RESTRICTED 內部文件

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八五年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1985

PURE MATHEMATICS (I)

MARKING SCHEME

This is a restricted document.

It is meant for use by markers of this paper for marking purposes only.

Reproduction in any form is strictly prohibited.

Special Note for Teacher Markers

It is highly undesirable that this marking scheme should fall into the hands of students. They are likely to regard it as a set of model answers, which it certainly is not.

Markers should therefore resist pleas from their students to have access to this document. Making it available would constitute misconduct on the part of the marker and is, moreover, in breach of the 1977 Hong Kong Examinations Authority Ordinance.

SOLUTIONS	MARKS	REMARKS

85 I

(a) (i) Let $\lim_{n\to\infty} a_n = \ell$.

Then $\lim_{n\to\infty} a_{n+p} = \ell$ .... (1)

$\therefore \lim_{n\to\infty} (a_n + a_{n+p})$ exists and equals

$\lim_{n\to\infty} a_n + \lim_{n\to\infty} a_{n+p}$ .... (2)

$= 2\ell$

(ii) Consider the sequence $\{c_n\}$ defined by $c_n = (-1)^n$. The (2)

sequence whose nth term is $c_n + c_{n+1}$ is the convergent

sequence 0, 0, 0, ... but $\{c_n\}$ itself is divergent. (2)

Total: 7

(b) Since $\lim_{n\to\infty} (a_n + a_{n+1}) = A$ and $\lim_{n\to\infty} (a_n + a_{n+2}) = B$,

$\lim_{n\to\infty}[(a_n + a_{n+1}) + (a_n + a_{n+2})] = A + B$. .... (2)

But $\lim_{n\to\infty} (a_{n+1} + a_{n+2}) = \lim_{n\to\infty} (a_n + a_{n+1})$

$= A$ .... (1)

$\therefore \lim_{n\to\infty} (2a_n)$ exists and equals B .... (1)

$\lim_{n\to\infty} a_n = \frac{B}{2}$ .... (1)

Further, $A = \lim_{n\to\infty} (a_n + a_{n+1})$

$= \lim_{n\to\infty} a_n + \lim_{n\to\infty} a_n$

$= B$ .... (2)

Total: 7

SOLUTIONS	MARKS	REMARKS

85 I

2. 2(a) (i)

$$\Delta = \begin{vmatrix} 0 & |a_1 - a_2| & |a_1 - a_3| \\ |a_2 - a_1| & 0 & |a_2 - a_3| \\ |a_3 - a_1| & |a_3 - a_2| & 0 \end{vmatrix}$$

$= |(a_1-a_2)(a_2-a_3)(a_3-a_1)| + |(a_2-a_1)(a_3-a_2)(a_1-a_3)|$ (1)

> 0 as a_1, a_2, a_3 are distinct. .... (1)

$\therefore$ (E) has a unique solution.

(ii) Assume that $a_1 > a_2 > a_3$ 設個 case

If $b_1 = b_2 = b_3\ (=b) \neq 0$,

$$x_1 = \frac{b}{\Delta}\begin{vmatrix} 1 & a_1 - a_2 & a_1 - a_3 \\ 1 & 0 & a_2 - a_3 \\ 1 & a_2 - a_3 & 0 \end{vmatrix}$$

$= \frac{b}{\Delta}[(a_1-a_2)(a_2-a_3) + (a_2-a_3)(a_1-a_3) - (a_2-a_3)(a_2-a_3)]$

$= \frac{2b}{\Delta}(a_1 - a_2)(a_2 - a_3) \neq 0$ ($\because$ all distinct) ..... (1)

$$x_2 = \frac{b}{\Delta}\begin{vmatrix} 0 & 1 & a_1 - a_3 \\ a_1 - a_2 & 1 & a_2 - a_3 \\ a_1 - a_3 & 1 & 0 \end{vmatrix}$$

$= \frac{b}{\Delta}[(a_1-a_3)(a_2-a_3) + (a_1-a_2)(a_1-a_3) - (a_1-a_3)(a_1-a_3)]$

$= 0$ .... (1)

$$x_3 = \frac{b}{\Delta}\begin{vmatrix} 0 & a_1 - a_2 & 1 \\ a_1 - a_2 & 0 & 1 \\ a_1 - a_3 & a_2 - a_3 & 1 \end{vmatrix}$$

$= \frac{b}{\Delta}[(a_1-a_2)(a_2-a_3) + (a_1-a_2)(a_1-a_3) - (a_1-a_2)(a_1-a_2)]$

$= \frac{2b}{\Delta}(a_1 - a_2)(a_2 - a_3)$ $\neq$ ($\because$ all distinct) ..... (1)

$\therefore$ only $x_2 = 0$.

Similarly, if $a_\ell > a_m > a_n$ where ℓ, m, n is any other permutation of 1, 2, 3, we have

$x_m = 0$ but $x_\ell = x_n \neq 0$ (1)

Total: 6

I

(b) Assume for contradiction that a_1, a_2, a_3 are not all distinct.

舉一個case

Let $a_1 = a_2$. Then the system (E) becomes

$$|a_1 - a_3|x_3 = b_1$$
$$|a_1 - a_3|x_3 = b_2$$
$$|a_3 - a_1|x_1 + |a_3 - a_1|x_2 = b_3 \qquad 2$$

If this system is consistent, the first two equations imply $b_1 = b_2$, contradicting the fact that b_1, b_2, b_3 are all distinct

∴ (E) is not consistent. 2

Similarly if $a_1 = a_3$ or $a_2 = a_3$, (E) cannot be consistent. 1

∴ (E) is consistent only if a_1, a_2, a_3 are all distinct. necessary but not sufficient conditions

5

(c) If $a_1 = a_2 = a_3$ the coefficient matrix of (E) is the zero matrix.

(E) is consistent iff $b_1 = b_2 = b_3 = 0$, 2

in which case the whole space $\mathbb{R}^3$ is the solution set. 1

3

I

3(a)
$$\frac{f(x)}{(x+a_1)(x+a_2)\dots(x+a_n)} = \frac{c_1}{x+a_1} + \frac{c_2}{x+a_2} + \dots + \frac{c_n}{x+a_n}$$

Combining the partial fractions of the R.S., the numerator is

$$c_1(x+a_2)(x+a_3)\dots(x+a_n) + c_2(x+a_1)(x+a_3)\dots(x+a_n) + \dots + c_n(x+a_1)(x+a_2)\dots(x+a_{n-1})$$
$$= (c_1 + c_2 + \dots + c_n)x^{n-1} + (\text{terms of degree} < n-1) \quad \dots\dots \quad 2$$

Since $f(x)$ is a polynomial of degree $< n-1$, ... 1

$c_1 + c_2 + \dots + c_n = 0$ .... Comparing coefficients 1

4

(b) $F(x) = \dfrac{px+q}{(x+a)(x+a+1)(x+a+2)} = \dfrac{b_1}{x+a} + \dfrac{b_2}{x+a+1} + \dfrac{b_3}{x+a+2}$

