9. (a) Let f(x) and g(x) be two functions continuous on the interval $[a, b]$. By considering the integral of the function $[\lambda f(x) + g(x)]^2$ on $[a, b]$, set up a quadratic inequality in the parameter λ. Hence show that

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leqslant \left(\int_a^b [f(x)]^2\, dx\right)\left(\int_a^b [g(x)]^2\, dx\right).$$

(b) Let f(x) be a non-constant function with continuous derivative on $[0, 1]$ satisfying $f(0) = 0$ and $f(1) = 0$.

(i) Show that

$$f(x) = \int_0^x f'(t)\, dt = -\int_x^1 f'(t)\, dt$$

for any $x \in [0, 1]$.

(ii) Use (i) and (a) to show that

$$[f(x)]^2 \leqslant x \int_0^{\frac{1}{2}} [f'(t)]^2\, dt \qquad \text{if } x \in [0, \tfrac{1}{2}]$$

and $$[f(x)]^2 \leqslant (1-x) \int_{\frac{1}{2}}^1 [f'(t)]^2\, dt \qquad \text{if } x \in [\tfrac{1}{2}, 1].$$

(iii) Use (ii) to show that $\int_0^1 [f(x)]^2\, dx \leqslant \frac{1}{8} \int_0^1 [f'(x)]^2\, dx$.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1986

純數學　試卷一

PURE MATHEMATICS　PAPER I

9.00 am–12.00 noon　(3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. Let $\mathscr{M}$ be the set of 2×2 real matrices. For any U in $\mathscr{M}$, let $\det U$ and U^T denote respectively the determinant and transpose of U.

Let $\mathscr{F} = \{ U \in \mathscr{M} : U^T = U^{-1} \text{ and } \det U > 0 \}$.

(a) (i) If $U \in \mathscr{F}$, show that $\det U = 1$.

(ii) Show that $U \in \mathscr{F}$ if and only if $U = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for some real a, b such that $a^2 + b^2 = 1$.

(b) Let $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ be in $\mathscr{M}$.

(i) Show that if there exist $U \in \mathscr{F}$ and $p, q \in \mathbf{R}$ such that

$$UBU^T = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

then $B = B^T$, i.e., $x = y$.

(ii) Show that if $B = B^T$, then there exists $U \in \mathscr{F}$ such that $UBU^T = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ for some $s, t \in \mathbf{R}$.

2. Consider the system of linear equations

$$(E) \begin{cases} x + y - z = a \\ -kx - y + kz = b \\ k^2x + y - kz = c \end{cases}.$$

(a) (i) Find all real values of k such that for any given values of a, b and c, (E) has a unique solution.

(ii) For $k = 0$, find all real values of a, b and c such that (E) is consistent.

(iii) For $a = b = c = 0$ and $k = 1$, find two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) of (E) which are linearly independent vectors.

(b) Let (x_0, y_0, z_0) be a solution of

$$\begin{cases} x - ky + k^2z = 0 \\ x - y + z = 0 \\ -x + ky - kz = 0 \end{cases}.$$

Show that if the scalar product $(a, b, c) \cdot (x_0, y_0, z_0) \neq 0$, then (E) is inconsistent.

3. The vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in $\mathbf{R}^3$ are said to be linearly dependent if there exist real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, not all zero, such that

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_n \mathbf{x}_n = \mathbf{0}.$$

(a) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be vectors in $\mathbf{R}^3$.

(i) If $n \geqslant 2$, show that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent if and only if one of the vectors can be expressed as a linear combination of the other vectors.

(ii) Show that if $k (1 \leqslant k \leqslant n)$ of the vectors are linearly dependent, then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly dependent.

(b) Let $\mathbf{x}_1 = (a_1, a_2, a_3)$, $\mathbf{x}_2 = (b_1, b_2, b_3)$, $\mathbf{x}_3 = (c_1, c_2, c_3)$ and $\mathbf{x}_4 = (d_1, d_2, d_3)$ be four vectors in $\mathbf{R}^3$.

(i) Show that if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent.

(ii) Show that if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then $\mathbf{x}_4$ can be expressed as a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.

Hence show that any four vectors in $\mathbf{R}^3$ are linearly dependent.

4. (a) (i) Let X be a non-empty set and $f : X \to X$ be a function such that $f \circ f = f$.

