SOLUTIONS	MARKS	REMARKS
1. (a) (i) If $U \in \mathcal{F}$, $U^TU = I$		
$U^T = U^{-1}$ $\Rightarrow \|U^T\|\|U\| = 1$		
$\|U^T\| = \|U^{-1}\|$	1	det
$\|U\| = \frac{1}{\|U\|}$ $\|U^T\| = \|U\|$ $\|U^TU\| = \|U^T\|\|U\|$		
$\|U\|^2 = 1$ $\|U^{-1}\| = \frac{1}{\|U\|}$		
$\therefore$ $\|U\| = 1$ as $\|U\| > 0$.	1	apply $\|U\|>0$
(ii) 'If' part		
If $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 + b^2 = 1$,		
$U^{-1} = \frac{1}{\|U\|}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$	1	U^{-1}
$= \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$		
$= U^T$		
$\therefore$ $U \in \mathcal{F}$	1	
'Only if' part		
Suppose $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{F}$, then $U^T = U^{-1}$		
$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{\|U\|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$	1	
$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ as $\|U\| = 1$ by (i)	1	
$\Rightarrow$ $a = d$ and $c = -b$. (1)		
$U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$		
and $\|U\| = a^2 + b^2 = 1$ (2)	1	
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(b) (i) Let $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathcal{F}$

$UBU^T = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$

$\Rightarrow \quad B = U^{-1}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}(U^T)^{-1}$ 1

$\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ 1

$= \begin{pmatrix} a^2p + b^2q & ab(p-q) \\ ab(p-q) & b^2p + a^2q \end{pmatrix}$

$\therefore \quad x = y$ 1

i.e. $B = B^T$

(ii) If $B = B^T$, let $B = \begin{pmatrix} w & x \\ x & z \end{pmatrix}$.

For $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathcal{F}$, $a^2 + b^2 = 1$

$UBU^{-1} = \begin{pmatrix} aw + bx & ax + bz \\ -bw + ax & -bx + az \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$= \begin{pmatrix} a^2w + 2abx + b^2z & (a^2-b^2)x + ab(z-w) \\ (a^2-b^2)x + ab(z-w) & b^2w - 2abx + a^2z \end{pmatrix}$ 1

If $x = 0$, $B = \begin{pmatrix} w & 0 \\ 0 & z \end{pmatrix}$, then $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ 作 satisfies the given condition. 1

If $x \neq 0$, equating $(a^2 - b^2)x + ab(z - w)$ to zero.

$\boxed{\dfrac{a^2 - b^2}{ab} = \dfrac{w - z}{x} = c}$ say 1

$\dfrac{a^2 - (1 - a^2)}{a\sqrt{1 - a^2}} = c$ X 非 0 所以要分[illegible]

$(4 + c^2)a^4 - (4 + c^2)a^2 + 1 = 0$

$a^2 = \dfrac{(4 + c^2) \pm \sqrt{c^2(4 + c^2)}}{2(4 + c^2)}$

Putting $a = \left[\dfrac{(4 + c^2) \pm \sqrt{c^2(4 + c^2)}}{2(4 + c^2)}\right]^{\frac{1}{2}}$, $0 < a < 1$

$b = \sqrt{1 - a^2}$ 1

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2. (a) (i) For any given a, b, c, (E) has a unique solution iff

$\begin{vmatrix} 1 & 1 & -1 \\ -k & -1 & k \\ k^2 & 1 & -k \end{vmatrix} \neq 0$ 1

$k(k - 1)^2 \neq 0$ 1

$\therefore \quad k \neq 0$ and $k \neq 1$ 1

(ii) For $k = 0$, the system becomes

$\Delta < 0$

$\frac{\Delta_i}{\Delta} = 0$ only by substitution

$\begin{cases} x + y - z = a \\ -y = b \\ y = c \end{cases}$ 1

$\therefore$ for any values of a, b, c with $b = -c$, the system is consistent. 1

(iii) For $a = b = c = 0$ and $k = 1$, the system reduces to

$x + y - z = 0$ 1

(1, 0, 1) and (1, -1, 0) are two linearly independent solutions as one is not a scalar multiple of the other. 1 + 1

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(b) Assuming, for contradiction, that (E) is consistent, then

$x + y - z = a$

$-kx - y + kz = b$

$k^2x + y - kz = c$

for some $x, y, z \in \mathbb{R}$ 1

$ax_0 + by_0 + cz_0 = (x+y-z)x_0 + (-kx-y+kz)y_0 + (k^2x+y-kz)z_0$ 2

$= (x_0-ky_0+k^2z_0)x + (x_0-y_0+z_0)y + (-x_0+ky_0-kz_0)z$ 1

$= 0 + 0 + 0$

$= 0$ (as (x_0, y_0, z_0) satisfies the 2nd system) 2

This contradicts the assumption that $(a,b,c)\cdot(x_0,y_0,z_0) \neq 0$.

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Alternatively

(b) As (x_0, y_0, z_0) satisfies the 2nd system,

$$\begin{cases} x_0 - ky_0 + k^2z_0 = 0 \\ x_0 - y_0 + z_0 = 0 \\ -x_0 + ky_0 - kz_0 = 0 \end{cases}$$
.... 1

For any real x, y, z,

$x(x_0 - ky_0 + k^2z_0) + y(x_0 - y_0 + z_0) + z(-x_0 + ky_0 - kz_0) = 0$ 2

$(x + y - z)x_0 + (-kx - y + kz)y_0 + (k^2x + y - kz)z_0 = 0$. 1

Since $(a, b, c) \cdot (x_0, y_0, z_0) \neq 0$, the system

$x + y - z = a$

$-kx - y + kz = b$

$k^2x + y - kz = c$

cannot hold simultaneously. 2

i.e. (E) is not solvable.

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3. (a) (i) 'If' part

If $\underline{x}_k$ $(1 \leq k \leq n)$ is a linear combination of the other vectors, let

$\underline{x}_k = \lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \dots + \lambda_n\underline{x}_n$ 1

Then $\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots - \underline{x}_k + \dots + \lambda_n\underline{x}_n = \underline{0}$

Since the coefficient of $\underline{x}_k \neq 0$, the vectors

$\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$ are linearly dependent. 1

'Only if' part

If $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$ are linearly dependent,

$\exists$ $\lambda_1, \lambda_2, \dots, \lambda_n$, not all zero, such that

$\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots + \lambda_n\underline{x}_n = \underline{0}$ 1

Let $\lambda_k \neq 0$ for some k.

Then

$\lambda_k\underline{x}_k = -[\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \dots + \lambda_n\underline{x}_n]$

or

$\underline{x}_k = -\frac{1}{\lambda_k}[\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots + \lambda_{k-1}\underline{x}_{k-1} + \lambda_{k+1}\underline{x}_{k+1} + \dots + \lambda_n\underline{x}_n]$ 1

(ii) Let $\underline{x}_{i_1}, \underline{x}_{i_2}, \dots, \underline{x}_{i_k}$ be k of the vectors which are linearly dependent, where $1 \leq i_1 < i_2 < \dots < i_k \leq n$.

