a(\frac{x + t}{2}) + b] =$ R.S. 1/2

(b) (i) Differentiating both sides with respect to x 1

$g'(x) = \frac{1}{2}(x - t)g''(\frac{x + t}{2}) + g'(\frac{x + t}{2})$ 1

$g''(x) = \frac{1}{4}(x - t)g'''(\frac{x + t}{2}) + g''(\frac{x + t}{2})$ 1

$\therefore g'''(\frac{x + t}{2}) = \frac{4(g''(x) - g''(\frac{x + t}{2}))}{x - t}$, $x \neq t$. 1

(ii) By L'Hospital's Rule

$\lim_{x \to t} g'''(\frac{x + t}{2}) = \lim_{x \to t} \frac{4[g''(x) - g''(\frac{x + t}{2})]}{x - t}$ ... $\frac{0}{0}$ 2

By the continuity of g''',

$g'''(t) = \lim_{x \to t} \frac{4[g'''(x) - \frac{1}{2}g'''(\frac{x + t}{2})]}{1}$ 1

$= 2g'''(t)$ 1

$\therefore g'''(t) = 0 \quad \forall t \in \mathbb{R}$. 1

Since $g'''(t) = 0$, $g''(t) = c_1$ 2

$g'(t) = c_1 t + c_2$

$g(t) = \frac{c_1}{2}t^2 + c_2 t + c_3$ 1

Hence g is a polynomial of degree ≤ 2.

12

6. (a) For $x > 0$, $f'(x) = \dfrac{2x(4x+3)}{3(x+1)^{\frac{1}{3}}}$, $f''(x) = \dfrac{2(20x^2+30x+9)}{9(x+1)^{\frac{4}{3}}}$ — 1+1

For $x < 0$, $x \neq -1$, $f'(x) = \dfrac{-2x(4x+3)}{3(x+1)^{\frac{1}{3}}}$, $f''(x) = \dfrac{-2(20x^2+30x+9)}{9(x+1)^{\frac{4}{3}}}$ — 1

At $x = 0$, $f'(x)$ exists and equals zero

$f''(x)$ doesn't exist ($f''_+(0) = 2$ but $f''_-(0) = -2$)

At $x = -1$, $f'(x)$ doesn't exist (As $x \to -1^-$, $f'(x) \to \infty$; As $x \to -1^+$, $f'(x) \to -\infty$) .... 1+1

$\therefore$ $f''(x)$ also doesn't exist there. — 5

(b) $f'(x) = 0$ iff $x = 0$ or $-\frac{3}{4}$.

$f''(x) = 0$ iff $x = \dfrac{-15 \pm 3\sqrt{5}}{20}$ (-0.41, -1.09) 1

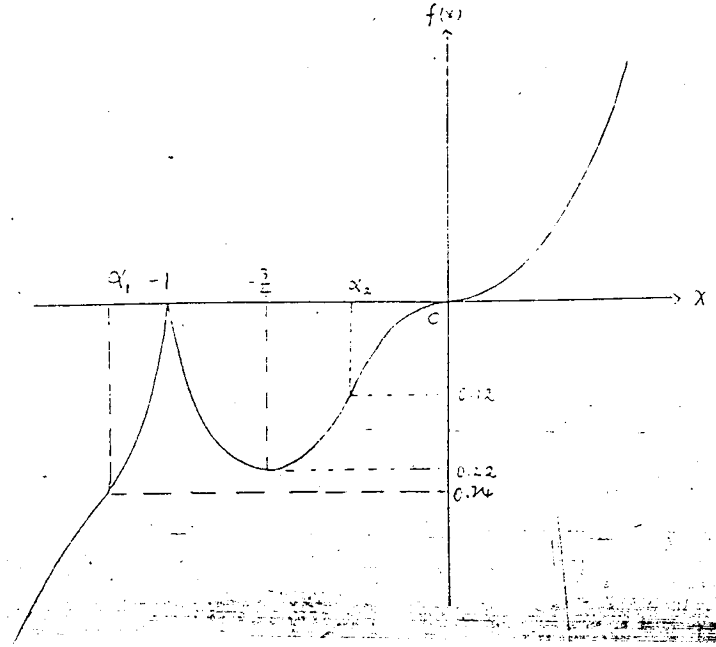
Noting that f is continuous in $\mathbb{R}$, the following could be concluded:

x		$\alpha_1 = \frac{-15-3\sqrt{5}}{20}$		-1		$-\frac{3}{4}$		$\alpha_2 = \frac{-15+3\sqrt{5}}{20}$		0	
y'	+	+	+	not defined	-	0	+	+	+	0	+
y''	-	0	+	not defined	+	+	+	0	-	not defined	+
y	↗	Infl pt.	↗	0 max	↘	min	↗	Infl pt.	↗	0 Infl pt.	↗