Prove that $f = i_X$, the identity function on X, if f is either injective or surjective.

(ii) Suppose $X = \{a, b\}$. Construct a function $f : X \to X$ such that $f \circ f = f$ but f is neither injective nor surjective.

(iii) Suppose X contains more than 2 elements. Construct a non-constant function $f : X \to X$ such that $f \circ f = f$ but f is neither injective nor surjective.

(b) Let B be a given subset of a set E. A function $h : \mathscr{P}(E) \to \mathscr{P}(E)$ is defined on the power set $\mathscr{P}(E)$ of E by

$$h(A) = A \cap B \quad \text{for all } A \subset E.$$

Using (a)(i), or otherwise, deduce that $B = E$ if h is injective or surjective.

5. (a) Let m and n be two positive integers with $m > n$.

(i) Show that $\sum_{r=1}^{n+1} C_{n+r}^{2n+1} = 2^{2n}$.

(ii) By considering the coefficient of x^k in $(1+x)^m (1+\frac{1}{x})^n$, show that $\sum_{r=0}^{m-k} C_{k+r}^{m} C_r^n = C_{n+k}^{m+n}$, $m - n \leqslant k \leqslant m$.

(b) A and B have $n+1$ and n fair coins respectively and they toss their coins simultaneously.

(i) Find the probability that A will obtain k more heads than B, where $1 \leqslant k \leqslant n+1$.

(ii) Show that the probability that A will obtain more heads than B is $\frac{1}{2}$.

6. (a) For $0 < x < 1$ and for any real number $p \geq 2$, show that

$$(1+x)^p + (1-x)^p \leq 2^{p-1}(1+x^p).$$

[Hint: Note that $\left(\frac{1-x}{1+x}\right)^{p-1} \leq \frac{1-x}{1+x}$ and $\left(\frac{2x}{1+x}\right)^{p-1} \leq \frac{2x}{1+x}$.]

(b) Let $h(\theta) = (1 + r^2 + 2r\cos\theta)^{\frac{p}{2}} + (1 + r^2 - 2r\cos\theta)^{\frac{p}{2}}$, where $r > 0$ and $p \geq 2$.

Prove that $h(\theta) \leq h(0)$ for any θ.

(c) Using (a) and (b), or otherwise, show that for any two complex numbers z_1 and z_2 and for any real number $p \geq 2$,

$$|z_1 + z_2|^p + |z_1 - z_2|^p \leq 2^{p-1}(|z_1|^p + |z_2|^p).$$

7. (a) Consider the equation (in z)

$$|z|^2 + qz + r\bar{z} + s = 0, \quad \ldots\ldots (*)$$

where $q, r \in \mathbf{C}$, $s \in \mathbf{R}$ with $s < 0$. By putting $z = x + iy$, or otherwise, show that equation (*) has at least two solutions.

(b) Let $u_1, u_2, v_1, v_2 \in \mathbf{C}$ such that $|u_1|^2 + |u_2|^2 = |v_1|^2 + |v_2|^2 = 1$ and $u_1\bar{u}_2 \neq v_1\bar{v}_2$.

For $0 < t < 1$, use (a) to show that the equation

$$(u_1 + zv_1)\overline{(u_2 + zv_2)} = [tu_1\bar{u}_2 + (1-t)v_1\bar{v}_2][|u_1 + zv_1|^2 + |u_2 + zv_2|^2]$$

has at least two solutions.

8. Let $f(x)$ and $g(x)$ be two non-zero polynomials. A polynomial $d(x)$ is said to be a Greatest Common Divisor (G.C.D) of $f(x)$ and $g(x)$ if $d(x)$ divides each of them and every common divisor of them also divides $d(x)$.

(a) Let $d_1(x)$ and $d_2(x)$ be two non-zero polynomials which divide each other. Show that $d_1(x) = k\,d_2(x)$ for some non-zero constant k.

(b) Let A be the set of non-zero polynomials $p(x)$, where $p(x) = m(x)f(x) + n(x)g(x)$ for some polynomials $m(x)$ and $n(x)$.

(i) Show that if a polynomial $s(x)$ divides both $f(x)$ and $g(x)$, then it divides every $p(x)$ in A.

