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香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八四年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1984

PURE MATHEMATICS (PAPER I)

MARKING SCHEME

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SOLUTION	MARK	REMARK
(a) $f(x) = \begin{vmatrix} a-x & 0 & 1 \\ 0 & b-x & 0 \\ 1 & 0 & c-x \end{vmatrix}$		
$= -x^3 + (a+b+c)x^2 - (ab+bc+ca-1)x + (abc-b)$	2	
$= -x^3 - (ab+bc+ca-1)x + (abc-b)$ as $a+b+c = 0$	1	
$A^3 = \begin{pmatrix} a^2+1 & 0 & a+c \\ 0 & b^2 & 0 \\ a+c & 0 & c^2+1 \end{pmatrix}\begin{pmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{pmatrix}$ $= \begin{pmatrix} a^3+2a+c & 0 & a^2+ac+c^2+1 \\ 0 & b^3 & 0 \\ a^2+ac+c^2+1 & 0 & a+2c+c^3 \end{pmatrix}$	2	
$\therefore f(A) = -A^3 - (ab+bc+ca-1)A + (abc-b)I$ $= \begin{pmatrix} -a^3-a-b-c-a^2b-a^2c & 0 & -a^2-2ac-c^2-ab-bc \\ 0 & -b^3-ab^2-b^2c & 0 \\ -a^2-2ac-c^2-ab-bc & 0 & -a-c-c^3-bc^2-ac^2-b \end{pmatrix}$	2	
$= \begin{pmatrix} -a^2(a+b+c) - (a+b+c) & 0 & -a(a+c+b) - c(a+c+b) \\ 0 & -b^2(b+a+c) & 0 \\ -a(a+c+b) - c(a+c+b) & 0 & -(a+c+b) - c^2(c+b+a) \end{pmatrix}$ $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	3	
	10	
(b) Since $-A^3 - (ab+bc+ca-1)A + (abc-b)I = 0$, $A^3 = \lambda A + \mu I$, where $\lambda = -(ab+bc+ca-1)$, $\mu = abc-b$	1	
For $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\lambda = 2$, $\mu = 1$.	1	
$A^9 = (\lambda A + \mu I)^3$ $= \lambda^3 A^3 + 3\lambda^2\mu A^2 + 3\lambda\mu^2 A + \mu^3 I$ $= \lambda^3(\lambda A + \mu I) + 3\lambda^2\mu A^2 + 3\lambda\mu^2 A + \mu^3 I$ $= 3\lambda^2\mu A^2 + (\lambda^4 + 3\lambda\mu^2)A + (\lambda^3\mu + \mu^3)I$	2	may sub. A either of these steps
$\therefore A^9 = 12A^2 + 22A + 9I$ $= \begin{pmatrix} 55 & 0 & 34 \\ 0 & -1 & 0 \\ 34 & 0 & 21 \end{pmatrix}$	3	-1 for each wrong entry
	7	

SOLUTION	MARK	REMARK
2. (a) For any $\alpha \in \mathbb{R}$, $\underline{u}, \underline{v} \in \mathbb{R}^3$, $g(\alpha\underline{u}) = \underline{a}\cdot(\alpha\underline{u}) = \alpha(\underline{a}\cdot\underline{u}) = \alpha g(\underline{u})$; $g(\underline{u}+\underline{v}) = \underline{a}\cdot(\underline{u}+\underline{v}) = \underline{a}\cdot\underline{u} + \underline{a}\cdot\underline{v} = g(\underline{u}) + g(\underline{v})$; $\therefore$ g is linear.	2+1	
	3	
(b) Since h is linear, for $\underline{u} = \alpha\underline{x} + \beta\underline{y} + \gamma\underline{z}$, $h(\underline{u}) = h(\alpha\underline{x} + \beta\underline{y} + \gamma\underline{z}) = h(\alpha\underline{x}) + h(\beta\underline{y}) + h(\gamma\underline{z}) = \alpha h(\underline{x}) + \beta h(\underline{y}) + \gamma h(\underline{z})$	2	
~~Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ be the usual base of $\mathbb{R}^3$. For any $\underline{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, $\underline{u} = u_1\underline{e}_1 + u_2\underline{e}_2 + u_3\underline{e}_3$.~~	2	$\underline{u}$ as a lin. comb. of basis vectors.
Since h is linear, $h(\underline{u}) = u_1 h(\underline{e}_1) + u_2 h(\underline{e}_2) + u_3 h(\underline{e}_3)$. Define $\underline{b} = (h(\underline{e}_1), h(\underline{e}_2), h(\underline{e}_3))$, then $h(\underline{u}) = \underline{b}\cdot\underline{u}$ for any $\underline{u} \in \mathbb{R}^3$.	3	
	7	
(c) Let $f: \mathbb{R}^3 \to \mathbb{R}$ be a linear function. By (b), there exists $\underline{b} = (f(\underline{e}_1), f(\underline{e}_2), f(\underline{e}_3)) \in \mathbb{R}^3$ s.t. $f(\underline{u}) = \underline{b}\cdot\underline{u}\ \forall\ \underline{u} \in \mathbb{R}^3$. Consider the set $H = \{\underline{u} \in \mathbb{R}^3 : f(\underline{u}) = 0\}$ $= \{\underline{u} \in \mathbb{R}^3 : f(\underline{e}_1)u_1 + f(\underline{e}_2)u_2 + f(\underline{e}_3)u_3 = 0\}$	3	
Since f is not identically zero, $\underline{b} \neq \underline{0}$. H is therefore a plane through the origin.	1	
Conversely if H is a plane through the origin, equation of H can be written as $Au_1 + Bu_2 + Cu_3 = 0$, where A, B, C are not all zero.	1	
Define $\underline{a} = (A, B, C)$, then by (a), the non-zero function $f: \mathbb{R}^3 \to \mathbb{R}$ defined by $f(\underline{u}) = \underline{a}\cdot\underline{u}\ \forall\ \underline{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ is linear. Further $H = \{\underline{u} \in \mathbb{R}^3 : f(\underline{u}) = 0\}$	2	
	7	