Marks: 1, 1, 3 — 6

Such great details not expected

(c)



3

7. (a) $\displaystyle\int_0^{2\pi} (\cos px\cos qx + \sin px\sin qx)dx = \int_0^{2\pi} \cos(p-q)x\,dx$ — 1

$= \begin{cases} 2\pi & \text{if } p = q. \\ 0 & \text{if } p \neq q. \end{cases}$ .... 1 — 2

(b) $\left|\sum_{p=0}^{n} a_p(\cos x + i\sin x)^p\right|^2 = \left|\sum_{p=0}^{n} a_p(\cos px + i\sin px)\right|^2$ — 1

$= \left|\sum_{p=0}^{n} a_p\cos px + i\sum_{p=0}^{n} a_p\sin px\right|^2$

$= \left(\sum_{p=0}^{n} a_p\cos px\right)^2 + \left(\sum_{p=0}^{n} a_p\sin px\right)^2$ 1

$\boxed{= \sum_{p=0}^{n}\sum_{q=0}^{n} a_pa_q\cos px\cos qx + \sum_{p=0}^{n}\sum_{q=0}^{n} a_pa_q\sin px\sin qx}$

$= \sum_{p=0}^{n}\sum_{q=0}^{n} a_pa_q(\cos px\cos qx + \sin px\sin qx)$ 1

$\therefore \displaystyle\int_0^{2\pi} |f(x)|^2dx = \int_0^{2\pi}\left[\sum_{p=0}^{n}\sum_{q=0}^{n} a_pa_q(\cos px\cos qx + \sin px\sin qx)\right]dx$

$= \displaystyle\sum_{p=0}^{n}\sum_{q=0}^{n} a_pa_q\int_0^{2\pi}(\cos px\cos qx + \sin px\sin qx)dx$ 1

$= 2\pi\sum_{p=0}^{n} a_p^2$ by (a) — 1 — 5

(c) $(1 + \cos x + i\sin x)^n = [1 + (\cos x + i\sin x)]^n = \sum_{p=0}^{n} C_p^n(\cos x + i\sin x)^p$ — 2

By (b) $\displaystyle\int_0^{2\pi}|g(x)|^2dx = 2\pi\sum_{p=0}^{n}(C_p^n)^2$ 1

Next, $(1 - \cos x - i\sin x)^n = \sum_{p=0}^{n} C_p^n(-1)^p(\cos x + i\sin x)^p$

$\therefore \displaystyle\int_0^{2\pi}|h(x)|^2dx = \int_0^{2\pi}|g(x)|^2dx$ .... 1 — 4

(d) The coefficient of x^n in the expansion of $(1+x)^{2n} = C_n^{2n}$.

On the other hand, $(1+x)^{2n} = (1+x)^n(1+x)^n = \left(\sum_{p=0}^{n} C_p^n x^p\right)\left(\sum_{p=0}^{n} C_p^n x^p\right)$ — 1

$= \left(\sum_{p=0}^{n} C_p^n x^p\right)\left(\sum_{p=0}^{n} C_{n-p}^n x^p\right)$ — 1

The coefficent of the term $x^n = \sum_{p=0}^{n}(C_p^n)^2$ 1

Hence the result. — 3

8. (a) Substitute x, y, z of L_2 into L_1, $1 + 2t + 1 = 0 \Rightarrow t = -1$

and $1 + 1 + t = 0 \Rightarrow t = -2$,

which are inconsistent. L_1 and L_2 therefore do not intersect each other. 1

Further, putting $x = t$ in L_1, $y = -t$, $z = t$, which are the parametric equations of L_1. 1

Comparing the direction numbers of L_1 and L_2, we see that they are not parallel.