$\exists$ $u_1, u_2, \dots, u_k$, not all zero, such that

$u_1\underline{x}_{i_1} + u_2\underline{x}_{i_2} + \dots + u_k\underline{x}_{i_k} = \underline{0}$ 1

Let $\lambda_l = \begin{cases} u_j & \text{if } l = i_j \\ 0 & \text{otherwise.} \end{cases}$

Then $\exists$ $\lambda_1, \lambda_2, \dots, \lambda_n$, not all zero, such that

$\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \dots + \lambda_n\underline{x}_n = \underline{0}$.

$\therefore$ the n given vectors are linearly dependent. 1

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(b) (i) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the system

$$\begin{cases} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{cases}$$

⇒ trivial soln (0, 0, 0)

|A| = 0 ⇒ non-trivial

has a non-trivial solution $(\lambda_1, \lambda_2, \lambda_3)$. …… 2

i.e. $\exists\ \lambda_1, \lambda_2, \lambda_3$, not all zero, such that

$\lambda_1(a_1, a_2, a_3) + \lambda_2(b_1, b_2, b_3) + \lambda_3(c_1, c_2, c_3) = (0,0,0)$ …… 1

$\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ are linearly dependent.

(ii) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

non-homogeneous equation

$\vec{x}_1, \vec{x}_2, \vec{x}_3$ are three linearly independent vector

a linear combination of vectors

has a unique solution $(\lambda_1, \lambda_2, \lambda_3)$ …… 2

i.e. $\underline{x}_4 = \lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 + \lambda_3\underline{x}_3$

by (a)(i), $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, $\underline{x}_4$ are linearly dependent. …… 1

For any four vectors $\underline{x}_1 = (a_1, a_2, a_3)$, $\underline{x}_2 = (b_1, b_2, b_3)$, $\underline{x}_3 = (c_1, c_2, c_3)$, $\underline{x}_4 = (d_1, d_2, d_3)$ in $\mathbb{R}^3$,

either $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

or $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ …… 1

In each case, the four vectors are linearly dependent …… 1

by (b)(i) and (ii)

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4. (a) (i) Suppose f is injective.

Since $f\circ f = f$, for any $x \in X$

$f\circ f(x) = f(x)$ …… 1

$\Rightarrow f(f(x)) = f(x)$

$\Rightarrow f(x) = x$ …… 1

i.e. $f = i_X$

Suppose f is surjective.

For any $y \in X$, let $y = f(x)$ for some $x \in X$ …… 1

$f(y) = f(f(x))$

$= f\circ f(x)$ …… 1

$= f(x)$

$= y$ …… 1

i.e. $f = i_X$

(ii) If $X = \{a, b\}$, consider $f: X \to X$ defined by

$f(a) = f(b) = a$.

Clearly $f\circ f = f$ but

f is neither injective

nor surjective. …… 2

If X contains more than two elements; let a, b be two distinct elements of X. Consider f defined by

$f(a) = f(b) = a$ and $f(x) = x$ for other $x \in X$.

f is neither injective nor surjective but it satisfies

$f\circ f = f$ and f is non-constant. …… 2

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(b) $\mathcal{P}(E) \neq \emptyset$.

For any $A \subset E$, $h\circ h(A) = h(h(A))$ …… 1

$= h(A \cap B)$

$= (A \cap B) \cap B$

$= A \cap B$

$= h(A)$ …… 1

$\therefore h\circ h = h$.

If h is injective or surjective, by (a)(i),

$h = i_{\mathcal{P}(E)}$. …… 1

$\therefore E = h(E)$ …… 1

$= E \cap B$

$= B$ …… 1

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5. (a) (i) Putting $x = 1$ in $(1+x)^{2n+1} = \sum_{r=0}^{2n+1} C_r^{2n+1} x^r$, 1

$$2^{2n+1} = \sum_{r=0}^{2n+1} C_r^{2n+1}.$$

Since $C_r^{2n+1} = C_{2n+1-r}^{2n+1}$, 1

$$\sum_{r=1}^{n+1} C_{n+r}^{2n+1} = \frac{1}{2}\cdot 2^{2n+1} = 2^{2n} \quad \ldots\ldots\ldots\ldots \quad 1$$

(ii) $(1+x)^m(1+\frac{1}{x})^n = \frac{1}{x^n}(1+x)^{m+n}$

$$= \frac{1}{x^n}\sum_{r=0}^{m+n} C_r^{m+n} x^r \quad 1$$

For $m-n \leq k \leq m$, the coefficient of x^k is C_{n+k}^{m+n}. 1

On the other hand,

$$(1+x)^m(1+\frac{1}{x})^n = (\sum_{s=0}^{m} C_s^m x^s)(\sum_{r=0}^{n} C_r^n \frac{1}{x^r}) \quad 1$$

The coefficient of $x^k = \sum_{r=0}^{m-k} C_{k+r}^m C_r^n$.

Hence the result. 1

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5. (b) (i) When B tosses n coins, the probability that he will obtain r heads ($r = 0, 1, \ldots, n$) is

$$C_r^n \frac{1}{2^r}\cdot\frac{1}{2^{n-r}} = \frac{C_r^n}{2^n}. \quad 1$$

When A and B toss their coins, the probability that A will obtain k ($1 \leq k \leq n+1$) more heads than B is

$$\sum_{r=0}^{n+1-k} \frac{C_{k+r}^{n+1}}{2^{n+1}}\cdot\frac{C_r^n}{2^n} = \frac{1}{2^{2n+1}}\sum_{r=0}^{n+1-k} C_{k+r}^{n+1} C_r^n \quad \ldots\ldots \quad 2$$

$r \leq n+1-k$

$$= \frac{1}{2^{2n+1}} C_{n+k}^{2n+1} \quad 1$$

(ii) The probability that A will obtain more heads than B is $\sum_{k=1}^{n+1} \frac{1}{2^{2n+1}} C_{n+k}^{2n+1}$ ... by (a i) 2

$$= \frac{1}{2^{2n+1}}\cdot 2^{2n}$$

$$= \frac{1}{2} \quad \ldots\ldots\ldots\ldots \quad 1$$

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SOLUTIONS 86 I	MARKS	REMARKS

6(a) Let $f(x) = 2^{p-1}(1+x^p) - (1+x)^p - (1-x)^p$

$f'(x) = p[(2x)^{p-1} - (1+x)^{p-1} + (1-x)^{p-1}]$ 1

For $0 \le x \le 1$, $p \ge 2$,

$\left(\frac{1-x}{1+x}\right)^{p-1} \le \frac{1-x}{1+x}$ and $\left(\frac{2x}{1+x}\right)^{p-1} \le \frac{2x}{1+x}$

$\left(\frac{1-x}{1+x}\right)^{p-1} + \left(\frac{2x}{1+x}\right)^{p-1} \le \frac{1-x}{1+x} + \frac{2x}{1+x}$

$= 1$ 1

i.e. $(1-x)^{p-1} + (2x)^{p-1} \le (1+x)^{p-1}$

$\therefore$ $f'(x) \le 0$ and f is decreasing in $[0, 1]$. 1

$f(x) \ge f(1)$

$= 0$

i.e. $(1+x)^p + (1-x)^p \le 2^{p-1}(1+x^p)$ 1

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(b) $h'(\theta) = pr\sin\theta[(1+r^2-2r\cos\theta)^{\frac{p}{2}-1} - (1+r^2+2r\cos\theta)^{\frac{p}{2}-1}]$ 1

For $r > 0$, $p \ge 2$ $h'(\theta) \le 0$ if $x \in [0, \frac{\pi}{2}]$ 1

and $h'(\theta) \ge 0$ if $x \in [\frac{\pi}{2}, \pi]$.