SOLUTION	MARK	REMARK
(a) When n is even,		
$A_n = (n-3)+(n-5)+\dots+1$		
$= \frac{1}{2}\left(\frac{n-2}{2}\right)(n-3+1)$	2	
$= \left(\frac{n-2}{2}\right)^2$		
	1	
when n is odd,		
$A_n = (n-3)+(n-1)+\dots+2$		
$= \frac{1}{2}\left(\frac{n-5}{2}+1\right)(n-3+2)$	1	
$= \left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)$	1	
	5	
(b) (i) To form a non-degenerate triangle with the longest side n, we have $y < x < t = n$, $x+y > t$. The number of such triangles is equal to the number of integral points in (a), i.e. A_n.	3	
(ii) The number of possible triangles formed is		
$B_{2k} = \sum_{i=4}^{2k} A_i$	2	
$= \sum_{i=2}^{k} A_{2i} + \sum_{i=2}^{k-1} A_{2i+1}$	1	
$= \sum_{i=2}^{k} (i-1)^2 + \sum_{i=2}^{k-1} i(i-1)$		
$= \sum_{i=1}^{k-1} i^2 + \sum_{i=2}^{k-1} i^2 - \sum_{i=2}^{k-1} i$		
$= 2 \times \frac{1}{6}(k-1)(k)(2k-1) - 1 - \frac{(k-2)(k+1)}{2}$		
$= \frac{k(k-1)(4k-5)}{6}$	2	
	3	
(c) $p(2k) = \frac{B_{2k}}{C_3^{2k}}$	2	
$= \frac{\frac{1}{6}k(k-1)(4k-5)}{\frac{(2k)(2k-1)(2k-2)}{3 \cdot 2}}$		
$= \frac{4k-5}{4(2k-1)}$	1	
$\therefore \lim_{k\to\infty} p(2k) = \lim_{k\to\infty} \frac{4k-5}{4(2k-1)}$		
$= \frac{1}{2}$	1	
	4	

SOLUTION	MARK	REMARK
(a) $z^m = 1$ iff $\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^m = 1$		
iff $\cos\frac{2\pi m}{n} + i\sin\frac{2\pi m}{n} = 1$		
iff $n \mid m$	2	
(i) If $n \mid m$, $z^m = 1$,		
$\sum_{r=0}^{n-1} z^{mr} = \sum_{r=0}^{n-1} 1 = n$	2	
(ii) If $n \nmid m$, $\sum_{r=0}^{n-1} z^{mr} = \frac{1-z^{mn}}{1-z^m}$ [*]	2	
$= 0$ as $z^{mn} = 1$	1	
	7	
(b) $\sum_{r=0}^{n-1} f(z^r) z^{(n-j)r} = \sum_{r=0}^{n-1}\left[\sum_{k=0}^{n-1} a_k (z^r)^k\right] z^{(n-j)r}$	1	
$= \sum_{r=0}^{n-1}\sum_{k=0}^{n-1} a_k z^{(n+k-j)r}$	1	
$= \sum_{k=0}^{n-1}\left[a_k \sum_{r=0}^{n-1} z^{(n+k-j)r}\right]$	1	
[*] As $0 \le k, j \le n-1$, $0 < n+k-j < 2n$.		
$n \mid (n+k-j)$ iff $k = j$	1	
By (a), $\sum_{r=0}^{n-1} z^{(n+k-j)r} = \begin{cases} n & \text{if } k = j \\ 0 & \text{if } k \ne j \end{cases}$	1	
$\therefore \sum_{r=0}^{n-1} f(z^r) z^{(n-j)r} = n a_j$	1	
	6	

SOLUTION	MARK	REMARK

4. (c) Let $f(x)=\sum_{j=0}^{n-1} a_j x^j$.

By (b), $a_j = \frac{1}{n}\sum_{r=0}^{n-1} f(z^r) z^{(n-j)r}$

∴ $f(x) = \frac{1}{n}\sum_{j=0}^{n-1}\left[\sum_{r=0}^{n-1} f(z^r) z^{(n-j)r}\right] x^j$ — **1**

$= \frac{1}{n}\sum_{j=0}^{n-1}\left[\sum_{r=0}^{n-1} (g(z^r) - (z^{nr}-1)h(z^r)) z^{(n-j)r}\right] x^j$ — **2**

$= \frac{1}{n}\sum_{j=0}^{n-1}\left[\sum_{r=0}^{n-1} g(z^r) z^{(n-j)r}\right] x^j$ since $z^{nr}-1=0$. — **1**

Total: **4**

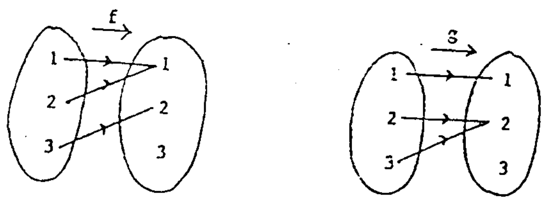
5. (a) (i) For any $x, y, z \in A$,

1. $f(x) = f(x)$ and $g(x) = g(x)$
   ∴ xRx . — **1**
2. If xRy , then $f(x) = f(y)$ and $g(x) = g(y)$
   ∴ $f(y) = f(x)$, $g(y) = g(x)$ and yRx . — **1**
3. If xRy and yRz , then $f(x) = f(y)$, $g(x) = g(y)$
   and $f(y) = f(z)$, $g(y) = g(z)$
   ∴ $f(x) = f(z)$ and $g(x) = g(z)$
   and xRz . — **1**

Thus R is an equivalence relation.