For $N > 3$,

$$\sum_{k=1}^{N} F(k) = \sum_{k=1}^{N}\frac{b_1}{k+a} + \sum_{k=1}^{N}\frac{b_2}{k+a+1} + \sum_{k=1}^{N}\frac{b_3}{k+a+2} \quad \dots\dots \quad 1$$

$$= \left[\frac{b_1}{1+a} + \frac{b_1}{2+a} + \sum_{k=3}^{N}\frac{b_1}{k+a}\right] + \left[\frac{b_2}{2+a} + \frac{b_2}{N+a+1} + \sum_{k=3}^{N}\frac{b_2}{k+a}\right]$$
$$+ \left[\frac{b_3}{N+a+1} + \frac{b_3}{N+a+2} + \sum_{k=3}^{N}\frac{b_3}{k+a}\right]$$

$$= \frac{b_1}{1+a} + \frac{b_1+b_2}{2+a} + \frac{b_2+b_3}{N+a+1} + \frac{b_3}{N+a+2} + \sum_{k=3}^{N}\frac{b_1+b_2+b_3}{k+a} \quad \dots\dots \quad 2$$

The last term vanishes since $px+q$ is of degree 1 and therefore $b_1 + b_2 + b_3 = 0$ by (a). 2

5

(c) Let $\dfrac{1}{(2k+1)(2k+3)(2k+5)} = \dfrac{b_1}{2k+1} + \dfrac{b_2}{2k+3} + \dfrac{b_3}{2k+5}$

$1 = b_1(2k+3)(2k+5) + b_2(2k+1)(2k+5) + b_3(2k+1)(2k+3)$ 1

Put $k = -\frac{1}{2}$, $1 = b_1(2)(4) \Rightarrow b_1 = \frac{1}{8}$

$k = -\frac{3}{2}, -\frac{5}{2} \Rightarrow b_2 = -\frac{1}{4}$, $b_3 = \frac{1}{8}$ 1

$$∴ \frac{1}{(2k+1)(2k+3)(2k+5)} = \frac{1}{8}\left(\frac{1}{2k+1}\right) - \frac{1}{4}\left(\frac{1}{2k+3}\right) + \frac{1}{8}\left(\frac{1}{2k+5}\right)$$

$$\frac{1}{8}\,\frac{1}{(k+\frac{1}{2})(k+\frac{3}{2})(k+\frac{5}{2})} = \frac{1}{8}\left(\frac{\frac{1}{2}}{k+\frac{1}{2}} - \frac{1}{k+\frac{3}{2}} + \frac{\frac{1}{2}}{k+\frac{5}{2}}\right) \quad \dots\dots \quad 1$$

Let $a = \frac{1}{2}$, by (b),

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{1}{(2k+1)(2k+3)(2k+5)}$$

$$= \lim_{N\to\infty}\frac{1}{8}\left[\frac{\frac{1}{2}}{1+\frac{1}{2}} + \frac{-\frac{1}{2}}{2+\frac{1}{2}} + \frac{-\frac{1}{2}}{N+\frac{3}{2}} + \frac{\frac{1}{2}}{N+\frac{5}{2}}\right]$$

$$= \frac{1}{60} \quad \dots\dots \quad 2$$

5

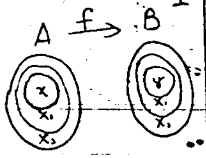
SOLUTIONS | 85 I | MARKS | REMARKS

4(a) (i) Let $X_1 \subset X_2$. For any y,

$y \in f[X_1] \Rightarrow y = f(x)$ for some $x \in X_1$

$\Rightarrow y = f(x)$ for some $x \in X_2$

$\Rightarrow y \in f[X_2]$



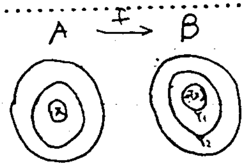
$\therefore f[X_1] \subset f[X_2]$ 2

(ii) Let $Y_1 \subset Y_2$. For any x,

$x \in f^{-1}[Y_1] \Rightarrow f(x) \in Y_1$

$\Rightarrow f(x) \in Y_2$

$\Rightarrow x \in f^{-1}[Y_2]$



$\therefore f^{-1}[Y_1] \subset f^{-1}[Y_2]$ .... 2 / 4

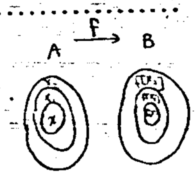
(b) 'Only if' part.

Let f be injective and let $X_1, X_2 \subset A$ s.t. $f[X_1] \subset f[X_2]$. For any $x \in X_1$,

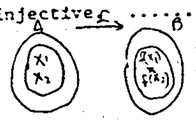
$f(x) \in f[X_1] \Rightarrow f(x) \in f[X_2]$

$\Rightarrow x \in X_2$ since f is injective ..... 2 — Explanation necessary.

$\therefore X_1 \subset X_2$



'If' part.



For any $x_1, x_2 \in A$, suppose $f(x_1) = f(x_2)$. 1

Let $X_1 = \{x_1\}$, $X_2 = \{x_2\}$. Then $f[X_1] \subset f[X_2]$ and $f[X_2] \subset f[X_1]$.

$\therefore X_1 = X_2$ i.e. $x_1 = x_2$

Hence f is injective. .... 2 / 5

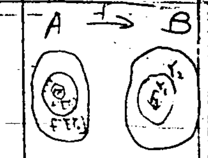
Alt. Solution:

'If' part. We shall show by contradiction that the given statement is not true. Suppose f is not injective. $\exists\, x, y \in A$ s.t.

$f(x) = f(y)$ but $x \neq y$. 1

Take $X_1 = \{x, y\}$, $X_2 = \{x\}$. Then $f[X_1] \subset f[X_2]$ but $X_1 \not\subset X_2$. 2

SOLUTIONS | 85 I | MARKS | REMARKS



4(c) 'Only if' part.

Let f be surjective and let $Y_1, Y_2 \subset B$ s.t. $f^{-1}[Y_1] \subset f^{-1}[Y_2]$. For any $y \in Y_1$, let $y = f(x)$ for some $x \in A$.

Then $x \in f^{-1}[Y_1] \Rightarrow x \in f^{-1}[Y_2]$

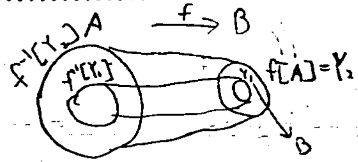
$\Rightarrow y \in Y_2$. $\therefore Y_1 \subset Y_2$ 2

'If' part.

Let $Y_1 = B$, $Y_2 = f[A]$ 1

Then $f^{-1}[Y_2] = A$.

$\therefore f^{-1}[Y_1] \subset f^{-1}[Y_2]$



$\Rightarrow Y_1 \subset Y_2$

$\Rightarrow B \subset f[A]$.

$\therefore f$ is surjective. .... 2 / 5

Alternative Solution:

'If' part.

Suppose f is not surjective.