$\therefore$ $h(\theta)$ is decreasing in $[0, \frac{\pi}{2}]$ and increasing in $[\frac{\pi}{2}, \pi]$.

$h(\theta) \le h(0) = h(\pi)$ if $\theta \in [0, \pi]$ 1

Noting that $h(\theta + n\pi) = h(\theta)$ $\forall n \in I$,

$h(\theta) \le h(0)$ for any $\theta \in \mathbb{R}$. 1

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(c) The inequality is trivial if either z_1 or z_2 equals zero.

Without loss of generality, suppose $|z_1| \ge |z_2| > 0$.

We shall prove $\left|1 + \frac{z_2}{z_1}\right|^p + \left|1 - \frac{z_2}{z_1}\right|^p \le 2^{p-1}\left(1 + \left|\frac{z_2}{z_1}\right|^p\right)$ 1

Let $\frac{z_2}{z_1} = r\,\text{cis}\,\theta$, $0 < r \le 1$. 1

$\left|1 + \frac{z_2}{z_1}\right|^2 = |1 + r\,\text{cis}\,\theta|^2$

$= (1 + r\cos\theta)^2 + (r\sin\theta)^2$

$= 1 + r^2 + 2r\cos\theta$ 1

$\therefore$ $\left|1 + \frac{z_2}{z_1}\right|^p + \left|1 - \frac{z_2}{z_1}\right|^p = (1+r^2+2r\cos\theta)^{\frac{p}{2}} + (1+r^2-2r\cos\theta)^{\frac{p}{2}}$

$\le (1+r)^p + (1-r)^p$ by (b) 1

$\le 2^{p-1}(1+r^p)$ by (a) 1

$= 2^{p-1}\left(1 + \left|\frac{z_2}{z_1}\right|^p\right)$ 1

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SOLUTIONS 86 I	MARKS	REMARKS

7. (a) Putting $z = x+iy$, $q = q_0+iq_1$, $r = r_0+ir_1$, $q_0, q_1, r_0, r_1 \in \mathbb{R}$,

(*) becomes $x^2+y^2+(q_0+iq_1)(x+iy)+(r_0+ir_1)(x-iy)+s = 0 + 0i$ 1

i.e. $\begin{cases} x^2 + y^2 + (q_0+r_0)x + (r_1-q_1)y + s = 0 & \text{......(i)} \\ (q_1+r_1)x + (q_0-r_0)y = 0 & \text{......(ii)} \end{cases}$ 1 1

As $s < 0$, (i) is a circle with positive radius. 1

Further the origin lies inside (i) 1

If $q_1 + r_1 = q_0 - r_0 = 0$, (i) and (ii) (and therefore (*))

has infinitely many solutions. 1

Otherwise (ii) is a straight line through the origin 1

intersecting (i) at two points. $\therefore$ (*) has 2 solutions. 1

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(b) $[tu_1\bar{u}_2+(1-t)v_1\bar{v}_2][(u_1+zv_1)(\bar{u}_1+\bar{z}\bar{v}_1)+(u_2+zv_2)(\bar{u}_2+\bar{z}\bar{v}_2)]$

$= (u_1+zv_1)(\bar{u}_2+\bar{z}\bar{v}_2)$ 1

Expanding and using $|u_1|^2 + |u_2|^2 = |v_1|^2 + |v_2|^2 = 1$, we obtain 1

$t(v_1\bar{v}_2 - u_1\bar{u}_2)|z|^2 + Pz + Q\bar{z} + (1-t)(u_1\bar{u}_2 - v_1\bar{v}_2) = 0$, $P, Q \in \mathbb{C}$. 1

Since $u_1\bar{u}_2 \ne v_1\bar{v}_2$, $t > 0$, this can be written as

$|z|^2 + pz + q\bar{z} + \frac{t-1}{t} = 0$ 1

where $\frac{t-1}{t} < 0$ as $0 < t < 1$. 1

By (a), this equation has at least two solutions. 1

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SOLUTIONS 86 I	MARKS	REMARKS
3. (a) If $d_1(x)$ and $d_2(x)$ divide each other, let		
$d_1(x) = h_1(x)\, d_2(x)$	1	
and $d_2(x) = h_2(x)\, d_1(x)$ for some polynomials $h_1(x)$, $h_2(x)$.		
Then $\deg d_1(x) = \deg h_1(x) + \deg d_2(x)$	1	
$= \deg h_1(x) + [\deg h_2(x) + \deg d_1(x)]$		
$\therefore \deg h_1(x) = \deg h_2(x) = 0$.		
$h_1(x) = k \neq 0$ (as $d_1(x)$ and $d_2(x) \neq 0$)	1	
$d_1(x) = k\, d_2(x)$.		
	3	
(b) (i) If $s(x)$ divides $f(x)$ and $g(x)$,		
let $f(x) = h_1(x)s(x)$, $g(x) = h_2(x)s(x)$ for some	1	
polynomials $h_1(x)$, $h_2(x)$.		
$\therefore p(x) = m(x)f(x) + n(x)g(x)$		
$= [m(x)h_1(x) + n(x)h_2(x)]s(x)$		
i.e. $s(x)$ divides $p(x)$ $\forall\, p(x) \in A$.	1	
(ii) Let $p(x) = m(x)f(x) + n(x)g(x)$ and suppose $p(x) \nmid f(x)$.		By Euclidean Algorithm
Let $f(x) = q(x)p(x) + r(x)$, where $r(x) \neq 0$ and $\deg r(x) < \deg p(x)$.	1	
Then $r(x) = f(x) - q(x)p(x)$		
$= f(x) - q(x)[m(x)f(x) + n(x)g(x)]$	1	
$= \underbrace{[1 - q(x)m(x)]}_{\text{non-zero}}f(x) + \underbrace{[-n(x)q(x)]}_{\text{non-zero}}g(x)$		
$\therefore r(x) \in A$. ← non-zero	1	
(iii) Let $d_1(x) = m_1(x)f(x) + n_1(x)g(x)$ and suppose $d_1(x) \nmid f(x)$.		
Let $f(x) = h(x)d_1(x) + r(x)$, where $r(x) \neq 0$ and $\deg r(x) < \deg d_1(x)$.		
Then by (ii), $r(x) \in A$.	1	
But $\deg r(x) < \deg d_1(x)$ contradicts the definition		
of $d_1(x)$.	1	
$\therefore d_1(x)$ divides $f(x)$.		
Similarly $d_1(x)$ divides $g(x)$.		
$\therefore d_1(x)$ is a common divisor of $f(x)$ and $g(x)$.	1	
By (i) $d_1(x)$ is a G.C.D. of $f(x)$ and $g(x)$.	1	
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SOLUTIONS 86 I	MARKS	REMARKS
(c) By (b), $d_1(x) = m_1(x)f(x) + n_1(x)g(x)$ is a G.C.D. of $f(x)$ and $g(x)$.	1	
Let $d(x)$ be a G.C.D. of $f(x)$ and $g(x)$.		
Since $d_1(x)$ and $d(x)$ divide each other, by (a), $d(x) = kd_1(x)$		
$= k(m_1(x)f(x) + n_1(x)g(x))$ for some non zero k		
$= m_0(x)f(x) + n_0(x)g(x)$, where $m_0(x) = km_1(x)$		
$n_0(x) = kn_1(x)$	1	
	2	