(ii) Let $A = B = \{1, 2, 3\}$. The following example of f and g does not define an equivalence relation S :

Here $1S2$ and $2S3$ but $1 \not S 3$.



— **4**

Total: **7**

SOLUTION	MARK	REMARK

5. (b) For any $a/R \in A/R$,
define $h(a/R) = f(a)$. — **2**

$a/R = a'/R \Rightarrow aRa' \Rightarrow f(a) = f(a')$
∴ h is a well-defined mapping. — **1**

Furthermore, for any $a \in A$, $h \circ u(a) = h(a/R)$
$= f(a)$

∴ $h \circ u = f$. — **1**

Let $h' : A/R \to B$ be any mapping such that $h' \circ u = f$.
Since u is surjective, any element, e.g. a/R , of A/R is the image of some element, e.g. a , in A .

∴ $h(a/R) = h \circ u(a)$
$= h' \circ u(a)$
$= h'(a/R)$ for any $a/R \in A/R$.

Hence $h = h'$, i.e. h is unique. — **3**

[If g is a constant mapping and if f is surjective, obviously h must be surjective.] — **1**
Furthermore, let a/R , $a'/R \in A/R$ such that $h(a/R) = h(a'/R)$.

Then $h \circ u(a) = h \circ u(a')$
$\Rightarrow f(a) = f(a')$.

As $g(a) = g(a')$, $aRa' \Rightarrow a/R = a'/R$.
∴ h is bijective — **2**

Total: **10**

(a) (i) $\bar{\alpha}\beta = 1 \Rightarrow \bar{\alpha}\beta\alpha\bar{\beta} = 1$

$\Rightarrow |\alpha| = \frac{1}{|\beta|}$

As $|\alpha|, |\beta| \leq 1$, the above is possible only if $|\alpha| = |\beta| = 1$.

∴ $\alpha\bar{\alpha} = 1$
and $\bar{\alpha}\beta = 1 \Rightarrow \frac{1}{\alpha}\beta = 1$

$\Rightarrow \alpha = \beta$ — **3**

(ii) By (i), $1 - \bar{\alpha}\beta \neq 0$.

$\frac{|\alpha-\beta|}{|1-\bar{\alpha}\beta|} \leq 1 \Leftrightarrow |\alpha-\beta| \leq |1-\bar{\alpha}\beta|$
$\Leftrightarrow (\alpha-\beta)(\bar{\alpha}-\bar{\beta}) \leq (1-\bar{\alpha}\beta)(1-\alpha\bar{\beta})$
$\Leftrightarrow \alpha\bar{\alpha} + \beta\bar{\beta} - \alpha\bar{\beta} - \bar{\alpha}\beta \leq 1 + \alpha\bar{\alpha}\beta\bar{\beta} - \alpha\bar{\beta} - \bar{\alpha}\beta$
$\Leftrightarrow (1-\alpha\bar{\alpha})(1-\beta\bar{\beta}) \geq 0$,

which is true as $|\alpha|, |\beta| \leq 1$. — **3**

If $|\alpha| < 1$, the equality holds iff $1 - \beta\bar{\beta} = 0$,

i.e. $|\beta| = 1$. — **1**

Total: **7**

SOLUTION | MARK | REMARK

6. (b) (i) $|f(1)| = |f(-1)| = 1$

$$\Rightarrow \left|\frac{1-a}{b-1}\right| = 1 \text{ and } \left|\frac{-1-a}{-b-1}\right| = 1$$

$$\Rightarrow |a-1| = |b-1| \text{ and } |a+1| = |b+1|$$

$\therefore$ a , b are equidistant from 1 and from −1 .

Hence $b = a$ or $\bar{a}$ 3

Alternatively:

$|f(1)| = |f(-1)| = 1$

$\Rightarrow$ $a\bar{a} - a - \bar{a} + 1 = b\bar{b} - b - \bar{b} + 1$

and $a\bar{a} + a + \bar{a} + 1 = b\bar{b} + b + \bar{b} + 1$

$\therefore$ $a + \bar{a} = b + \bar{b}$

and $a\bar{a} = b\bar{b}$

i.e. $Re(a) = Re(b)$

and $|a| = |b|$

Hence $b = a$ or $\bar{a}$

$|f(i)| = 1 \Rightarrow \left|\frac{i-a}{ib-1}\right| = i$

$\Rightarrow |i - a| = |i + b|$

Together with the above result, this implies $b = \bar{a}$ 2

Writing $f(z) = \frac{z-a}{\bar{a}z - 1}$

If $|z| = 1$,

$$|f(z)|^2 = \frac{(z-a)(\bar{z}-\bar{a})}{(\bar{a}z-1)(a\bar{z}-1)}$$

$$= \frac{z\bar{z} + a\bar{a} - a\bar{z} - \bar{a}z}{a\bar{a}z\bar{z} - a\bar{z} - \bar{a}z + 1}$$

$$= \frac{1 + a\bar{a} - a\bar{z} - \bar{a}z}{a\bar{a} - a\bar{z} - \bar{a}z + 1}$$

$$= 1 .$$ 2

(ii) If $|a| = 1$,

$$f(z) = \frac{z-a}{\bar{a}z - 1}$$

$$= \frac{1}{\bar{a}}\left(\frac{z-a}{z-a}\right)$$

$= \frac{1}{\bar{a}}(-a)$ which is constant. 3

10

SOLUTION | MARK | REMARK

7. (a) $(x_1\ x_2\ x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= (a_{11}x_1 + a_{21}x_2 + a_{31}x_3\quad a_{12}x_1 + a_{22}x_2 + a_{32}x_3\quad a_{13}x_1 + a_{23}x_2 + a_{33}x_3)$