$\exists\, y \in B$ s.t. $y \notin f[A]$. 1

Let $Y_1 = B$, $Y_2 = B \setminus \{y\}$

Then $f^{-1}[Y_1] \subset f^{-1}[Y_2]$ but $Y_1 \not\subset Y_2$. 2

SOLUTIONS 85 I	MARKS	REMARKS

7(a) Let $f(x) = x^k - kx + k - 1$

$f'(x) = k(x^{k-1} - 1)$

$$\begin{cases} < 0 & \text{if } 0 \leq x < 1 \\ = 0 & x = 1 \\ > 0 & x > 1 \end{cases}$$

$\therefore$ $f(x)$ has an absolute minimum at $x = 1$. 2

But $f(1) = 0$

$\therefore$ $x^k - kx + k - 1 \geq 0$ 1

i.e. $x^k + k - 1 \geq kx$ ...... (*)

The equality holds iff $x = 1$. 1

$\frac{1}{4}$

(b) (i) $\left(\frac{G_m}{G_{m-1}}\right)^m = \frac{(G_m)^m}{(G_{m-1})^{m-1}\, G_{m-1}}$

$= \frac{\prod_{i=1}^{m} a_i}{\left(\prod_{i=1}^{m-1} a_i\right) G_{m-1}}$

$= \frac{a_m}{G_{m-1}}$

$= \frac{mA_m - (m-1)A_{m-1}}{G_{m-1}}$ ....(**) 2

$A_m = \frac{1}{m}\sum_{i=1}^{m} a_i$

$mA_m = a_1 + a_2 + \cdots + a_m$ —(1)

$A_{m-1} = \frac{1}{m-1}\sum_{i=1}^{m-1} a_i$

$(m-1)A_n = a_1 + a_2 + \cdots + a_{m-1}$ —(2)

(1) - (2)

(ii) Putting $x = \frac{G_m}{G_{m-1}}$ in (*), for $m = 2, 3, \ldots, n$,

$\left(\frac{G_m}{G_{m-1}}\right)^m + m - 1 \geq m\left(\frac{G_m}{G_{m-1}}\right)$ 1

By (**), $\frac{mA_m - (m-1)A_{m-1}}{G_{m-1}} + (m-1) \geq m\left(\frac{G_m}{G_{m-1}}\right)$ 1

$\therefore$ $m(A_m - G_m) + (m-1)(G_{m-1} - A_{m-1}) \geq 0$

$A_m - G_m \geq \frac{m-1}{m}(A_{m-1} - G_{m-1})$ 1

SOLUTIONS 85 I	MARKS	REMARKS

5(b) (iii) By (ii), $A_n - G_n \geq \frac{n-1}{n}(A_{n-1} - G_{n-1})$ 1

$\geq$ ....

$\geq \frac{(n-1)(n-2)\ldots(2)(1)}{n(n-1)\ldots(3)(2)}(A_1 - G_1)$

$= 0$ ....(***) 1

$\therefore$ $A_n \geq G_n$

Next, if $a_1 = a_2 = \ldots = a_n = a$, then both A_n, G_n equal a. 1

Conversely, if $A_n = G_n$, then all equalities in (***) hold.

By (a), this is true iff $x = \frac{G_m}{G_{m-1}} = 1$, ... 1

i.e. $(a_1 a_2 \ldots a_m)^{\frac{1}{m}} = (a_1 a_2 \ldots a_{m-1})^{\frac{1}{m-1}}$

$a_m = (a_1 a_2 \ldots a_{m-1})^{\frac{1}{m-1}}$ $(1 < m \leq n)$.

$\therefore$ $a_2 = a_1$

$a_3 = (a_1 a_2)^{\frac{1}{2}} = a_1$

....

Hence $a_n = a_{n-1} = \ldots = a_1$ .... $\frac{1}{10}$

(c)

SOLUTIONS	MARKS	REMARKS
6(a) For $r = 0$ or 1, the statement is trivially true.	1	
Consider $r > 1$.		
(i) When $k \geq r$, $P_r(k) = \frac{k(k-1)\ \dots\ k-r+1}{r!} = C^k_r$, which is an integer.	1	
(ii) When $0 \leq k < r$, $P_r(k) = 0$ since $(x-k)$ is a factor of $P_r(k)$.	1	
(iii) When $k < 0$, putting $k = -\ell$, $P_r(k) = \frac{(-1)^r \ell(\ell+1)\ \dots\ (\ell+r-1)}{r!} = (-1)^r C^{\ell+r-1}_r$, which is also an integer.	1 / 4	
(b) Let $0 < m \leq n$. $P_r(m) = C^m_r$		
$P(m) = a_0P_0(m) + a_1P_1(m) + \dots + a_mP_m(m) + \dots + a_nP_n(m)$. $\longrightarrow 0$		
By (a) $P_r(m) = 0$ for $m < r$ and $P_r(m) = C^m_r$ for $m \geq r$		
$\therefore P(m) = a_0C^m_0 + a_1C^m_1 + \dots + a_mC^m_m$	2	
If $a_0, a_1, \dots, a_{m-1}$ (given) are integers while a_m is not, $P(m) = (a_0C^m_0 + a_1C^m_1 + \dots + a_{m-1}C^m_{m-1}) + a_m$ cannot be an integer.	1	
(i) First $a_0 = P(0)$ is an integer. Assume for contradiction that $a_0, a_1, \dots, a_n$ are not all integers. Let m be the least suffix such that a_m is not an integer $(0 < m \leq n)$. By the above result, $P(m)$ cannot be an	1	
integer, which contradicts the given conditions.	1	
(ii) $P(k) = \sum_{r=0}^{n} a_rP_r(k)$.		
Since $a_0, a_1, \dots, a_n$ are integers by b(i), and $P_r(k)$ are integers by (a), $P(k)$ is therefore an integer for all k.	2 / 7	
(c) Let $Q(x) = P_2(x) = \frac{x(x-1)}{2} = \frac{1}{2}x^2 - \frac{1}{2}x$.		
$Q(k)$ is an integer $\forall k$ by (a), but the coefficients of $Q(x)$ are not all integral.	3 / 3	

SOLUTIONS	MARKS	REMARKS
7(a) (i) If $X, Y \in M_1$, $AX = k_1X$ and $AY = k_1Y$		
For any $a, b \in \mathbb{R}$, $(aX + bY) \in M$ (3×1 matrics) and		
$A(aX + bY) = A(aX) + A(bY)$ $= aAX + bAY$ $= ak_1X + bk_1Y$	1	
$= k_1(aX + bY)$	1	
$\therefore aX + bY \in M_1$. Similarly for $i = 2$.		
(ii) Obviously $0 \in M_1 \cap M_2$	1	
If $X \in M_1 \cap M_2$, $AX = k_1X$ and $AX = k_2X$ $\therefore k_1X = k_2X \Rightarrow (k_1 - k_2)X = 0$ $\Rightarrow X = 0$ as $k_1 \neq k_2$	1	
$\therefore M_1 \cap M_2 = \{0\}$.	4	
(b) For any $X \in M$, $(A - k_2I)X \in M$ and $A[(A - k_2I)X] = (A^2 - k_2A)X$	1	$A^2-(k_1+k_2)A+k_1k_2I=0$
$= (k_1A - k_1k_2I)X$	1	$A^2-k_2A=k_1A-k_1k_2$
$= k_1[(A - k_2I)X]$	1	
$\therefore (A - k_2I)X \in M_1$. Similarly $(A - k_1I)X \in M_2$.	1 / 4	
(c) For any $X \in M$, by (b), (2)		
$(A - k_2I)X \in M_1$ and $(A - k_1I)X \in M_2$ 存在		
By (a) $X_1 = \frac{1}{k_1 - k_2}(A - k_2I)X \in M_1$ and	1	
$X_2 = \frac{-1}{k_1 - k_2}(A - k_1I)X \in M_2$	1	
and $X_1 + X_2 = \frac{1}{k_1 - k_2}(A - k_2I)X + \frac{-1}{k_1 - k_2}(A - k_1I)X$ $= X$	1	
uniqueness (1) If $\exists X_1' \in M_1$ and $X_2' \in M_2$ such that $X = X_1' + X_2'$	1	
then $X_1 + X_2 = X_1' + X_2'$		$X = M_1$
$X_1 - X_1' = X_2' - X_2 \in M_1 \cap M_2$ by (a)(i).	1	$y = M_2$
$\therefore X_1 - X_1' = X_2' - X_2 = 0$ by (a)(ii)	1	if $x = y$,
i.e. $X_1 = X_1'$, $X_2 = X_2'$	6	$x = y \in M_1 \cap$

SOLUTIONS	85 I MARKS	REMARKS

8(a) We shall prove the first part by induction. The case where $n = 1$ is trivial.