SOLUTIONS 86 II	MARKS	REMARKS
(a) (i) For $0 \le \theta \le \frac{\pi}{4}$, $0 \le \tan\theta \le 1$.	1	
$0 \le \tan^{n+1}\theta \le \tan^{n}\theta$		
$\therefore\ 0 \le I_{n+1} \le I_n$ for $n \ge 0$	1	
(ii) For $n \ge 2$, $I_n = \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta(\sec^2\theta - 1)d\theta$	1	
$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \, d\tan\theta - I_{n-2}$	1	
$= \left.\frac{\tan^{n-1}\theta}{n-1}\right\|_0^{\frac{\pi}{4}} - I_{n-2}$		
$= \frac{1}{n-1} - I_{n-2}$		
$\therefore\ I_n + I_{n-2} = \frac{1}{n-1}$	1	
(iii) For $n \ge 2$, $\frac{1}{n-1} = I_n + I_{n-2}$		
$\ge 2I_n$ by (i)		
$\therefore\ \frac{1}{2(n-1)} \ge I_n$	1	
Further $\frac{1}{n+1} = I_{n+2} + I_n$		
$\le 2I_n$		
$\therefore\ \frac{1}{2(n+1)} \le I_n$	1 / 7	

SOLUTIONS 86 II	MARKS	REMARKS
1. (b) For $n \ge 1$,		
$I_{2n+1} = \frac{1}{2n} - I_{2n-1}$	1	
$= \frac{1}{2n} - \frac{1}{2n-2} + I_{2n-3}$ $\quad \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2}\right)$		
$= \ldots$		
$= \frac{1}{2n} - \frac{1}{2n-2} + \frac{1}{2n-4} - \ldots + (-1)^{n-1}\frac{1}{2} + (-1)^n I_1$	1	
$= \frac{(-1)^{n-1}}{2}(a_n - 2I_1)$	1	
Now $I_1 = \int_0^{\frac{\pi}{4}} \tan\theta \, d\theta$		
$= -\int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} d\cos\theta$		
$= [-\ln\cos\theta]_0^{\frac{\pi}{4}}$		
$= \frac{1}{2}\ln 2$	1	
$\therefore\ I_{2n+1} = \frac{(-1)^{n-1}}{2}(a_n - \ln 2)$	1	
From (a)(iii) $\frac{1}{2(2n+2)} \le I_{2n+1} \le \frac{1}{2(2n)}$, Since both $\lim_{n\to\infty}\frac{1}{2(2n+2)}$ and $\lim_{n\to\infty}\frac{1}{2(2n)}$ equal zero	1	
$\lim_{n\to\infty} I_{2n+1} = 0$		
i.e. $\lim_{n\to\infty} a_n = \ln 2$	1 / 7	
$\therefore\ a_n = \frac{2I_{2n+1}}{(-1)^{n-1}} + \ln 2$		
$\therefore\ \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left[\frac{2I_{2n+1}}{(-1)^{n-1}} + \ln 2\right]$		

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2. (a) For $C \neq 0$, let the equations of the two lines be

$y - m_1x - C_1 = 0$ and $y - m_2x - C_2 = 0$.

Then the given equation can be written as

$*(y - m_1x - C_1)(y - m_2x - C_2) = 0$ 1

$m_1m_2x^2 - (m_1+m_2)xy + y^2 + (m_1C_2+m_2C_1)x - (C_1+C_2)y + C_1C_2 = 0$

$C \neq 0$ Comparing coefficients, we have

$m_1m_2 = \frac{A}{C}$, $m_1 + m_2 = -\frac{B}{C}$ 1

(i) If $A + C = 0$, $m_1m_2 = -1$.

$A = -C$

$\therefore \alpha = \frac{\pi}{2}$ 1

(ii) If $A + C \neq 0$, $\tan^2\alpha = \left(\frac{m_1 - m_2}{1 + m_1m_2}\right)^2$ 1

$= \frac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$

$= \frac{\left(-\frac{B}{C}\right)^2 - \frac{4A}{C}}{\left(1 + \frac{A}{C}\right)^2}$

$= \frac{B^2 - 4AC}{(A+C)^2}$ 1

*Note: Candidates may also consider $Ax^2 + Bxy + Cy^2 = 0$.

For $C = 0$,

Case 1

$Ax^2 + Bxy = 0$

$x(Ax + By) = 0$

If $B \neq 0$, then the pair of straight lines are given by

$(y - m_1x - C_1)(x - C_2) = 0$

$-m_1x^2 + xy - (C_1 - m_1C_2)x - C_2y + C_1C_2 = 0$

$\therefore m_1 = -\frac{A}{B}$

$x - C_2 = 0$ is a vertical line,

If $A + C = 0$, then $A = 0$ and therefore $m_1 = 0$.

i.e. the two lines intersect at right angles. 1

If $A + C \neq 0$, then $A \neq 0$.

$\tan^2\alpha = \tan^2\left(\frac{\pi}{2} - \beta\right)$, (where $\tan\beta = m_1$)

$= \cot^2\beta = \frac{1}{m_1^2} = \frac{B^2}{A^2} = \frac{B^2 - 4AC}{(A+C)^2}$ (as $C = 0$) 1

Case 2

$C = 0$, $B = 0$, $A \neq 0$

If $B = 0$ (in which case $A \neq 0$ and $A + C \neq 0$), then both straight lines are vertical.

$\therefore \alpha = 0 \Rightarrow \tan^2\alpha = 0 = \frac{B^2 - 4AC}{(A+C)^2}$ (as $B = C = 0$) 1

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2. (b) Let $y = mx + c$ be the equation of a line through P.

Substituting in (E) 1

$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$

$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$. 1

For tangency, $4a^4m^2c^2 = 4a^2(c^2 - b^2)(b^2 + a^2m^2)$

$c^2 - b^2 - a^2m^2 = 0$ 1

Since $y = mx + c$ passes through (h, k)

$c = k - mh$

$(k - mh)^2 - b^2 - a^2m^2 = 0$.