$\therefore x'_1 + x'_2 + x'_3 = (a_{11}+a_{12}+a_{13})x_1 + (a_{21}+a_{22}+a_{23})x_2 + (a_{31}+a_{32}+a_{33})x_3$

$= x_1 + x_2 + x_3 \quad (\because A \in \mathcal{D} \Rightarrow \sum_{j=1}^{3} a_{ij} = 1)$ 2

Similarly $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ a_{31}y_1 + a_{32}y_2 + a_{33}y_3 \end{pmatrix}$

$y'_1 + y'_2 + y'_3 = (a_{11}+a_{21}+a_{31})y_1 + (a_{12}+a_{22}+a_{32})y_2 + (a_{13}+a_{23}+a_{33})y_3$

$= y_1 + y_2 + y_3 \quad (\because \sum_{i=1}^{3} a_{ij} = 1)$ 1

3

(b) Let $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \in \mathcal{D}$

(i) The ij-entry of $JB = \frac{1}{3}\sum_{i=1}^{3} b_{ij}$

$= \frac{1}{3}$ 1

The ij-entry of $BJ = \frac{1}{3}\sum_{j=1}^{3} b_{ij}$

$= \frac{1}{3}$ 1

$\therefore JB = J = BJ \quad \forall B \in \mathcal{D}$.

(ii) By (i) $BJ = J \quad \forall B \in \mathcal{D}$

$\therefore J \in S(B) \Rightarrow S(B) \neq \emptyset$ 1

(iii) By (i) $JB = J \quad \forall B \in \mathcal{D}$

$\therefore \mathcal{D} \subset S(J)$

$\subset \mathcal{D}$ (By definition) 1

$\Rightarrow S(J) = \mathcal{D}$ 1

5

SOLUTION	MARK	REMARK
(c) If A is invertible, $A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.		
Let $(x_1\ x_2\ x_3)$ be the first row of A^{-1}. By (a), $x_1 + x_2 + x_3 = 1 + 0 + 0 = 1$.		
Similarly for other rows and columns. $\therefore A^{-1} \in \mathscr{D}$	2	
$X \in S(A) \Rightarrow AX = J \Rightarrow X = A^{-1}J = J$ By (b)(i)	1	
Since $J \in S(A)$ by (b)(ii), $\therefore S(A) = \{J\}$	1	
	4	
(d) If A is singular, the system of equations $A\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a non-zero solution $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$	2	
with $b_1 + b_2 + b_3 = 0$ by (a)	1	
Let $C = \begin{pmatrix} b_1 & b_1 & -2b_1 \\ b_2 & b_2 & -2b_2 \\ b_3 & b_3 & -2b_3 \end{pmatrix}$, then $C \neq 0$ but C has zero row and column sums and $AC = 0$.	1	
Further $J + C \in \mathscr{D}$ and $A(J + C) = AJ + AC = J \Rightarrow J + C \in S(A)$ $\therefore S(A) \neq \{J\}$	1	
	5	

SOLUTION	MARK	REMARK
(a) Since a non-zero polynomial has only a finite number of zeroes, $g(n) = a_n f(n) = 0$ and hence $a_n = 0$ for only a finite number of n.	2	
Assume, for contradition, that $\deg f(x) > \deg g(x)$.		
Then if n is sufficiently large, $g(n)$, a_n, $f(n) \neq 0$ and $\|f(n)\| > \|g(n)\|$.		
This contradicts the fact the $g(n) = a_n f(n)$. Hence $\deg f(x) \leqslant \deg g(x)$.	3	
	5	
(b) As $\deg f(x) \leqslant \deg g(x)$, by the Euclidean algorithm, there exist polynomials $h(x)$ and $r(x)$ with $\deg r(x) < \deg f(x)$ such that $g(x) = h(x)f(x) + r(x)$.	3	
It is obvious that the coefficients of $h(x)$ and $r(x)$ are rational.		
Since for any $n \in \mathbb{N}$, $\exists\ a_n \in \mathbb{Z}$ such that $g(n) = a_n f(n)$,	1	
$r(n) = a_n f(n) - h(n)f(n) = [\,a_n - h(n)\,]f(n)$ ....(*)	2	
Let M be a common multiple of the denominators of coefficients of $r(x)$ and $h(x)$, then the polynomials $Mr(x)$ and $Mh(x)$ have integral coefficients.	1	
From (*), for any $n \in \mathbb{N}$, $\exists\ b_n = [\,Ma_n - Mh(n)\,] \in \mathbb{Z}$ such that $Mr(n) = [\,Ma_n - Mh(n)\,]f(n)$		
By (a), $\deg f(x) \leqslant \deg r(x)$ which contradicts the definition of $r(x)$ unless $r(x) \equiv 0$, i.e. $g(x) = f(x)h(x)$	2	
	9	
(c) If $\deg f(x) = \deg g(x)$, since $g(x) = f(x)h(x)$, $h(x)$ is constant.	1	
By (a), there is an n such that $g(n) = a_n f(n)$ and a_n, $g(n)$, $f(n)$ are non-zero integers.		
Then $h(n) = \dfrac{g(n)}{f(n)} = a_n$	2	
	3	

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PURE MATHEMATICS (PAPER II)
MARKING SCHEME