Hence they are not coplanar. 1

3

(b) The system of planes containing L_1 is given by

$x + y + \lambda(y + z) = 0$ or $x + (1 + \lambda)y + \lambda z = 0$. 1

For one of these planes to be parallel to L_2,

$(1, 1+\lambda, \lambda) \cdot (2, 0, 1) = 0$ 1

$\lambda = -2$

$\therefore \pi_1$ is given by $x - y - 2z = 0$. 1

3

(c) Let the equation of π_2 be $Ax + By + Cz + D = 0$.

Since it contains L_2, $(2, 0, 1) \cdot (A, B, C) = 0$. 1

or $2A + C = 0$.

As π_1 and π_2 are perpendicular, $(1, -1, -2) \cdot (A, B, C) = 0$

or $A - B - 2C = 0$. 1

Solving these two equations, $A : B : C = 1 : 5 : -2$. 1

Writing π_2 as $x + 5y - 2z + D = 0$ and putting $(x, y, z) = (1, 1, 1)$,

which is a point on L_2, $D = -4$, $\therefore \pi_2$ is given by $x + 5y - 2z - 4 = 0$. 1

4

(d) The vector $(1, -1, -2)$ is perpendicular to π_1. 1

Let $O = (1+t, 1-t, 1-2t)$ 1

Substituting in π_1, $(1 + t) - (1 - t) - 2(1 - 2t) = 0$

$t = \frac{1}{3}$

$\therefore O = (\frac{4}{3}, \frac{2}{3}, \frac{1}{3})$ 1

The shortest distance between L_1 and L_2

$= \sqrt{(1 - \frac{4}{3})^2 + (1 - \frac{2}{3})^2 + (1 - \frac{1}{3})^2}$

$= \sqrt{\frac{2}{3}}$ 1

4

9. (a) If f is Lipschitz-continuous, $0 \le |f(x_1) - f(x_2)| \le k|x_1 - x_2|$ for any $x_1, x_2 \in I$.

Since $\lim_{x_1 \to x_2} k|x_1 - x_2| = 0$, $\lim_{x_1 \to x_2} |f(x_1) - f(x_2)| = 0$. 1

$\therefore \lim_{x_1 \to x_2} f(x_1) = f(x_2)$ and f is continuous at any $x_2 \in I$. 1

For any $x_1, x_2 \in (0, 1]$, $|g(x_1) - g(x_2)|$

$= |\sqrt{x_1} - \sqrt{x_2}|$

$= \frac{1}{\sqrt{x_1} + \sqrt{x_2}} |x_1 - x_2|$ 1

If x_1 and x_2 tend to zero, $\frac{1}{\sqrt{x_1} + \sqrt{x_2}}$ increases without bound. Hence the

Lipschitz condition cannot be satisfied. 1

4

(b) For any $x_1, x_2 \in I$, we may let $x_1 \ge x_2$. As $f'(x)$ is continuous, — May use mean value theorem

$|f(x_1) - f(x_2)| = \left|\int_{x_2}^{x_1} f'(t)dt\right|$ 1

$\le \int_{x_2}^{x_1} |f'(t)|dt$ 1

$\le \int_{x_2}^{x_1} M dt$

$= M|x_1 - x_2|$ 1

i.e. f is Lipschitz-continuous on I.

3

(c) (i) Consider the function $h(x) = x - f(x)$. Since f is continuous, by (a),

h is also also continuous. 1

If $f(a) = a$ or $f(b) = b$, we are through. 1

Otherwise, let $a < f(a)$, $f(b) < b$. Then $h(a) < 0$ and $h(b) > 0$. 1

$\therefore \exists\, x_0 \in (a, b)$ such that $h(x_0) = 0$ i.e. $x_0 - f(x_0) = 0$. 1

(ii) Let x_0' be in (a, b) such that $x_0' - f(x_0') = 0$. 1

Since f is Lipschitz-continuous with $0 < k < 1$, $k|x_0' - x_0| \ge |f(x_0') - f(x_0)|$

$= |x_0' - x_0|$ 1

The inequality holds only if $x_0' = x_0$. $\therefore$ the solution is unique. 1

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