$(h^2 - a^2)m^2 - 2hkm + (k^2 - b^2) = 0$ 1

If the tangents are perpendicular

$\frac{k^2 - b^2}{h^2 - a^2} = -1$ 1

$h^2 + k^2 = a^2 + b^2$

$\therefore$ P lies on the circle $x^2 + y^2 = a^2 + b^2$ 1

6

* Alternatively

(b) The pair of tangents through P(h, k) is

$\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} + \frac{ky}{b^2} - 1\right)^2$ 2

$\left[\frac{1}{a^2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \frac{h^2}{a^4}\right]x^2 + \left[\frac{1}{b^2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right) - \frac{k^2}{b^4}\right]y^2$

$-\frac{2hk}{a^2b^2}xy + \frac{2h}{a^2}x + \frac{2k}{b^2}y - \left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right) = 0$ 1

If the tangents are perpendicular, by (a) (i) and (ii)

$\left(\frac{k^2}{a^2b^2} - \frac{1}{a^2}\right) + \left(\frac{h^2}{a^2b^2} - \frac{1}{b^2}\right) = 0$. 2

$k^2 + h^2 = a^2 + b^2$

$\therefore$ P(h, k) lies on the circle $x^2 + y^2 = a^2 + b^2$. 1

6

*Note: There are a number of alternative methods used in common textbooks

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3\. (a)

$$f'(0) = \lim_{\Delta x \to 0} \frac{(\Delta x + 0)^{n+1}|\Delta x + 0| - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (\Delta x)^n |\Delta x|$$

$$= 0$$

$$f'(x) = \begin{cases} \frac{d}{dx} x^{n+1} \cdot x & x > 0 \\ 0 & x = 0 \\ \frac{d}{dx} x^{n+1} (-x) & x < 0 \end{cases}$$

$$= \begin{cases} (n+2)\, x^n \cdot x & x > 0 \\ 0 & x = 0 \\ -(n+2)\, x^n \cdot x, & x < 0 \end{cases}$$

$$= (n+2)x^n \cdot |x|, \quad \forall x \in \mathbb{R}$$

$$\therefore \int x^n \cdot |x|\, dx = \frac{1}{n+2} x^{n+1} \cdot |x| + c$$

	Marks
$f'(0)$ limit	1
$= 0$	1
$f'(x)$ cases	1
$(n+2)x^n\cdot\lvert x\rvert$	1
integral	1
Total	5

(b)

$$\int_a^b \left|\sum_{j=1}^{2n} c_j x^j\right| dx \le \int_a^b \sum_{j=1}^{2n} \left|c_j x^j\right| dx$$

$$= \sum_{j=1}^{2n} \int_a^b |c_j|\,|x^j|\, dx$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \int_a^b |x^{2j-1}|\, dx + |c_{2j}| \int_a^b |x^{2j}|\, dx \right\}$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \int_a^b x^{2j-2} |x|\, dx + |c_{2j}| \int_a^b x^{2j}\, dx \right\}$$

$$= \sum_{j=1}^{n} \left\{ \left[|c_{2j-1}| \frac{1}{2j} x^{2j-1} |x| + |c_{2j}| \frac{1}{2j+1} x^{2j+1} \right]_a^b \right\}$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \frac{1}{2j} [b^{2j-1}|b| - a^{2j-1}|a|] + |c_{2j}| \frac{1}{2j+1} [b^{2j+1} - a^{2j+1}] \right\}$$

$$= \sum_{j=1}^{n} \left\{ |c_{2j-1}| \frac{1}{2j} [b^{2j} + |a|^{2j}] + |c_{2j}| \frac{1}{2j+1} [b^{2j+1} + |a|^{2j+1}] \right\}$$

$$= \sum_{j=1}^{2n} |c_j| \left(\frac{b^{j+1} + |a|^{j+1}}{j+1} \right)$$

Step	Marks
$\le \int_a^b \sum \lvert c_j x^j\rvert\, dx$	1
split into odd and even terms	1
$x^{2j-2}\lvert x\rvert$	1
integrate	1
final result	1
Total	5

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Alternatively

3\. (b)

$$\int_a^b \left|\sum_{j=1}^{2n} c_j x^j\right| dx$$

$$= \int_a^0 \left|\sum_{j=1}^{2n} c_j x^j\right| dx + \int_0^b \left|\sum_{j=1}^{2n} c_j x^j\right| dx$$

$$\le \int_a^0 \sum_{j=1}^{2n} |c_j x^j|\, dx + \int_0^b \sum_{j=1}^{2n} |c_j x^j|\, dx$$

$$= \int_a^0 \sum_{j=1}^{2n} |c_j| (-1)^j x^j\, dx + \int_0^b \sum_{j=1}^{2n} |c_j| x^j\, dx$$

$$= \sum_{j=1}^{2n} \left[|c_j| (-1)^j \frac{x^{j+1}}{j+1} \right]_a^0 + \sum_{j=1}^{2n} \left[|c_j| \frac{x^{j+1}}{j+1} \right]_0^b$$

$$= \sum_{j=1}^{2n} |c_j| (-1)^{j+1} \frac{(a)^{j+1}}{j+1} + \sum_{j=1}^{2n} |c_j| \frac{b^{j+1}}{j+1}$$

$$= \sum_{j=1}^{2n} |c_j| \frac{|a|^{j+1}}{j+1} + \sum_{j=1}^{2n} |c_j| \frac{b^{j+1}}{j+1}$$

$$= \sum_{j=1}^{2n} |c_j| \frac{b^{j+1} + |a|^{j+1}}{j+1}$$

Step	Marks
split at 0	1
$\le$	1
$(-1)^j$	1
integrate	1
final result	1
Total	5

3\. (c)

$$\int_{-1}^1 \left| \sum_{j=1}^{2n} \frac{(-1)^j (j+1) x^j}{2^j} \right| dx$$

$$= 2 \int_0^1 \left| \sum_{j=1}^{2n} \frac{(-1)^j (j+1) x^j}{2^j} \right| dx$$

$$\le 2 \int_0^1 \sum_{j=1}^{2n} \frac{(j+1) x^j}{2^j}\, dx$$

$$= 2 \sum_{j=1}^{2n} \left[\frac{x^{j+1}}{2^j} \right]_0^1$$

$$= 2 \sum_{j=1}^{2n} \frac{1}{2^j}$$

$$= 2 \left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^{2n}}\right)}{1 - \frac{1}{2}} \right)$$

$$< 2$$

Step	Marks
$= 2\int_0^1$	1
$\le$	1
integrate	1
< 2	[illegible]

p.7

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4. (a) (i) $f(0) = f(0+0)$ ....

$= [f(0)]^2$ — 1

$\Rightarrow f(0) = 1$ as $f(0) \neq 0$ 1

(ii) We shall prove by induction:

$\forall x \geq 0,\ f(x) \geq f(0) = 1$ as f is increasing)

For $k = 0$, R.S. $= [f(x)]^0 = 1 = f(0\cdot x) =$ L.S.) ... 1

Assume that $f(kx) = [f(x)]^k$ for some $k \geq 0$.

$$f((k+1)x) = f(kx + x) = f(kx)f(x) = [f(x)]^{k+1}$$

$\therefore f(kx) = [f(x)]^k \quad \forall$ integers $k \geq 0$ 1

(iii) Let $a = f(1) \geq f(0) = 1$. ← Indispensable because of, otherwise, a may be ≤ 1. By given, f is increasing 1

For any non-negative integer n, 1

$f(n) = f(n \cdot 1)$

$= [f(1)]^n$ by (ii)

$= a^n$ 1

6

(b) (i) For any $x \geq 0$, let n be the non-negative integer such that $n \leq x < n+1$.