Assume that $g(kx) = kg(x)$ for some integer $k \geq 1$.

Then
$$\begin{aligned} g((k+1)x) &= g(kx + x) \\ &= g(kx) + g(x) \\ &= kg(x) + g(x) \\ &= (k+1)g(x) \end{aligned}$$

Hence $g(nx) = ng(x) \quad \forall\, n \geq 1$. **2**

Next assume for contradiction that $g(x) \not\equiv 0$.

Let $|g(x_0)| = b > 0$ for some $x_0 \in \mathbb{R}$. **1**

Since g is bounded on $\mathbb{R}$, let $|g(x)| \leq M \;\; \forall\, x \in \mathbb{R}$ and let n be an integer greater than $\frac{M}{b}$. **1**

Then
$$\begin{aligned} |g(nx_0)| &= |ng(x_0)| \\ &= n\,|g(x_0)| \\ &> \frac{M}{b}\,|g(x_0)| \\ &= M, \end{aligned}$$

which contradicts the boundedness of g. **1**

$\therefore g(x) \equiv 0$ **5**

(b) (i) For any $x, y \in \mathbb{R}$,
$$\begin{aligned} h(x+y) &= f(x+y) - \frac{f(a)}{a}(x+y) \\ &= f(x) + f(y) - \frac{f(a)}{a}x - \frac{f(a)}{a}y \\ &= h(x) + h(y) \end{aligned}$$
.... **2**

$\therefore h$ is additive.

Let f be bounded by M on $[0, a]$.

For any $x \in [0, a]$,
$$\begin{aligned} |h(x)| &= \left|f(x) - \frac{f(a)}{a}x\right| \\ &\leq |f(x)| + \left|\frac{f(a)}{a}\right||x| \\ &\leq |f(x)| + |f(a)| \\ &\leq 2M \end{aligned}$$

$\therefore h$ is bounded on $[0, a]$. **2**

(ii) For any $x \in \mathbb{R}$,
$$\begin{aligned} h(x+a) &= h(x) + h(a) \\ &= h(x) + f(a) - \frac{f(a)}{a}a \\ &= h(x) \end{aligned}$$
.... **1**

Next for any $x \in \mathbb{R}$, since h is periodic of period a, let y be in $[0, a]$ such that $h(x) = h(y)$. **1**

$\therefore |h(x)| = |h(y)| \leq 2M$ by (i)

$\therefore h$ is bounded on $\mathbb{R}$. **1**

(iii) As h is additive and bounded on $\mathbb{R}$, by (a)

$h(x) \equiv 0$, i.e. $f(x) = \frac{f(a)}{a}x \;\; \forall\, x \in \mathbb{R}$ **2**

SOLUTIONS	85 I MARKS	REMARKS

9(a) (i) $(1+x)^{m-p}(1+x)^m = \left(\sum_{r=0}^{m-p} C_r^{m-p} x^r\right)\left(\sum_{r=0}^{m} C_r^m x^r\right)$

The coefficient of $x^m = \sum_{s=0}^{m-p} C_s^{m-p} C_{m-s}^m$ **1**

$$= \sum_{r=p}^{m} C_{m-r}^{m-p} C_r^m$$

$$= \sum_{r=p}^{m} C_{r-p}^{m-p} C_r^m$$ **1**

But the coefficient of x^m in $(1+x)^{2m-p}$ is C_m^{2m-p} **1**

Hence the result.

(ii) $\sum_{r=0}^{m} r(C_r^m)^2 = \sum_{r=0}^{m} (r)(C_r^m)(C_r^m)$

$$= m\sum_{r=1}^{m} C_{r-1}^{m-1} C_r^m$$ **1**

$$= m \cdot C_m^{2m-1}, \text{ from (i) } (p = 1)$$ **1**

$$\sum_{r=0}^{m} r^2 (C_r^m)^2 = m^2 \sum_{r=1}^{m} (C_{r-1}^{m-1})^2$$

$$= m^2 \sum_{r=0}^{m-1} (C_r^{m-1})^2$$

$$= m^2 \cdot C_{m-1}^{2m-2}, \text{ from (i) } (p = 0)$$ **1**

6

SOLUTIONS	MARKS	REMARKS
(b) Suppose bag A contains r $(0 \le r \le m)$ white balls. If a ball is drawn from each bag, the probability that they have the same colour is		
$\frac{r}{m}\cdot\frac{m-r}{m}+\frac{m-r}{m}\cdot\frac{r}{m}$ $= \frac{2r(m-r)}{m^2}$	1	
If m balls are selected at random and put into each bag, the probability that bag A contains r white balls is		
$\frac{C^m_r \cdot C^m_{m-r}}{C^{2m}_m}$	2	
$= \frac{(C^m_r)^2}{C^{2m}_m}$		
$\therefore$ the probability that the two balls subsequently drawn have the same colour is		
$\frac{(C^m_r)^2}{C^{2m}_m}\cdot\frac{2r(m-r)}{m^2}$	1	
The required probability is		
$\sum_{r=0}^{m}\frac{(C^m_r)^2}{C^{2m}_m}\cdot\frac{2r(m-r)}{m^2}$	2	
$= \frac{2}{C^{2m}_m}\left[\frac{1}{m}\sum_{r=0}^{m} r(C^m_r)^2 - \frac{1}{m^2}\sum_{r=0}^{m} r^2(C^m_r)^2\right]$		
$= \frac{2}{C^{2m}_m}\left[C^{2m-1}_m - C^{2m-2}_{m-1}\right]$	1	
$= \frac{2(m!)^2}{(2m)!}\left[\frac{(2m-1)!}{m!\,(m-1)!} - \frac{(2m-2)!}{[(m-1)!]^2}\right]$		
$= \frac{m-1}{2m-1}$	$\frac{1}{8}$	

85 I

RESTRICTED 内部文件

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八五年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1985

PURE MATHEMATICS (II)

MARKING SCHEME

This is a restricted document.

It is meant for use by markers of this paper for marking purposes only.

Reproduction in any form is strictly prohibited.

Special Note for Teacher Markers

It is highly undesirable that this marking scheme should fall into the hands of students. They are likely to regard it as a set of model answers, which it certainly is not.

Markers should therefore resist pleas from their students to have access to this document. Making it available would constitute misconduct on the part of the marker and is, moreover, in breach of the 1977 Hong Kong Examinations Authority Ordinance.