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SOLUTION	MARK	REMARK
(a) $u_{k+2} = \int_0^{\pi} \frac{\sin(k+2)x}{\sin x}\,dx$		
$= \int_0^{\pi} \frac{\sin kx \cos 2x + \cos kx \sin 2x}{\sin x}\,dx$		
$= \int_0^{\pi} \frac{\sin kx\,(1-2\sin^2 x) + 2\cos kx \sin x \cos x}{\sin x}\,dx$		
$= \int_0^{\pi} \frac{\sin kx}{\sin x}\,dx + \int_0^{\pi} (-2\sin kx \sin x + 2\cos kx \cos x)\,dx$		
$= u_k + 2\int_0^{\pi} \cos(k+1)x\,dx$		
$= u_k + \frac{2}{k+1}\left[\sin(k+1)x\right]_0^{\pi}$		
$= u_k$	3	
$\therefore u_k = u_{k-2}$		
$= \text{etc}$		
$= \begin{cases} u_0 & \text{if } k \text{ is even} \\ u_1 & \text{if } k \text{ is odd.} \end{cases}$	1 1	
$= \begin{cases} 0 & \text{if } k \text{ is even} \\ \pi & \text{if } k \text{ is odd.} \end{cases}$	1	
	6	

SOLUTION	MARK	REMARK

(b) (i) Put $u = \cos^{m-1}\theta$, $dv = \cos\theta \sin^n\theta \, d\theta$,

then $du = -(m-1)\cos^{m-2}\theta \cdot \sin\theta \, d\theta$, $v = \frac{1}{n+1}\sin^{n+1}\theta$.

$$I(m, n) = \left[\frac{1}{n+1}\cos^{m-1}\theta \sin^{n+1}\theta\right]_0^{\frac{\pi}{2}} + \frac{m-1}{n+1}\int_0^{\frac{\pi}{2}} \cos^{m-2}\theta \sin^{n+2}\theta \, d\theta$$

$$= \left(\frac{m-1}{n+1}\right) I(m-2, n+2) , \quad m \geq 2$$

3

(ii) For $n \geq 0$, $I(1, n) = \int_0^{\frac{\pi}{2}} \cos\theta \sin^n\theta \, d\theta$

$$= \frac{1}{n+1}\sin^{n+1}\theta \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{n+1}$$

2

(iii) Put $u = \sin^{n-1}\theta$, $dv = \sin\theta \, d\theta$,

then $du = (n-1)\sin^{n-2}\theta \cos\theta \, d\theta$, $v = -\cos\theta$.

For $n \geq 2$, $I(0, n) = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$

$$= \left[-\sin^{n-1}\theta \cos\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \cos^2\theta \, d\theta$$

$$= (n-1)\left[\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta\right]$$

$\therefore I(0, n) = \left(\frac{n-1}{n}\right) I(0, n-2)$.

3

(iv) $I(6, 4) = \frac{5}{5} I(4, 6)$ by (i)

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} I(0, 10)$$

1

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} \cdot \frac{9}{10} I(0, 8) \quad \text{by (iii)}$$

1

$$= \frac{5 \cdot 3 \cdot 1}{5 \cdot 7 \cdot 9} \cdot \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} I(0, 0)$$

$$= \frac{3\pi}{512}$$

1

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II

SOLUTION	MARK	REMARK

(a) Integrating by parts,

Let $u = \frac{(h-t)^{n-1}}{(n-1)!}$, $dv = f^{(n)}(t)dt$

$du = \frac{-(n-1)(h-t)^{n-2}}{(n-1)!}dt$, $v = f^{(n-1)}(t)$

$$R_n = \frac{1}{(n-1)!}\left\{\left[(h-t)^{n-1} f^{(n-1)}(t)\right]_0^h + (n-1)\int_0^h (h-t)^{n-2} f^{(n-1)}(t)\,dt\right\}$$

$$= R_{n-1} - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) \quad (2 \leq n \leq n) .$$

4

(b) $R_1 = \int_0^h f'(t)\,dt$

$= f(h) - f(0)$

$R_2 = R_1 - hf'(0)$

$= f(h) - f(0) - hf'(0)$

$$R_n = R_{n-1} - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)$$

$$= R_{n-2} - \frac{h^{n-2}}{(n-2)!} f^{(n-2)}(0) - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)$$

= etc.

$$= R_1 - hf'(0) - \frac{h^2}{2!} f''(0) - \ldots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)$$

Subst. $R_1 = f(h) - f(0)$, we have

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

3

6

SOLUTION	MARK	REMARK
2. (c) Putting $f(x) = \ln(1+x)$ which is infinitely differentiable on $(-1, 1)$,		
$f'(x) = \frac{1}{1+x}$	1	
$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.	1	
By (b), for any $0<h<1$, $\ln(1+h) = \ln 1 + h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + R_5$	1	
Since $R_5 = \frac{1}{4!}\int_0^h (h-t)^4 \cdot \frac{(-1)^4(4!)}{(1+t)^5}\,dt$		
> 0 as $(h-t)^4$, $(1+t)^5 > 0$ for $t \in (0, h)$.		
$\therefore \ln(1+h) - h + \frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{4} = R_5 > 0$	1	
Similarly, $R_6 = \frac{1}{5!}\int_0^h \frac{(h-t)^5(-1)^5(5!)}{(1+t)^6}\,dt$	1	
< 0	1	
$\therefore \ln(1+h) - h + \frac{h^2}{2} - \frac{h^3}{3} + \frac{h^4}{4} = R_6 + \frac{h^5}{5}$ $\leq \frac{h^5}{5}$	1	
	7	