Then $a^n \leq a^x \leq a^{n+1}$ and $\frac{1}{a^{n+1}} \leq \frac{1}{a^x} \leq \frac{1}{a^n}$ ....(i) 1

Since $f(x)$ is increasing, $f(n) \leq f(x) \leq f(n+1)$

$\therefore a^n \leq f(x) \leq a^{n+1}$ ....(ii) 1

Since all quantities involved are positive, (i) and (ii) give $\frac{a^n}{a^{n+1}} \leq \frac{f(x)}{a^x} \leq \frac{a^{n+1}}{a^n}$

i.e. $\frac{1}{a} \leq \frac{f(x)}{a^x} \leq a$ ....(*) 1

(ii) As (*) is independent of x, replacing x by kx, we have

$\frac{1}{a} \leq \frac{f(kx)}{a^{kx}} \leq a \quad \forall k > 0$ 1

$\frac{1}{a} \leq \left(\frac{f(x)}{a^x}\right)^k \leq a$ 1

$\frac{1}{a^{\frac{1}{k}}} \leq \frac{f(x)}{a^x} \leq a^{\frac{1}{k}}$ 1

As both $\lim_{k\to\infty} a^{\frac{1}{k}}$ and $\lim_{k\to\infty} \frac{1}{a^{\frac{1}{k}}}$ equal 1

$\lim_{k\to\infty} \frac{f(x)}{a^x} = 1$ 1

i.e. $f(x) = a^x \quad \forall x \geq 0$. 1

p.8

SOLUTIONS	86 II	MARKS	REMARKS

5. (a) Let x, y be the cartesian coordinates of P, $\tan\phi$ be the gradient of the curve at P

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$\frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$

$\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$ 1

$$\therefore \tan\phi = \frac{dy}{dx} = \frac{\sin\theta\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\frac{dr}{d\theta} - r\sin\theta} = \frac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}$$ 1

$\tan\psi = \tan(\phi - \theta)$ 1

$$= \frac{\dfrac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta} - \tan\theta}{1 + \dfrac{\tan\theta\frac{dr}{d\theta} + r}{\frac{dr}{d\theta} - r\tan\theta}\tan\theta}$$

$$= \frac{r(1 + \tan^2\theta)}{\frac{dr}{d\theta}(1 + \tan^2\theta)}$$

$$= \frac{r}{\left(\frac{dr}{d\theta}\right)}$$ 1

4

SOLUTIONS 86 II	MARKS	REMARKS

5. (b) (i) Substituting $r = 2$ in C_1, $2 = 2(1-\cos\theta)$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$ — 1

$\therefore$ the points of intersection are $(2, \frac{\pi}{2})$ and $(2, \frac{3\pi}{2})$.

At $\theta = \frac{\pi}{2}$, the tangent to C_2 is parallel to the x-axis.

For C_1, $\frac{dr}{d\theta} = 2\sin\theta$

$= 2$ ....

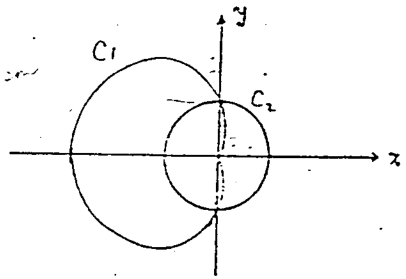
$\therefore \tan\psi = \dfrac{r}{\frac{dr}{d\theta}} = \frac{2}{2} = 1$

The angle between C_1 and C_2 is $= \frac{\pi}{2} - \tan^{-1} 1$

$= \frac{\pi}{4}$

By symmetry, the angle between C_1 and C_2 at $\theta = \frac{3\pi}{2}$ is also $\frac{\pi}{4}$ .... — 1

(ii)



1+1

For any point (r, θ) of C_1 lying inside C_2

$r = 2(1-\cos\theta)$, $r < 2$

$\therefore 2(1-\cos\theta) < 2$

$0 < \cos\theta$

$0 < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{2} < \theta < 2\pi$ .... — 1 — or for lim. of integra. belo

Length of C_1 inside C_2

$= \int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta + \int_{\frac{3\pi}{2}}^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$ — 1

$= 2\int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta$ by symmetry

$= 2\int_0^{\frac{\pi}{2}} \sqrt{(\cos\theta\frac{dr}{d\theta} - r\sin\theta)^2 + (\sin\theta\frac{dr}{d\theta} + r\cos\theta)^2}\, d\theta$

$= 2\int_0^{\frac{\pi}{2}} \sqrt{(\frac{dr}{d\theta})^2 + r^2}\, d\theta$ .... — 1

$= 2\int_0^{\frac{\pi}{2}} \sqrt{4\sin^2\theta + 4(1-\cos\theta)^2}\, d\theta$

SOLUTIONS 86 II	MARKS	REMARKS

5. (b) (ii) $= 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\cos\theta}\, d\theta$ .... — 1

$= 8\int_0^{\frac{\pi}{2}} \sin\frac{\theta}{2}\, d\theta$

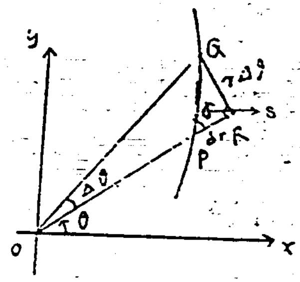
$= 16\left[-\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= 8(2-\sqrt{2})$ (≈ 4.69) .... — 1

10

Alternatively

(a) From the diagram



$PR \doteq \Delta r$ — 1

$QR \doteq r\Delta\theta$

$\tan\phi \doteq \frac{r\Delta\theta}{\Delta r}$ — 1

$\therefore \tan\psi = \lim_{\Delta\theta \to 0} r\frac{\Delta\theta}{\Delta r}$ — 1

$= \dfrac{r}{\frac{dr}{d\theta}}$ .... — 1

4

(b) (ii) $ds^2 = (rd\theta)^2 + (dr)^2$ .... — 1

Length of C_1 inside $C_2 = \int ds$

$= \int \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$ — 1

$=$ etc

etc.