SOLUTIONS | MARKS | REMARKS

1 (a) The tangents to (E) at P and Q are given by

$$\frac{\cos\theta}{a}x + \frac{\sin\theta}{b}y = 1$$

$$\frac{\cos\phi}{a}x + \frac{\sin\phi}{b}y = 1 \quad \ldots\ldots \quad 1$$

Solving these equations, the coordinates of T are :

$$x = \frac{a(\sin\phi - \sin\theta)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$= \frac{a(\sin\phi - \sin\theta)}{\sin(\phi - \theta)}$$

$$y = \frac{b(\cos\theta - \cos\phi)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$= \frac{b(\cos\theta - \cos\phi)}{\sin(\phi - \theta)}$$

by Cramer's rule

The tangents to (K) at M and N are

$$x\cos\theta + y\sin\theta = a$$

$$x\cos\phi + y\sin\phi = a \quad \ldots\ldots \quad 1$$

The coordinates of R are

$$x = \frac{a(\sin\phi - \sin\theta)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad 1$$

$$y = \frac{a(\cos\theta - \cos\phi)}{\sin\phi\cos\theta - \cos\phi\sin\theta} \quad \ldots\ldots \quad \frac{1}{6}$$

(which could also be obtained by putting b = a in the first part)

(b) (i) Substituting the coordinates of T in (F),

$$\frac{a^2(\sin\phi - \sin\theta)^2}{a^2\sin^2(\phi - \theta)} + \frac{b^2(\cos\theta - \cos\phi)^2}{b^2\sin^2(\phi - \theta)} = 2 \quad \ldots\ldots \quad 2$$

$$\sin^2\phi + \sin^2\theta - 2\sin\phi\sin\theta + \cos^2\phi + \cos^2\theta - 2\cos\phi\cos\theta = 2\sin^2(\phi - \theta)$$

$$\cos(\phi - \theta) = \sin^2(\phi - \theta) \quad \ldots\ldots \quad 1$$

$$\cos(\phi - \theta)[\cos(\phi - \theta) - 1] = 0$$

$\therefore \cos(\phi - \theta) = 0$ as θ and ϕ are distinct 1

(ii) Squaring and adding the coordinates of R,

$$x^2 + y^2 = \frac{a^2(\sin\phi - \sin\theta)^2}{\sin^2(\phi - \theta)} + \frac{a^2(\cos\theta - \cos\phi)^2}{\sin^2(\phi - \theta)} \quad \ldots\ldots \quad 2$$

$$= a^2\,\frac{\sin^2\phi + \sin^2\theta - 2\sin\theta\sin\phi + \cos^2\theta + \cos^2\phi - 2\cos\theta\cos\phi}{\sin^2(\phi - \theta)}$$

$$= 2a^2\,\frac{[1 - \cos(\theta - \phi)]}{\sin^2(\phi - \theta)}$$

$= 2a^2$ since $\cos(\theta - \phi) = 0$ 1

$\therefore$ R moves on a circle with radius $a\sqrt{2}$ ...... $\frac{1}{8}$

(Note: This part can also be deduced by geometric method.)

SOLUTIONS | MARKS | REMARKS

2 (a) The direction numbers of a line normal to π are A, B, C. 1

If this line passes through P, its equations are

$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt \qquad t \in \mathbb{R} \quad \ldots (*) \\ z = z_0 + Ct \end{cases} \quad \ldots\ldots \quad 1$$

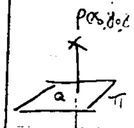
Substituting in π,

$$A(x_0 + At) + B(y_0 + Bt) + C(z_0 + Ct) + D = 0 \quad \ldots\ldots \quad 1$$

$$Ax_0 + By_0 + Cz_0 + D + (A^2 + B^2 + C^2)t = 0$$

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2} \quad \ldots\ldots \quad 1$$

Putting this value of t in (*), we obtain the coordinates of Q. $\frac{1}{5}$



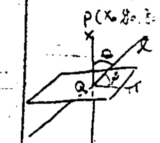
(b) The direction numbers of ℓ are p, q, r. 1

The acute angle θ between ℓ and the normal to π is given by

$$\cos\theta = \frac{|Ap + Bq + Cr|}{\sqrt{A^2 + B^2 + C^2}\sqrt{p^2 + q^2 + r^2}} \quad \ldots\ldots \quad 2$$

$\therefore$ the angle ϕ between ℓ and π is given by

$$\phi = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \cos^{-1}\frac{|Ap + Bq + Cr|}{\sqrt{A^2 + B^2 + C^2}\sqrt{p^2 + q^2 + r^2}} \quad \ldots\ldots \quad \frac{1}{4}$$



$\vec{a}\cdot\vec{n} = |\vec{a}||\vec{n}|\cos\theta$

$\cos\theta = \frac{\vec{a}\cdot\vec{n}}{|\vec{a}||\vec{n}|}$

(c) Substituting the coordinates of a point on ℓ into π, ... 1

$$A(a + tp) + B(b + tq) + C(c + tr) + D = 0$$

$$(Aa + Bb + Cc + D) + (Ap + Bq + Cr)t = 0 \quad \ldots\ldots \quad 1$$

ℓ lies on π if and only if the above equation holds for all real t. 1

i.e. $Aa + Bb + Cc + D = 0$

$Ap + Bq + Cr = 0$ ...... $\frac{2}{5}$

[Note: Candidates may use vectors in their solution.]

3(a) For $t \in (-1, 1)$, the sum of the GP

$$1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} = \frac{1-(-t)^n}{1+t} \leftarrow n \text{ terms}$$

$$\therefore \frac{1}{1+t} = 1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}$$

Putting $u = 1 + t$, for $x \in (-1, 1)$,

$$\ln(1+x) = \int_1^{1+x} \frac{1}{u}\,du \qquad \begin{matrix} u = 1+t \\ du = dt \\ 1+x = 1+t \Rightarrow t = x \quad 1 = 1+t \Rightarrow t = 0 \end{matrix}$$

$$= \int_0^x \frac{1}{1+t}\,dt \quad \dots\dots \quad 1$$

$$= \int_0^x \left[1 - t + t^2 - \dots + (-1)^{n-1} t^{n-1} + \frac{(-1)^n t^n}{1+t}\right] dt$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-1)^n t^n}{1+t}\,dt \quad 1$$

Similarly, $\frac{1}{1-t} = 1 + t + t^2 + \dots + t^{n-1} + \frac{t^n}{1-t}$ 1

$$\ln(1-x) = \int_1^{1-x} \frac{1}{u}\,du \qquad \begin{matrix} u = 1-t \\ du = -dt \\ 1-x = 1-t \Rightarrow t = x \quad 1 = 1-t \Rightarrow t = 0 \end{matrix}$$

$$= -\int_0^x \frac{1}{1-t}\,dt$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} - \int_0^x \frac{t^n}{1-t}\,dt \quad \dots\dots \quad 1$$

5

(or putting $x = -u$ in the first result)