SOLUTION	MARK	REMARK
(a) Since $a\theta = b\phi$,		
if $P = (x, y)$,		
$x = (a-b)\cos\theta + b\sin[\pi - \phi - (\frac{\pi}{2} - \theta)]$	2	
$= (a-b)\cos\theta + b\sin[\frac{\pi}{2} - (\phi - \theta)]$		
$= (a-b)\cos\theta + b\cos(\phi - \theta)$	1	
$= (a-b)\cos\theta + b\cos(\frac{a-b}{b})\theta$	2	
$y = (a-b)\sin\theta - b\cos(\frac{\pi}{2} - (\phi - \theta))$		
$= (a-b)\sin\theta - b\sin(\frac{a-b}{b})\theta$.	2	
	7	
(b) If $b = \frac{a}{4}$,		
$x = \frac{3}{4}a\cos\theta + \frac{a}{4}\cos 3\theta$		
$= \frac{3}{4}a\cos\theta + \frac{a}{4}[4\cos^3\theta - 3\cos\theta]$		
$= a\cos^3\theta$	2	
$y = \frac{3}{4}a\sin\theta - \frac{a}{4}\sin 3\theta$		
$= \frac{3}{4}a\sin\theta - \frac{a}{4}[3\sin\theta - 4\sin^3\theta]$		
$= a\sin^3\theta$	2	
$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}\cos^2\theta + a^{\frac{2}{3}}\sin^2\theta$ $= a^{\frac{2}{3}}$	1	
	5	
(c) Length of hypocycloid		
$= \int_0^{2\pi}\sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\,d\theta$		
$= \int_0^{2\pi}\sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\,d\theta$	2	
$= 4\int_0^{\frac{\pi}{2}} 3a\sin\theta\cos\theta\,d\theta$		
$= 6a[\sin^2\theta]_0^{\frac{\pi}{2}}$		
$= 6a$	3	
	5	

SOLUTION	MARK	REMARK

(a) Equation of tangent at (x', y') is $\frac{x'x}{a^2} + \frac{y'y}{b^2} = 1$.

Comparing coefficients with $\ell x + my = 1$, $\ell = \frac{x'}{a^2}$, $m = \frac{y'}{b^2}$

∴ $\ell x + my = 1$ is tangent to (E) iff

$$a^2\ell^2 + b^2m^2 = a^2\frac{x'^2}{a^4} + b^2\frac{y'^2}{b^4} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

since (x', y') lies on (E). **4**

Alternatively

Let $\ell \neq 0$, substituting $x = \frac{1 - my}{\ell}$ in (E),

$$\frac{(1-my)^2}{a^2\ell^2} + \frac{y^2}{b^2} = 1$$

or $(a^2\ell^2 + b^2m^2)y^2 - 2mb^2y + (b^2 - a^2b^2\ell^2) = 0$

$\ell x + my = 1$ is tangent iff $4m^2b^4 - 4(a^2\ell^2 + b^2m^2)(b^2 - a^2b^2\ell^2) = 0$

iff $a^2\ell^2(a^2\ell^2 + b^2m^2 - 1) = 0$

iff $a^2\ell^2 + b^2m^2 = 1$

It is noted that this still holds if $\ell = 0$.

(b) Let $\ell x + my = 1$ be a common tangent to (E) and (F).

By (a) $a^2\ell^2 + b^2m^2 = 1$ and

$b^2\ell^2 + a^2m^2 = 1$

∴ $\ell^2 = m^2 = \frac{1}{a^2 + b^2}$ **1**

Equations of common tangents to (E) and (F) are

$x \pm y = \pm\sqrt{a^2 + b^2}$ **2**

3

SOLUTION	MARK	REMARK

4. (c) Equation of tangent at $S(s_1, s_2)$ is

$$\frac{s_1x}{a^2} + \frac{s_2y}{b^2} = 1.$$

If R(h, k) lies on the tangent,

$$\frac{s_1h}{a^2} + \frac{s_2k}{b^2} = 1$$ **2**

Similarly, for the tangent at $T(t_1, t_2)$, we have

$$\frac{t_1h}{a^2} + \frac{t_2k}{b^2} = 1$$

∴ the straight line $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ passes through S and T. **3**

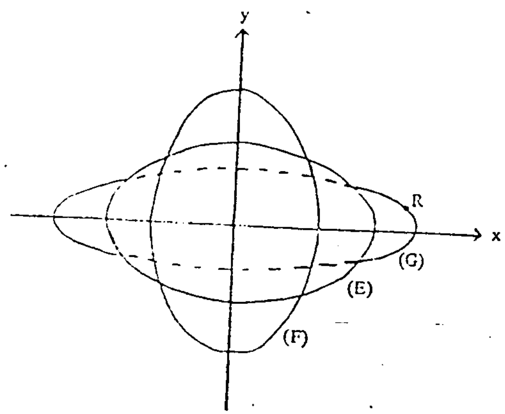
5

(d) By (a), the chord ST is tangent to (F) iff

$\frac{h^2b^2}{a^4} + \frac{k^2a^2}{b^4} = 1$, where (h, k) lies outside (E) **2**

∴ the locus of R consists of the parts of the ellipse

(G) : $\frac{x^2}{(\frac{a^2}{b})^2} + \frac{y^2}{(\frac{b^2}{a})^2} = 1$ which lie outside (E). **1**



(G), locus of R **2**

5

SOLUTION | MARK | REMARK

5\. (a) If $a \geq 0$, let N be an integer such that $N > a$.

$$\frac{a^{N+1}}{(N+1)!} = \frac{a}{N+1} \cdot \frac{a^N}{N!} < \frac{N}{N+1} \cdot \frac{a^N}{N!}$$

$$\frac{a^{N+2}}{(N+2)!} = \frac{a}{(N+2)} \cdot \frac{a}{(N+1)} \cdot \frac{a^N}{N!} < \left(\frac{N}{N+1}\right)^2 \frac{a^N}{N!} \quad \text{, etc.}$$

$$0 < \frac{a^n}{n!} < \left(\frac{N}{N+1}\right)^{n-N} \cdot \frac{a^N}{N!} \quad \text{for every } n > N.$$