(ii) Let $p(x)$ be in A. Show that when $f(x)$ is divided by $p(x)$, the remainder $r(x)$ is either zero or a polynomial in A.

(iii) Let $d_1(x)$ be in A with $\deg d_1(x) \leq \deg p(x)$ for all $p(x)$ in A. Show that $d_1(x)$ is a G.C.D. of $f(x)$ and $g(x)$.

(c) Show that if $d(x)$ is a G.C.D. of $f(x)$ and $g(x)$, then there exist polynomials $m_0(x)$ and $n_0(x)$ such that

$$d(x) = m_0(x)f(x) + n_0(x)g(x).$$

9. For any positive integers m and n, let

$$T(m, n) = (1 - x^m)(1 - x^{m+1}) \dots (1 - x^{m+n-1}),$$

$$D(n) = (1 - x)(1 - x^2) \dots (1 - x^n).$$

Let $P(m, n)$ denote the statement

"$T(m, n)$ is divisible by $D(n)$."

(a) Show that for any positive integers m and n,

(i) $P(m, 1)$ and $P(1, n)$ are true,

(ii) if $P(m, n+1)$ and $P(m+1, n)$ are true, then $P(m+1, n+1)$ is also true.

[Hint: Consider $T(m+1, n+1) - T(m, n+1)$.]

(b) For any positive integer $r \geq 2$, let $Q(r)$ denote the statement

"$P(r-n, n)$ is true for any $n = 1, 2, \dots, r-1$."

(i) Show that $Q(2)$ is true.

(ii) If $Q(r)$ $(r \geq 2)$ is true, show that $Q(r+1)$ is also true.

(c) Using (b), or otherwise, show that for any positive integers m and n,

$$(1 - x^m)(1 - x^{m+1}) \dots (1 - x^{m+n-1})$$

is divisible by

$$(1 - x)(1 - x^2) \dots (1 - x^n).$$

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1986

純數學 試卷二

PURE MATHEMATICS PAPER II

2.00 pm–5.00 pm (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.

1. For any non-negative integer n, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n \theta \, d\theta .$$

(a) Prove that (i) $I_n > I_{n+1} > 0$ for $n \geq 0$,

(ii) $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$,

(iii) $\frac{1}{2(n+1)} \leq I_n \leq \frac{1}{2(n-1)}$ for $n \geq 2$.

(b) For $n = 1, 2, \ldots$, let

$$a_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} .$$

Using (a)(ii), or otherwise, express I_{2n+1} in terms of a_n.

Hence use (a)(iii) to evaluate $\lim_{n \to \infty} a_n$.

2. (a) Let $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ (A, B, C not all zero) represent a pair of straight lines and α be the angle between them. Show that

(i) $\alpha = \frac{\pi}{2}$ if $A + C = 0$,

(ii) $\tan^2 \alpha = \frac{B^2 - 4AC}{(A+C)^2}$, if $A + C \neq 0$.

[Note: The cases $C \neq 0$ and $C = 0$ should be considered separately.]

(b) Let (E) be the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

$P(h, k)$ is a point outside (E). If the two tangents drawn from P to (E) are perpendicular, show that P lies on a fixed circle.

3. (a) Let $f(x) = x^{n+1}|x|$, where n is a positive integer.

Show, from first principles, that $f'(0)$ exists and find its value.

Show that $f'(x) = (n+2)x^n|x|$. Hence find $\int x^n |x| \, dx$.

(b) Let $c_1, c_2, \ldots, c_{2n}$ be real numbers and let $a < 0 < b$.

Using (a), or otherwise, show that

$$\int_a^b \left| \sum_{j=1}^{2n} c_j x^j \right| dx \leq \sum_{j=1}^{2n} |c_j| \left(\frac{b^{j+1} + |a|^{j+1}}{j+1} \right) .$$

[Hint: Note that $|x|^{2k-1} = x^{2k-2}|x|$ and $|x|^{2k} = x^{2k}$.]

(c) Using (b), or otherwise, deduce that for any positive integer n,

$$\int_{-1}^{1} \left| \sum_{j=1}^{2n} \frac{(-1)^j (j+1) x^j}{2^j} \right| dx < 2 .$$

4. Let f be an increasing function such that $f(0) \neq 0$ and $f(x+y) = f(x)f(y)$ for any $x, y \geq 0$.

(a) (i) Prove that $f(0) = 1$.