3(b) Putting $n = 2k + 1$,

$$\ln\left(\frac{1+x}{1-x}\right) \qquad \frac{t^{2k+1}+t^{2k+2}-t^{2k+1}+t^{2k+2}}{1-t^2}$$

$$= \ln(1+x) - \ln(1-x) \quad \dots\dots \quad 1$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right] + \int_0^x \left[\frac{t^{2k+1}}{1-t} - \frac{t^{2k+1}}{1+t}\right] dt$$

$$= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right] + \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \quad \dots\dots \quad 1$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) - 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2k+1}}{2k+1}\right] = \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt$$

from the $\int_0^x$

For $0 < t < x < 1$, $0 < \frac{t^{2k+2}}{1-t^2} < \frac{t^{2k+2}}{1-x^2}$

$$\therefore \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \geq 0 \quad \dots\dots \quad 1$$

$$\text{and } \int_0^x \frac{2t^{2k+2}}{1-t^2}\,dt \leq \frac{1}{1-x^2}\int_0^x 2t^{2k+2}\,dt \quad \dots\dots \quad 1$$

$$= \frac{2}{1-x^2}\left(\frac{x^{2k+3}}{2k+3}\right) \quad \dots\dots \quad 1$$

The result follows. 5

(c) Putting $x = \frac{1}{2}$,

$$0 \leq \ln 3 - 2\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right] \leq \frac{8}{3}\cdot\frac{1}{2k+3}\left(\frac{1}{2}\right)^{2k+3} \quad 1$$

As $\lim_{k\to\infty} \frac{8}{3}\cdot\frac{1}{2k+3}\left(\frac{1}{2}\right)^{2k+3} = 0$, ... 1

By Sandwich theorem,

$$\lim_{k\to\infty}\left[\ln 3 - 2\left(\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right)\right] = 0 \quad 1$$

$\therefore \lim_{k\to\infty}\left[\frac{1}{2} + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \dots + \frac{1}{2k+1}\left(\frac{1}{2}\right)^{2k+1}\right]$ exists and

equals $\frac{1}{2}\ln 3$ 1

4

SOLUTIONS | 85 II | MARKS | REMARKS

4. (a) The coordinates of the centre of the circle are $(r\phi, r)$. arc length … 1

The coordinates of Q with respect to the centre are $(-r\sin\phi, -r\cos\phi)$ ……… 1

$\therefore$ the locus of Q is given by

$$\begin{cases} x = r\phi - r\sin\phi & \text{……… 1} \\ y = r - r\cos\phi, \quad 0 \le \phi \le 2\pi & \text{……… 1} \end{cases}$$

i.e. the locus of P is given by

$$\begin{cases} x = r(\phi - \sin\phi) \\ y = r(\cos\phi - 1), \quad 0 \le \phi \le 2\pi \end{cases}$$ ……… 1

greatest when P reaches P_L

When P reaches P_L, the value of $\phi = \pi$. ……… 1

6

(b) $dx = r(1 - \cos\phi)\, d\phi$, $\quad dy = -r\sin\phi\, d\phi$

$$\frac{dy}{dx} = \frac{-\sin\phi}{1 - \cos\phi} \quad \text{……… 1}$$

$$T = \int_{\phi_s}^{\pi} \sqrt{\frac{1 + \left(\frac{\sin\phi}{1 - \cos\phi}\right)^2}{2gr(\cos\phi_s - \cos\phi)}}\, r(1 - \cos\phi)\, d\phi$$

$$= \sqrt{\frac{r}{g}} \int_{\phi_s}^{\pi} \sqrt{\frac{1 - \cos\phi}{\cos\phi_s - \cos\phi}}\, d\phi \quad \text{……… 2}$$

$$= \sqrt{\frac{r}{g}} \int_{\phi_s}^{\pi} \sqrt{\frac{2\sin^2\frac{\phi}{2}}{(2\cos^2\frac{\phi_s}{2} - 1) - (2\cos^2\frac{\phi}{2} - 1)}}\, d\phi \quad \text{……… 2}$$

$$= \sqrt{\frac{r}{g}} \int_{\phi_s}^{\pi} \frac{-2d\left(\cos\frac{\phi}{2}\right)}{\sqrt{\cos^2\frac{\phi_s}{2} - \cos^2\frac{\phi}{2}}}$$

$$= 2\sqrt{\frac{r}{g}} \left[-\sin^{-1}\left(\frac{\cos\frac{\phi}{2}}{\cos\frac{\phi_s}{2}}\right) \right]_{\phi_s}^{\pi} \quad \text{……… 2}$$

$$= 2\sqrt{\frac{r}{g}}\,(-\sin^{-1}0 + \sin^{-1}1)$$

$$= \pi\sqrt{\frac{r}{g}}, \quad \text{……… 1}$$

which is independent of ϕ_s.

8

SOLUTIONS | 85 II | MARKS | REMARKS

5. (a) Since $x_1 + x_2 = 2t$, $x_1x_2 = 1$, for $n \ge 1$, ……… 1

sum of roots, product of root

$$x_1^{n+1} + x_2^{n+1} = (x_1^n + x_2^n)(x_1 + x_2) - x_1x_2(x_1^{n-1} + x_2^{n-1}) \quad \text{……… 1}$$

$$= 2F_n(t)(2t) - 2F_{n-1}(t)$$

$$\therefore F_{n+1}(t) = 2tF_n(t) - F_{n-1}(t) \quad \text{……… 1}$$

Now $F_1(t) = \frac{1}{2}(x_1 + x_2) = t$ is a polynomial in t of degree 1 and with leading coefficient 1. ……… 1

For all positive integers k less than or equal to some positive n, assume that $F_k(t)$ is a polynomial in t of degree k and with leading coefficient 2^{k-1}. $k \le n$

Then $F_{n+1}(t) = 2tF_n(t) - F_{n-1}(t)$ is also a polynomial in t of degree $(n + 1)$. Further the leading coefficient of $F_{n+1}(t) = 2 \times 2^{n-1} = 2^n$. The result follows by induction. second principle

2

6

(b) As $-1 \le t \le 1$, we may let $\cos\theta = t$ ……… 1

$F_0(t) = 1 = \cos 0$, $F_1(t) = t = \cos[\cos^{-1}t]$. ……… 1

Assume that $F_k(t) = \cos[k\cos^{-1}t]$, where $0 \le k \le n$, $n \ge 1$.

$$F_{n+1}(t) = 2t\,F_n(t) - F_{n-1}(t)$$

$$= 2\cos\theta\cos n\theta - \cos(n - 1)\theta$$

$$= 2\cos\theta\cos n\theta - \cos n\theta\cos\theta - \sin n\theta\sin\theta$$

$$= \cos(n + 1)\theta$$

$$= \cos[(n + 1)\cos^{-1}t]$$

$$\therefore F_n(t) = \cos[n\cos^{-1}t] \quad \forall\ n \ge 0. \quad \text{……… 2}$$

$$\int_0^{\pi} F_m(\cos\theta)\, F_n(\cos\theta)\, d\theta = \int_0^{\pi} \cos m\theta\cos n\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} [\cos(m + n)\theta + \cos(m - n)\theta]\, d\theta \quad \text{……… 1}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m + n} + \frac{\sin(m-n)\theta}{m - n}\right]_0^{\pi} & \text{if } m \ne n \\ [\theta]_0^{\pi} & \text{if } m = n = 0 \\ \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m + n} + \theta\right]_0^{\pi} & \text{if } m = n > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } m \ne n \quad \text{……… 1} \\ \pi & \text{if } m = n = 0 \quad \text{……… 1} \\ \frac{\pi}{2} & \text{if } m = n > 0 \quad \text{……… 1} \end{cases}$$

8

SOLUTIONS 85 II	MARKS	REMARKS

Alternative Solution :

5(a) Solving $x^2 - 2tx + 1 = 0$, $x = \frac{2t \pm \sqrt{4t^2 - 4}}{2}$

$= t \pm \alpha$, where $\alpha = \sqrt{t^2 - 1}$ — 1

Let $x_1 = t + \alpha$, $x_2 = t - \alpha$.