$\therefore \lim_{n\to\infty} \frac{a^n}{n!} = 0$ as $0 < \frac{N}{N+1} < 1$. — 6

If $a < 0$, the same result follows from the inequalities

$$-\frac{|a|^n}{n!} \leq \frac{a^n}{n!} \leq \frac{|a|^n}{n!}$$ — 1

[7]

(b) (i) $f(1-x) = (1-x)^n(1-(1-x))^n$
$= (1-x)^n x^n$
$= f(x) \quad \forall x \in \mathbb{R}$

$f'(x) = -f'(1-x)$
$f''(x) = f''(1-x)$
etc — 1
$f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$ — 1

(ii) $f(x) = x^n \sum_{i=0}^{n} C_i^n (-1)^i x^i$

$= C_0^n x^n - C_1^n x^{n+1} + \ldots + C_n^n (-1)^n x^{2n}$.

For $k > 2n$, $f^{(k)}(x) \equiv 0$

For $0 \leq k < n$, $f^{(k)}(0) = 0$ — 1

For $n \leq k \leq 2n$, $f^{(k)}(x) = \sum_{i=0}^{n} C_i^n (-1)^i \frac{d^k}{dx^k}(x^{n+i})$ — 1

At $x = 0$, $f^{(k)}(0) = C_{k-n}^n (-1)^{k-n} \frac{d^k}{dx^k}(x^k)$

$= (-1)^{k-n} C_{k-n}^n \cdot k!$

$= (-1)^{k-n} C_{2n-k}^n \, k!$

Now $\frac{k!n!}{(k-n)!(2n-k)!} = k! C_{k-n}^n \in \mathbb{Z}$

and $C_{2n-k}^k = \frac{k!}{(2n-k)!(2k-2n)!}$

$= \frac{k!}{(2n-k)![2(k-n)]!} \in \mathbb{Z}$

It follows that $\frac{k!}{(k-n)!(2n-k)!}$ is an integer and hence $f^{(k)}(0)$ is divisible by $n!$. — 5

That $f^{(k)}(1)$ is also divisible by $n!$ follows from the fact that $f^{(k)}(1-x) = (-1)^k f^{(k)}(x)$. — 1

[10]

SOLUTION | MARK | REMARK

6\. (a) That $e^{-x^2} > 0 \ \forall x$ is trivial.
Since the exponential function is strictly increasing,

$e^{x^2} \geq e^0 = 1$

$\frac{1}{e^{x^2}} \leq 1$.

$\therefore 0 < f(x) \leq 1 \quad \forall x \in \mathbb{R}$. — 1+2

ALTERNATIVELY,

Since $\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$ $\begin{cases} 0 \geq \text{ if } x \leq 0 \\ 0 < \text{ if } x > 0, \end{cases}$

$f(x)$ has an absolute maximum at $x = 0$ and $f(0) = 1$.

Since $0 < [f(x)]^n \leq 1$,

$I_n = [\int_{-1}^{1} [f(x)]^n dx]^{\frac{1}{n}}$

$\leq [\int_{-1}^{1} dx]^{\frac{1}{n}}$

$= 2^{\frac{1}{n}}$ — 2

[5]

(b) $r \leq e^{-x^2}$

$\Leftrightarrow \ln r \leq -x^2$

$\Leftrightarrow \ln \frac{1}{r} \geq x^2$

$\Leftrightarrow -\alpha \leq x \leq \alpha$, where $\alpha = \sqrt{\ln\left(\frac{1}{r}\right)}$ — 2

For $\frac{1}{e} < r < 1$, $0 < \alpha < 1$

$0 < r \leq f(x)$

$0 < r^n \leq [f(x)]^n$

$\therefore \int_{-\alpha}^{\alpha} r^n dx \leq \int_{-\alpha}^{\alpha} [f(x)]^n dx$

$\leq \int_{-1}^{1} [f(x)]^n dx$ — 3

Hence $2r^n\alpha \leq \int_{-1}^{1} [f(x)]^n dx$

$\Rightarrow r[2\sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} \leq I_n$ — 2

[7]

SOLUTION	MARK	REMARK

(c) By (a) and (b)

$r[2\sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} \le I_n \le 2^{\frac{1}{n}}, \quad \frac{1}{e} < r < 1$. 1

Now $\lim_{n\to\infty} 2^{\frac{1}{n}} = 1$ and $\lim_{n\to\infty} [\sqrt{\ln(\frac{1}{r})}]^{\frac{1}{n}} = 1$. 1+1

As $\lim_{n\to\infty} I_n$ exists

$r \le \lim_{n\to\infty} I_n \le 1$ 1

Since r is an arbitrary number between $\frac{1}{e}$ and 1 ,

$\lim_{n\to\infty} I_n = 1$. 1

5

(a) (i) $f(-x) = (-x)^{\frac{2}{3}} - ((-x)^2 - 1)^{\frac{1}{3}} = f(x)$.

∴ f is even.

$f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}} > 0$

$\Leftrightarrow x^{\frac{2}{3}} > (x^2 - 1)^{\frac{1}{3}}$

$\Leftrightarrow x^2 > x^2 - 1$, which is true for all x .