(ii) Prove that $f(kx) = [f(x)]^k$ for all $x \geq 0$ and for any non-negative integer k.

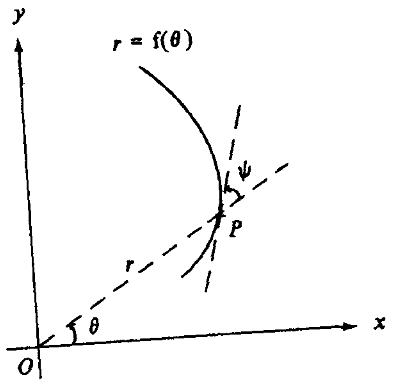
(iii) Show that there exists a real number $a \geq 1$ such that $f(n) = a^n$ for any non-negative integer n.

(b) (i) For any $x \geq 0$, show that $\frac{1}{a} \leq \frac{f(x)}{a^x} \leq a$.

[Hint: If n is the integer such that $n \leq x < n+1$, then $a^n \leq a^x \leq a^{n+1}$.]

(ii) Using (b)(i), or otherwise, show that $f(x) = a^x$ for all $x \geq 0$.

5.



(a) Let r and θ be the polar coordinates of a point P on a curve $r = f(\theta)$ in the plane, where f is a non-negative and continuously differentiable function. Let O be the origin and ψ be the angle from the line OP to the tangent line at P. Show that

$$\tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)}.$$

(b) Consider the two curves

C_1 : $r = 2(1-\cos\theta)$ $\quad 0 \leqslant \theta < 2\pi$,

C_2 : $r = 2$.

(i) Find the points of intersection of C_1 and C_2 and the angle between the curves at each point (i.e. the angle between the tangent lines at the intersection point).

(ii) Draw C_1 and C_2 on the same diagram and find the length of the part of C_1 inside C_2.

6. Let $f(x)$ and $u(x)$ be two functions defined by

$$f(x) = x^3 - 3x^2 + 4,$$

$$u(x) = \begin{cases} 0 & \text{when } x < 2, \\ 1 & \text{when } x \geqslant 2. \end{cases}$$

(a) Find the maximum, minimum and inflexion points of the graph of $f(x)$.

(b) Sketch the graph of the function $h(x)$ which is defined by

$$h(x) = f(x-2)\cdot u(x).$$

(c) Let $I_n = \int_0^n e^{-x}h(x)\,dx$, where n is a positive integer.

Evaluate $\lim_{n\to\infty} I_n$.

7. The position vector of a point $R(x, y, z)$ is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} .$$

(a) Consider the vector equations of a plane

$$\pi : \mathbf{r} \cdot \mathbf{n} = \rho ,$$

and of a line

$$\ell : \mathbf{r} = \mathbf{a} + t\mathbf{b} , \quad t \in \mathbf{R} .$$

(i) If $\mathbf{b} \cdot \mathbf{n} \neq 0$, prove that ℓ and π intersect at a point with position vector $\mathbf{a} + \left(\frac{\rho - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\right) \mathbf{b}$.

(ii) Find the position vector of the foot of the perpendicular from a point $R_0(x_0 , y_0 , z_0)$ to the plane π .

(b) The image by reflection of a point P with respect to a plane π is a point P' such that π bisects the line segment PP' perpendicularly.

Using (a), or otherwise, find the coordinates of the image P' of the point $P(\alpha , \beta , \gamma)$ with respect to the plane

$$\pi : x + y + z - 1 = 0 .$$

Hence, or otherwise, find the equation of the locus of P' as P moves along the line

$$\ell : \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} .$$

8. Let $g : \mathbf{R} \to \mathbf{R}$ be a differentiable function such that its derivative g' is increasing.

(a) Let a and λ be real constants such that $0 < \lambda < 1$. Show that the function

$$F(x) = g(\lambda x + (1 - \lambda)a) - \lambda g(x) - (1 - \lambda) g(a)$$

attains its greatest value when $x = a$.

(b) Let $\lambda_1 , \lambda_2 , \ldots, \lambda_m$ $(m \geqslant 2)$ be m positive real numbers such that $\lambda_1 + \lambda_2 + \ldots + \lambda_m = 1$.