$F_{n+1}(t) - 2t\,F_n(t) + F_{n-1}(t)$

$= \frac{1}{2}[(t+\alpha)^{n+1} + (t-\alpha)^{n+1}] - \frac{2t}{2}[(t+\alpha)^n + (t-\alpha)^n]$
$+ \frac{1}{2}[(t+\alpha)^{n-1} + (t-\alpha)^{n-1}]$ — 1

$= \frac{1}{2}(t+\alpha)^{n-1}[(t+\alpha)^2 - 2t(t+\alpha) + 1]$
$+ \frac{1}{2}(t-\alpha)^{n-1}[(t-\alpha)^2 - 2t(t-\alpha) + 1]$

$= \frac{1}{2}(t+\alpha)^{n-1}[t^2 + (t^2-1) + 2\alpha t - 2t^2 - 2\alpha t + 1]$
$+ \frac{1}{2}(t-\alpha)^{n-1}[t^2 + (t^2-1) - 2\alpha t - 2t^2 + 2\alpha t + 1]$
$= 0$ — 1

Next, for each given n,

$F_n(t) = \frac{1}{2}[(t+\alpha)^n) + (t-\alpha)^n]$ — 1

$= \frac{1}{2}[\sum_{r=0}^{n} C^n_r t^{n-r}\alpha^r + \sum_{r=0}^{n}(-1)^r t^{n-r}\alpha^r]$

$= C^n_0 t^n + C^n_2 t^{n-2}\alpha^2 + C^n_4 t^{n-4}\alpha^4 + \ldots$

We see that each term is in the form $C^n_{2k} t^{n-2k}(t^2-1)^k$.

$\therefore$ $F_n(t)$ is a polynomial in t, of degree n and with leading coefficient $(C^n_0 + C^n_2 + C^n_4 + \ldots) = 2^{n-1}$. — $\frac{2}{6}$

SOLUTIONS 85 II	MARKS	REMARKS

6. (a) Put $u = f_n(t)$, $dv = (x-t)^{m-1}\,dt$

$du = f_{n-1}(t)\,dt$, $v = -\frac{(x-t)^m}{m}$

Integrating by parts, we have

$\int_0^x (x-t)^{m-1} f_n(t)\,dt = \left[-\frac{(x-t)^m f_n(t)}{m}\right]_0^x + \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt$ — 2

$= \frac{x^m f_n(0)}{m} + \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt$

$= \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)\,dt$ .... $\frac{2}{4}$

(b) Applying (a) repeatedly, for n = 1, 2, 3, ...,

$f_n(x) = \int_0^x f_{n-1}(t)\,dt$

$= \int_0^x (x-t) f_{n-2}(t)\,dt$ 2

$= \frac{1}{2}\int_0^x (x-t)^2 f_{n-3}(t)\,dt$

$= \ldots\ldots$

$= \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f_0(t)\,dt$ .... $\frac{2}{4}$

(c) For $0 \le t \le x \le 1$, $0 \le (x-t)^{n-1} \le 1$. 1

$0 \le |f_n(x)| = \frac{1}{(n-1)!}\left|\int_0^x (x-t)^{n-1} f_0(t)\,dt\right|$ 2

$\le \frac{M}{(n-1)!}\int_0^x (x-t)^{n-1}\,dt$ ← NB

$= \frac{M}{(n-1)!}\left[-\frac{(x-t)^n}{n}\right]_0^x$

$= \frac{Mx^n}{n!}$ 1

$\le \frac{M}{n!}$ as $0 \le x \le 1$ 1

But $\lim_{n\to\infty}\frac{M}{n!} = 0$,

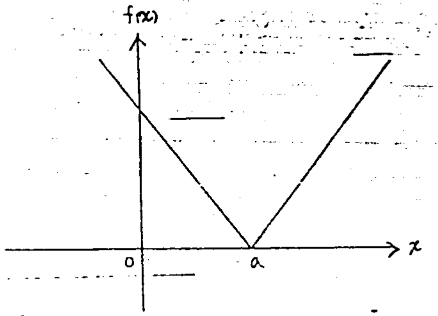
$\therefore \lim_{n\to\infty}|f_n(x)| = 0$ by Sandwich theorem

i.e. $\lim_{n\to\infty} f_n(x) = 0$ 1

SOLUTIONS 85 II | MARKS | REMARKS

(a) $f(x) = |x - a|$

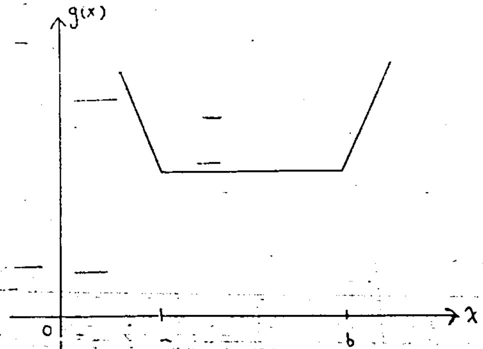
Value of x	x < a	x = a	x > a	Marks
Value of f	a − x	0	x − a	1
Value of f'	−1	f' doesn't exist ($f'_+ \neq f'_-$)	1	1
Behaviour of f	↘	minimum	↗	1



$\frac{1}{4}$

(b) $g(x) = |x - a| + |x - b|$.

Value of x	x < a	x = a	a<x<b	x = b	x>b	Marks
Value of g	(a+b)−2x	b − a	b − a	b − a	2x−(a+b)	1
Value of g'	−2	Doesn't exist	0	Doesn't exist	2	1
Behaviour of g	↘	minimum			↗	1

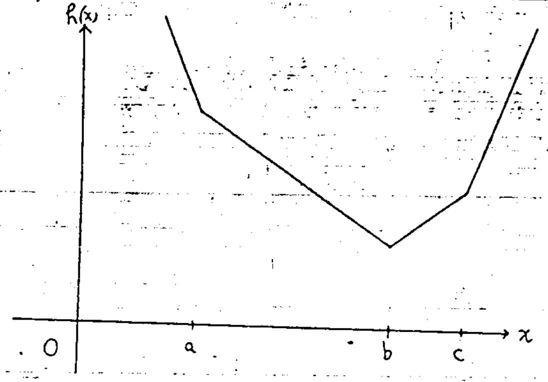


$\frac{2}{5}$

SOLUTIONS 85 II | MARKS | REMARKS

(c) $h(x) = |x - a| + |x - b| + |x - c|$

Value of x	x< a	x = a	a<x<b	x = b	b<x<c	x = c	x>c	Marks
Value of h	(a+b+c)−3x	b+c−2a	(b+c−a)−x	c − a	x−(a+b−c)	2c−(a+b)	3x−(a+b+c)	1
Value of h'	−3	Doen't exist	−1	Doesn't exist	1	Doesn't exist	3	1
Behaviour of h	↘	↘	↘	minimum	↗	↗	↗	1



$\frac{2}{5}$

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Differentiable iff $f'_+(a) = f'_-(a) = l$

$f'(a) = l$

SOLUTIONS | 85 II | MARKS | REMARKS

8 (a) (i) Put $u = \pi - x$ 1

$$\int_{\frac{\pi}{2}}^{\pi-2r} \ln \sin x \, dx = -\int_{\frac{\pi}{2}}^{2r} \ln \sin(\pi - x) \, dx$$