∴ $f(x) > 0$ for all x . 2

(ii) Put $x^2 = \frac{1}{y}$, then $x^{\frac{2}{3}} - (x^2-1)^{\frac{1}{3}} = (\frac{1}{y})^{\frac{1}{3}} - (\frac{1}{y} - 1)^{\frac{1}{3}}$

$= \frac{1-(1-y)^{\frac{1}{3}}}{y^{\frac{1}{3}}}$

$\lim_{x\to\infty} f(x) = \lim_{y\to 0^+} \frac{1-(1-y)^{\frac{1}{3}}}{y^{\frac{1}{3}}}$

$= \lim_{y\to 0^+} \frac{-\frac{1}{3}(1-y)^{-\frac{2}{3}}(-1)}{\frac{1}{3}y^{-\frac{2}{3}}}$ (by L'Hopital's rule)

$= \lim_{y\to 0^+} (\frac{1}{(\frac{1}{y}-1)^{\frac{2}{3}}})$

$= 0$ 3

5

(b) (i) For $x \ne 0$, 1 or -1 ,

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{1}{3}(x^2-1)^{-\frac{2}{3}}(2x)$

$= \frac{2}{3} \cdot \frac{(x^2-1)^{\frac{2}{3}} - x^{\frac{4}{3}}}{x^{\frac{1}{3}}(x^2-1)^{\frac{2}{3}}}$ 2

SOLUTION	MARK	REMARK

(b) (ii) For $x > 0$, $x \ne \pm 1$, the denominator $x^{\frac{1}{3}}(x^2-1)^{\frac{2}{3}} > 0$.

$(x^2-1)^{\frac{2}{3}} - x^{\frac{4}{3}} \gtrless 0$ according as $(x^2-1)^2 \gtrless x^4$

$-2x^2 + 1 \gtrless 0$

$\frac{1}{\sqrt{2}} \gtrless x \quad (x > 0)$

∴ the required sets are

$\{x : x = \frac{1}{\sqrt{2}}\}$ 2

$\{x : x > \frac{1}{\sqrt{2}}\}$ 1

$\{x : 0 < x < \frac{1}{\sqrt{2}}\}$ 1

x	$x = 0$	$0 < x < \frac{1}{\sqrt{2}}$	$x = \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} < x < 1$	$x = 1$	$1 < x$
f'	does not exist	+	0	−	does not exist	−
f	min.	↗	max.	↘		↘

At $x = 0$, 1 and -1 , $f(x) = 1$

At $x = \pm\frac{1}{\sqrt{2}}$, $f(x) = 4^{\frac{1}{3}}$ ($\doteq 1.5876$)

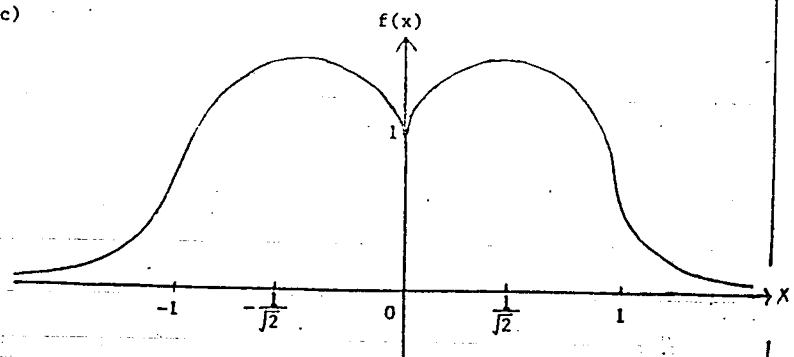
($\doteq \pm 0.7071$)

$(0, 1)$ is a minimum 1

$(-\frac{1}{\sqrt{2}}, 4^{\frac{1}{3}})$, $(\frac{1}{\sqrt{2}}, 4^{\frac{1}{3}})$ are maxima. 1+1

9

(c)



3

SOLUTION | MARK | REMARK

(a) $g'(t) = b - f(t)$.

Since f is strictly increasing,

$g'(t) \gtreqless 0$ according as $t \lesseqgtr f^{-1}(b)$.

As g is continuously differentiable on $[0, \infty)$,

g attains its greatest value at $f^{-1}(b)$. 3

(b) (i) Integrating by parts,

$$\int_0^{f^{-1}(b)} xf'(x)dx = xf(x)\Big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} f(x)dx$$

$$= g(f^{-1}(b)).$$ 2

(ii) Putting $y = f(x)$, $dy = f'(x)dx$.

$x = f^{-1}(y)$ as f is strictly increasing.

$$\therefore \int_0^{f^{-1}(b)} xf'(x)dx = \int_0^b f^{-1}(y)dy$$

$$= \int_0^b f^{-1}(x)dx$$ 3

5

By (b), $\int_0^b f^{-1}(x)dx = \int_0^{f^{-1}(b)} xf'(x)dx$

$= g(f^{-1}(b))$

$\geq bt - \int_0^t f(x)dx \quad \forall t \in [0, \infty)$ by (a) 1

Hence $\int_0^b f^{-1}(x)dx \geq ba - \int_0^a f(x)dx$, 1

i.e. $\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab$

Geometrically, 1

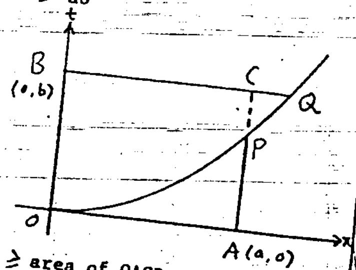
$\int f(x)dx$ = area of region OAP

$\int f^{-1}(x)dx$ = area of region OBQ

ab = area of OACB.

Sum of areas of the two regions $\geq$ area of OACB

(The equality holds when $f(a) = b$.)



SOLUTION | MARK | REMARK

(d) Consider the function $f(x) = x^{p-1}$ ($x \in [0, \infty)$, $p > 2$) which satisfies the given conditions. 1

Since $\frac{1}{p} + \frac{1}{q} = 1$,

$f^{-1}(x) = x^{\frac{1}{p-1}}$

$= x^{q-1}$ 1

By (c), $\int_0^a x^{p-1}dx + \int_0^b x^{q-1}dx \geq ab$

$\therefore \frac{1}{p}a^p + \frac{1}{q}b^q \geq ab$. 1

3