(i) Prove by mathematical induction, or otherwise, that

$$g(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m) \leqslant \lambda_1 g(x_1) + \lambda_2 g(x_2) + \ldots + \lambda_m g(x_m)$$

for any real numbers $x_1 , x_2 , \ldots, x_m$.

(ii) By considering $g(x) = e^x$, or otherwise, deduce that

$$a_1^{\lambda_1} a_2^{\lambda_2} \ldots a_m^{\lambda_m} \leqslant \lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_m a_m$$

for any positive numbers $a_1 , a_2 , \ldots, a_m$.

9. Let $f(x)$ be a function defined and continuously differentiable on the interval $I = [0, a]$, and satisfying the following conditions:

(1) $f(0) = 0$ and $f(x) > 0$ for all $x \in I$,

(2) $f'(x)$ is increasing on I.

(a) By considering $\int_0^x f'(t)\,dt$, or otherwise, show that $f(x) < x f'(x)$ for all $x \in I$.

(b) Let $F(x) = f(x)\sqrt{x^2 + [f(x)]^2}$,

$G(x) = 2f(x)\sqrt{1 + [f'(x)]^2}$.

(i) Show that $G(x) > 2f(x)f'(x)$.

(ii) Using (a), or otherwise, show that $[F'(x)]^2 - [G(x)]^2 > 0$ for all $x \in I$. Hence deduce that $F'(x) > G(x)$ for all $x \in I$.

(c) Let S be the area of the surface generated by rotating the graph of $f(x)$ about the x-axis. Using (b), or otherwise, show that

$$\pi[f(a)]^2 \leqslant S \leqslant \pi f(a)\sqrt{a^2 + [f(a)]^2}.$$

END OF PAPER

The following are outlines of solutions extracted from the annual reports of past Hong Kong Advanced Level examinations. Readers should note that they are not meant to be model answers.

Outline of Solutions

1981

Paper I

Q.1 (a) Solution set of (I) is

$\{(3t+2,\ 2t-1,\ t) : t \in \mathbb{R}\}$.

(b) (i) For $p \neq -5$ and $q \in \mathbb{R}$ (II) is solvable.
(ii) For $p = -5$ and $q = 1$ (II) is solvable.

(c) (i) If $p \neq -5$ the only solution for the first three equations of (III) is $(2, -1, 0)$ which does not satisfy the 4th equation. So (III) has no solution for $p \neq -5$.

(ii) If $p = -5$, the solution set for the first three equations of (III) is $\{(3t+2,\ 2t-1,\ t) : t \in \mathbb{R}\}$. Substituting it into the 4th equation we get

$$7t^2 + 4t - 3 = 0$$

and hence

$$t = -1 \text{ or } \frac{3}{7}.$$

Thus $(-1, -3, -1)$ and $(\frac{23}{7}, -\frac{1}{7}, \frac{3}{7})$ are the solutions of (III).

Q.2 (a) Consider the sequence $\{-b_n\}$.

Since $-b_n \leqslant -b_{n+1}$ $\forall n$ and
$-b_n \leqslant -M$ $\forall n$,

$\{-b_n\}$ converges.

$\lim_{n\to\infty} b_n = -\lim_{n\to\infty}(-b_n)$, since the last limit exists.

$\therefore \{b_n\}$ converges.

(b) Since G.M. $\leqslant$ A.M. and $x_1 < y_1$,
$x_n \leqslant y_n$ for $n = 1, 2, \ldots$.

It is obvious that $x_n, y_n > 0$ $\forall n$.

Further, for $n \geqslant 1$

$$x_{n+1} = \sqrt{x_n y_n} \geqslant \sqrt{x_n x_n} = x_n,$$

$$y_{n+1} = \frac{x_n + y_n}{2} \leqslant \frac{y_n + y_n}{2} = y_n.$$

Thus $\{x_n\}$ is increasing and bounded above by b, and $\{y_n\}$ is decreasing and bounded below by a. Therefore both $\{x_n\}$, $\{y_n\}$ are convergent.

Let $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$.

Since $y_{n+1} = \frac{x_n + y_n}{2}$

$$\therefore \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty}\frac{x_n + y_n}{2}$$

$$= \frac{1}{2}(\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n)$$

$\therefore x = y$.