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6. (a) $f(x) = x^3 - 3x^2 + 4$

$f'(x) = 3x^2 - 6x = 3x(x-2)$

$f''(x) = 6x - 6 = 6(x-1)$

$\therefore$ the stationary points are at $x = 0$ and $x = 2$. **1**

x	-1	0	1	2	3
f(x)	0	4	2	0	4
f'(x)	+	0	−	0	+
f''(x)	−		0	+	

1

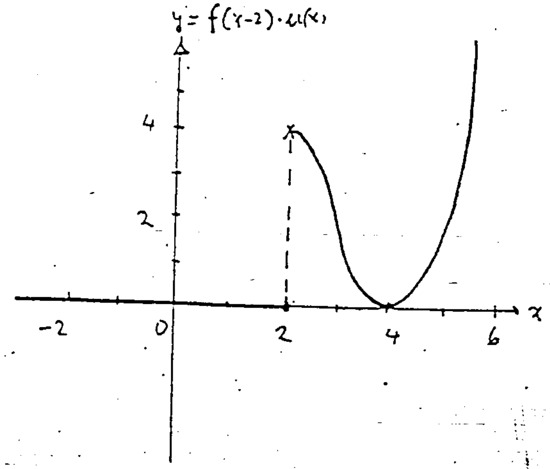
$\therefore$ (0, 4) is a maximum point,

(2, 0) is a minimum point,

and (1, 2) is the only point of inflexion. **1** — **3**

(b) $h(x) = f(x-2)\cdot u(x) = \begin{cases} 0 & \text{when } x < 2 \\ f(x-2) & \text{when } x \geq 2. \end{cases}$ **1**

Translating the graph of $f(x)$ $(x \geq 0)$ horizontally to the right by 2 units, one obtains the graph of $h(x)$ for $x \geq 2$.



2 — **3**

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6. (c) $I_n = \int_0^n e^{-x} h(x)\,dx$

$= \int_2^n e^{-x} f(x-2)\,dx \quad (n \geq 2)$ ………… **1**

$= \int_0^{n-2} e^{-(t+2)} f(t)\,dt$ $\quad t = x-2$

$= e^{-2}\int_0^{n-2} e^{-t}(t^3 - 3t^2 + 4)\,dt$ ………… **1**

Now $\int te^{-t}dt = -te^{-t} + \int e^{-t}dt$ **1**

$= -(t+1)e^{-t} + c$

$\int t^2e^{-t}dt = -t^2e^{-t} + 2\int te^{-t}dt$

$= -(t^2 + 2t + 2)e^{-t} + c$

$\int t^3e^{-t}dt = -t^3e^{-t} + 3\int t^2e^{-t}dt$

$= -(t^3 + 3t^2 + 6t + 6)e^{-t} + c$

$\therefore I_n = -e^{-2}e^{-t}[(t^3 + 3t^2 + 6t + 6) - 3(t^2 + 2t + 2) + 4]\Big|_0^{n-2}$ **1**

$= -e^{-2}e^{-t}(t^3 + 4)\Big|_0^{n-2}$ **1**

$= -e^{-2}[e^{-(n-2)}((n-2)^3 + 4) - 4]$ ………… **1**

By L'Hospital's rule,

$\lim_{t\to\infty}\frac{t^3}{e^t} = \lim_{t\to\infty}\frac{3t^2}{e^t} = \lim_{t\to\infty}\frac{6t}{e^t} = \lim_{t\to\infty}\frac{6}{e^t} = 0$ ………… **1**

$\therefore \lim_{n\to\infty} I_n = -e^{-2}[\lim_{n\to\infty} e^{-(n-2)}(n-2)^3 + \lim_{n\to\infty} e^{-(n-2)}4 - \lim_{n\to\infty} 4]$

$= 4e^{-2}$ ………… **1** — **8**

SOLUTIONS	MARKS	REMARKS

7. (a) (i) $\pi : \vec{r}\cdot\vec{n} = p$ ....(1)   passing through the pt   position vector $\perp \pi$

$l : \vec{r} = \vec{a} + t\vec{b}$ ....(2)

Putting (2) in (1) $(\vec{a} + t\vec{b})\cdot\vec{n} = p$ **1**

$t(\vec{b}\cdot\vec{n}) = p - \vec{a}\cdot\vec{n}$

$t = \dfrac{p - \vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}$ $(\vec{b}\cdot\vec{n} \neq 0)$ Solving for t

$\therefore$ l intersects π at the point with position vect

$\vec{a} + \left(\dfrac{p - \vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}}\right)\vec{b}$ **1**

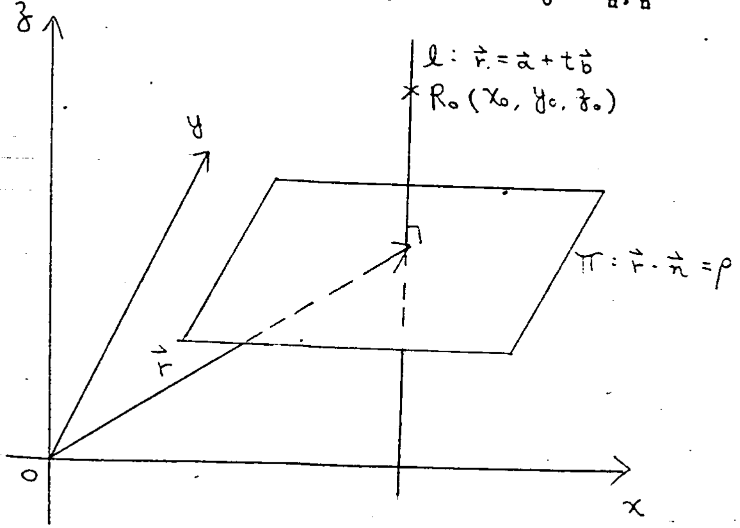
(ii) Let the position vector of R_0 be $\vec{r_0} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$.

The equation of the line through R_0 and perpendicular to π is $\vec{r} = \vec{r_0} + t\vec{n}$, $t \in \mathbb{R}$ .... **2**   passing through $\vec{r_0}$

By (i), the position vector of the foot of the perpendicular from R_0 to π is $r_0 + \dfrac{p - \vec{r_0}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$ **1**

Total: **5**

Remarks: Candidate: take $\vec{n}\cdot\vec{r}$



SOLUTIONS	MARKS	REMARKS

7. (b) $\pi : x + y + z - 1 = 0$ can be written as

$(x, y, z)\cdot(1, 1, 1) = 1$????? **1**

By (a)(ii), PP' intersects π ~~at the point M~~ with position vector $\vec{m} = \vec{p} + \dfrac{1 - \vec{p}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}$, where $\vec{n} = (1, 1, 1)$ **1**

Since M is the mid-point of PP',

P' is given by , $\dfrac{\vec{p}+\vec{p'}}{2} = \vec{m}$

$\vec{p'} = 2\vec{m} - \vec{p}$ **1**

$= \vec{p} + \dfrac{2(1 - \vec{p}\cdot\vec{n})}{\vec{n}\cdot\vec{n}}\vec{n}$

$= (\alpha, \beta, \gamma) + \left[\dfrac{2(1 - (\alpha + \beta + \gamma))}{(1, 1, 1)\cdot(1, 1, 1)}\right](1, 1, 1)$ **1**

i.e., the coordinates of P' are given by

$$\begin{cases} x = \alpha + \frac{2}{3}(1 - \alpha - \beta - \gamma) \\ y = \beta + \frac{2}{3}(1 - \alpha - \beta - \gamma) \\ z = \gamma + \frac{2}{3}(1 - \alpha - \beta - \gamma) \end{cases}$$ ....(3) **1**