NB $= \int_{\frac{\pi}{2}}^{2r} \ln \sin(\pi-u)\, d(\pi-u)$

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx \quad \text{............ } 1$$

Remarks: $x = \pi - 2r \Rightarrow u = \pi - \pi + 2r = 2r$; $x = \frac{\pi}{2} \Rightarrow u = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

(ii) Put $u = 2x$

$x = \frac{u}{2}$, $dx = \frac{1}{2}du$

$$\int_r^{\frac{\pi}{2}-r} \ln \sin 2x \, dx = \frac{1}{2}\int_{2r}^{\pi - 2r} \ln \sin u \, du \quad \text{............ } 1$$

Remarks: $x = \frac{\pi}{2} - r \Rightarrow u = \pi - 2r$; $x = r \Rightarrow u = 2r$

$$= \frac{1}{2}\left[\int_{2r}^{\frac{\pi}{2}} \ln \sin u \, du + \int_{\frac{\pi}{2}}^{\pi - 2r} \ln \sin u \, du\right]$$

by (i)

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx \quad \text{............ } 1$$

$$\int_r^{\frac{\pi}{2}-r} (\ln \sin x + \ln \cos x) dx = \int_r^{\frac{\pi}{2}-r} \ln \sin x \cos x \, dx \quad \text{...... } 1$$

$$= \int_r^{\frac{\pi}{2}-r} \ln \frac{\sin 2x}{2} \, dx \quad \text{............ } 1$$

$$= \int_r^{\frac{\pi}{2}-r} \ln \sin 2x \, dx - \int_r^{\frac{\pi}{2}-r} \ln 2 \, dx \quad \text{............ } 1$$

$$= \int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx - \left(\frac{\pi}{2} - 2r\right) \ln 2 \quad \text{............ } \frac{1}{8}$$

SOLUTIONS | 85 II | MARKS | REMARKS

8. (b) Put $u = \frac{\pi}{2} - x$ 1

$$\int_r^{\frac{\pi}{2}-r} \ln \sin x \, dx = -\int_{\frac{\pi}{2}-r}^{r} \ln \sin\left(\frac{\pi}{2} - u\right) du$$

$$= \int_r^{\frac{\pi}{2}-r} \ln \cos x \, dx \quad \text{............ } 1$$

By (*),

$$\lim_{r\to 0} \int_r^{\frac{\pi}{2}-r} (\ln \sin x + \ln \cos x) dx$$

$$= \lim_{r\to 0}\left[\int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx - \left(\frac{\pi}{2} - 2r\right) \ln 2\right] \quad \text{............ } 1$$

$$\left(\lim_{r\to 0}\overset{A}{\int_r^{\frac{\pi}{2}-r} \ln \sin x \, dx}\right) + \left(\lim_{r\to 0}\overset{A}{\int_r^{\frac{\pi}{2}-r} \ln \cos x \, dx}\right)$$

$$= \left(\lim_{r\to 0}\overset{A}{\int_{2r}^{\frac{\pi}{2}} \ln \sin x \, dx}\right) - \lim_{r\to 0}\left[\left(\frac{\pi}{2} - 2r\right) \ln 2\right] \quad \text{............ } 1$$

(Note that all limits involved exist)

$$\therefore A = \lim_{r\to 0} \int_r^{\frac{\pi}{2}-r} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2 \quad \text{............ } \frac{2}{6}$$

	MARKS	REMARKS
85 II		

9\. (a) Since $[\lambda f(x) + g(x)]^2 \geq 0$ for any λ and x,

$$\int_a^b [\lambda f(x) + g(x)]^2\,dx \geq 0 \quad \text{.......... } 1$$

$$\int_a^b [\lambda^2 [f(x)]^2 + 2\lambda f(x)\,g(x) + [g(x)]^2]\,dx \geq 0$$

$$\therefore\ \lambda^2 \int_a^b [f(x)]^2 dx + 2\lambda \int_a^b f(x)g(x)dx + \int_a^b [g(x)]^2 dx \geq 0 \text{ for any } \lambda. \quad 1$$

Since the inequality holds for any λ, the discriminant ≤ 0. 1

D ≤ 0

$$\therefore\ [2\int_a^b f(x)g(x)dx]^2 - 4\left(\int_a^b [f(x)]^2 dx\right)\left(\int_a^b [g(x)]^2 dx\right) \leq 0$$

i.e. $$[\int_a^b f(x)g(x)dx]^2 \leq \left(\int_a^b [f(x)]^2 dx\right)\left(\int_a^b [g(x)]^2 dx\right) \quad \frac{1}{4}$$

(b) (i) As $f(0) = f(1) = 0$,

$$\int_0^x f'(t)\,dt = f(x) - f(0)$$
$$= f(x) \quad \text{.......... } 1$$

and $$-\int_x^1 f'(t)\,dt = -f(1) + f(x)$$
$$= f(x) \quad \text{.......... } 1$$

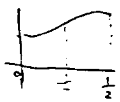
(ii) If $x \in [0, \frac{1}{2}]$, from (i),

$$[f(x)]^2 = [\int_0^x f'(t)\,dt]^2$$

$$\leq \left(\int_0^x dt\right)\left(\int_0^x [f'(t)]^2\,dt\right) \quad \text{(by (a)) .... } 1$$

$$= x\int_0^x [f'(t)]^2\,dt$$

$$\leq x\int_0^{\frac{1}{2}} [f'(t)]^2\,dt \quad (\because [f'(t)]^2 \geq 0) \text{ .... } 1$$



If $x \in [\frac{1}{2}, 1]$, $[f(x)]^2 = [-\int_x^1 f'(t)\,dt]^2$

$$\leq \left(\int_x^1 dt\right)\left(\int_x^1 [f'(t)]^2\,dt\right) \quad \text{.......... } 1$$

$$= (1-x)\int_x^1 [f'(t)]^2\,dt$$

$$\leq (1-x)\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt \quad \text{.......... } 1$$

	MARKS	REMARKS
85 II		

(b) (iii)

By (ii), $$\int_0^{\frac{1}{2}} [f(x)]^2\,dx \leq \left(\int_0^{\frac{1}{2}} [f'(t)]^2\,dt\right)\left(\int_0^{\frac{1}{2}} x\,dx\right) \quad 1$$

$$= \frac{1}{8}\int_0^{\frac{1}{2}} [f'(t)]^2\,dt \quad \text{.......... } 1$$

and $$\int_{\frac{1}{2}}^1 [f(x)]^2\,dx \leq \left(\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt\right)\left(\int_{\frac{1}{2}}^1 (1-x)\,dx\right)$$

$$= \frac{1}{8}\int_{\frac{1}{2}}^1 [f'(t)]^2\,dt \quad \text{.......... } 1$$

Adding these two results

$$\int_0^{\frac{1}{2}} [f(x)]^2 dx + \int_{\frac{1}{2}}^1 [f(x)]^2 dx \leq \frac{1}{8}\left[\int_0^{\frac{1}{2}} [f'(t)]^2 dt + \int_{\frac{1}{2}}^1 [f'(t)]^2 dt\right]$$

$$\int_0^1 [f(x)]^2\,dx \leq \frac{1}{8}\int_0^1 [f'(t)]^2\,dt \quad \text{.......... } \frac{1}{10}$$