Now $l : \dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z-3}{3}$ can be written as

$\vec{r} = (1, 2, 3) + t(1, 2, 3)$ **1**

If $P(\alpha, \beta, \gamma)$ lies on l, $\alpha = 1 + t$

$\beta = 2 + 2t$

$\gamma = 3 + 3t$ **1**

Substituting in (3)

$x = (1 + t) + \frac{2}{3}(1 - 1 - t - 2 - 2t - 3 - 3t) = -\frac{7}{3} - 3t$

$y = (2 + 2t) + \frac{2}{3}(-5 - 6t) = -\frac{4}{3} - 2t$

$z = (3 + 3t) + \frac{2}{3}(-5 - 6t) = -\frac{1}{3} - t$

$\therefore$ the locus of P' is $l' : \dfrac{x + \frac{7}{3}}{3} = \dfrac{y + \frac{4}{3}}{2} = \dfrac{z + \frac{1}{3}}{1}$ **1**

[or $\vec{r} = (-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}) + t(3, 2, 1)$, $t \in \mathbb{R}$]

Total: **9**

SOLUTIONS 86 II | MARKS | REMARKS

8. (a) $F'(x) = \lambda g'(\lambda x + (1-\lambda)a) - \lambda g'(x)$ — 1

If $x = a$,

$\lambda x + (1-\lambda)a = x$

$\therefore F'(a) = 0$ 1

$\forall\ x < a$, $\lambda x + (1-\lambda)a > x$ (as $0 < \lambda < 1$)

$\Rightarrow F'(x) \geq 0$ as g' is increasing 1

Similarly, $\forall\ x > a$, $F'(x) \leq 0$.

$\therefore F(x)$ attains its greatest value at $x = a$.

$\frac{1}{4}$

(b) (i) By (a), introduction $F(x) \leq F(a)$

$= g(\lambda a + (1-\lambda)a) - \lambda g(a) - (1-\lambda)g(a)$

$= 0$ .... $\lambda_1 + \lambda_2 = 1$ 2

For $m = 2$, let $\lambda = \lambda_1$, $1 - \lambda = \lambda_2$, we have from (a)

$g(\lambda_1 x_1 + \lambda_2 x_2) - [\lambda_1 g(x_1) + \lambda_2 g(x_2)] \leq 0$ see introduction

i.e. $g(\lambda x_1 + \lambda_2 x_2) \leq \lambda_1 g(x_1) + \lambda_2 g(x_2)$ — 1

Suppose the given statement is true for $2 \leq m < k$.

Let $\lambda_1 + \lambda_2 + \dots + \lambda_{k-1} = \lambda$, $1 - \lambda = \lambda_k$

$x = \frac{1}{\lambda}(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_{k-1}x_{k-1})$ 2

Then $g(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k)$

$= g(\lambda x + (1-\lambda)x_k)$

$\leq \lambda g(x) + (1-\lambda)g(x_k)$

$= \lambda g(\frac{\lambda_1}{\lambda}x_1 + \frac{\lambda_2}{\lambda}x_2 + \dots + \frac{\lambda_{k-1}}{\lambda}x_{k-1}) + (1-\lambda)g(x_k)$

$\leq \lambda_1 g(x_1) + \lambda_2 g(x_2) + \dots + \lambda_{k-1}g(x_{k-1}) + \lambda_k g(x_k)$

$\therefore$ the statement is true for $m=k$ and hence $\forall\ m \geq 2$ 2

(ii) Let $g(x) = e^x$, which is differentiable and

$g'(x) = e^x$ is increasing. — 1

For any positive numbers $a_1, a_2, \dots, a_m$,

let $a_i = e^{x_i}$ $(1 \leq i \leq m)$ 1

Then by (i),

$e^{\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m} \leq \lambda_1 e^{x_1} + \lambda_2 e^{x_2} + \dots + \lambda_m e^{x_m}$

i.e. $a_1^{\lambda_1} \cdot a_2^{\lambda_2} \cdots a_m^{\lambda_m} \leq \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_m a_m$ — 1

SOLUTIONS 86 II | MARKS | REMARKS

9. (a) For any $x \in I$

$f(x) = f(x) - f(0)$

$= \int_0^x f'(t)\,dt$ 1

$\leq \int_0^x f'(x)\,dt$ as f' is increasing $\therefore f'(x) > f'(t)\ \forall t \in [0,x]$ — 1

$= x f'(x)$ $[f'(x)t]_0^x$ independent of t .... $\frac{1}{3}$

(b) (i) $G(x) = 2f(x)\sqrt{1 + f'(x)^2}$

$\geq 2f(x)\,|f'(x)|$

$\geq 2f(x)f'(x)$ as $f'(x) \geq 0$. — 1

(ii) $F(x) = f(x)\sqrt{x^2 + f(x)^2}$

$F'(x) = f'(x)\sqrt{x^2 + f(x)^2} + \frac{f(x)}{2}\frac{(2x + 2f(x)f'(x))}{\sqrt{x^2 + f(x)^2}}$

$= \frac{xf(x) + x^2f'(x) + 2f(x)^2f'(x)}{\sqrt{x^2 + f(x)^2}}$ 1

$[F'(x)]^2 - [G(x)]^2$

$= \frac{x^2f(x)^2 + x^4f'(x)^2 + 4f(x)^4f'(x)^2 + 2x^3f(x)f'(x) + 4xf(x)^3f'(x) + 4x^2f(x)^2f'(x)^2}{x^2 + f(x)^2}$

$- 4f^2(x)(1 + f'(x)^2)$

$= \frac{1}{x^2 + f(x)^2}[-3x^2f(x)^2 + x^4f'(x)^2 + 2x^3f(x)f'(x) + 4xf(x)^3f'(x) - 4f(x)^4]$

$\geq \frac{1}{x^2 + f(x)^2}[-3x^2f(x)^2 + x^2f(x)^2 + 2x^2f(x)^2 + 4f(x)^3f(x) - 4f(x)^4]$ — 1

$f(x) \leq xf'(x)$

$= 0$ (as $0 \leq f(x) \leq xf'(x)$) 1

From (a) $f'(x) \geq 0$, $\therefore F'(x) \geq 0$ as all quantities involved are non-negative. — 1

$\therefore F'(x) \geq G(x)$ .... $\frac{1}{6}$

(c) $S = 2\pi\int_0^a f(x)\sqrt{1 + (f'(x))^2}\,dx$ 2

$= \pi\int_0^a G(x)\,dx$ $(G(x) \geq 2f(x)f'(x))$ $(F'(x) \geq G(x))$

$\therefore 2\pi\int_0^a f(x)f'(x)dx \leq S \leq \pi\int_0^a F'(x)dx$ 1

$\pi[f(x)^2]_0^a \leq S \leq \pi[F(x)]_0^a$

$\pi[f(a)]^2 \leq S \leq \pi f(a)\sqrt{a^2 + [f(a)]^2}$ .... $\frac{1+1}{5}$