8. Let $L_i : a_ix + b_iy + c_i = 0$ $(i = 1, 2, 3)$ be three distinct straight lines which meet pairwise as shown in Figure 2. Suppose the three points of intersection P_1, P_2 and P_3 are non-collinear.

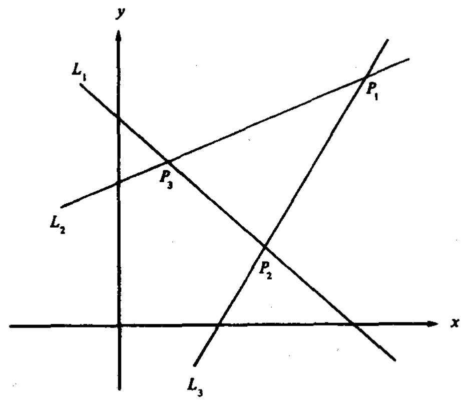
For any non-zero real constants λ_1, λ_2 and λ_3, consider the equations

$$C(\lambda_1, \lambda_2, \lambda_3) : \lambda_3(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \lambda_1(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + \lambda_2(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) = 0$$

and $T_k : \lambda_j(a_ix + b_iy + c_i) + \lambda_i(a_jx + b_jy + c_j) = 0$,

where (i, j, k) is any permutation of the indices 1, 2 and 3.

Figure 2



(a) Show that $C(\lambda_1, \lambda_2, \lambda_3)$ represents a conic passing through the points P_1, P_2 and P_3 and that T_k is a tangent to $C(\lambda_1, \lambda_2, \lambda_3)$ at P_k $(k = 1, 2, 3)$.

(b) Let the three lines L_i now be given by

$L_1 : x + y - 2 = 0$
$L_2 : x - y + 2 = 0$
$L_3 : 2x - y = 0$.

Consider all the conics which are of the form $C(\lambda_1, \lambda_2, \lambda_3)$ and whose axes are parallel to the coordinate axes. Find the equation of the locus of the point of intersection of the tangents T_1 and T_2.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1984

純數學 試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of eight questions all carrying equal marks.

Answer any SIX questions.

1. The matrix $A = \begin{pmatrix} a & 0 & 1 \\ 0 & b & 0 \\ 1 & 0 & c \end{pmatrix}$ satisfies the condition $a + b + c = 0$.

(a) A polynomial $f(x)$ is defined by

$$f(x) = \det(A - xI) = c_0x^3 + c_1x^2 + c_2x + c_3 .$$

Write down the polynomial $f(x)$ with coefficients expressed in terms of a, b and c.

Evaluate the matrix $f(A) = c_0A^3 + c_1A^2 + c_2A + c_3I$.

(b) Using (a), or otherwise, express A^3 in the form $\lambda A + \mu I$, where λ and μ are real numbers.

Hence find A^9 for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

2. A function $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ is said to be *linear* if

(1) $f(\alpha \mathbf{u}) = \alpha f(\mathbf{u})$ for any $\alpha \in \mathbf{R}$ and $\mathbf{u} \in \mathbf{R}^3$;

(2) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$.

(a) For $\mathbf{a} = (a_1, a_2, a_3) \in \mathbf{R}^3$, a function $g : \mathbf{R}^3 \longrightarrow \mathbf{R}$ is defined by the scalar product

$$g(\mathbf{u}) = \mathbf{a} \cdot \mathbf{u} \quad \text{for any } \mathbf{u} \in \mathbf{R}^3 .$$

Show that g is linear.

(b) Given a linear function $h : \mathbf{R}^3 \longrightarrow \mathbf{R}$. Show that

$$h(\mathbf{u}) = \alpha h(\mathbf{x}) + \beta h(\mathbf{y}) + \gamma h(\mathbf{z}) \quad \text{for } \mathbf{u} = \alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} .$$

Find a vector $\mathbf{b} \in \mathbf{R}^3$ so that

$$h(\mathbf{u}) = \mathbf{b} \cdot \mathbf{u} \quad \text{for any } \mathbf{u} \in \mathbf{R}^3 .$$

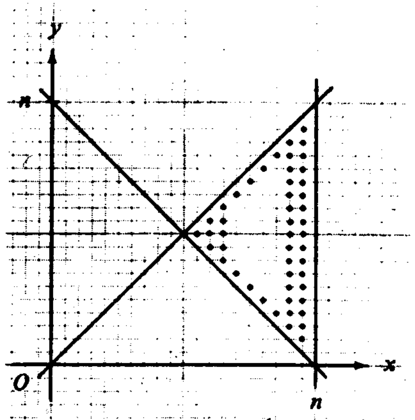
(c) Show that a subset H of $\mathbf{R}^3$ is a plane passing through the origin if and only if there is a linear function $f : \mathbf{R}^3 \longrightarrow \mathbf{R}$ which is not identically zero so that

$$H = \{\mathbf{u} \in \mathbf{R}^3 : f(\mathbf{u}) = 0\} .$$

3. (a) Let n be a positive integer and Δ the triangle bounded by

$x = y$,
$x + y = n$ and
$x = n$.

Find the number A_n of integral points (i.e. points whose coordinates are integers) in the interior of Δ for both cases where n is even and n is odd.



(n is even)

(b) Three different numbers $y < x < t$ are taken from the $2k$ positive integers $1, 2, 3, \ldots, 2k$ to form the sides of a triangle (non-degenerate).

Let B_{2k} be the number of all possible triangles formed.

(i) Show that A_n is the number of triangles with the longest side $t = n$.

(ii) Show that

$$B_{2k} = \frac{k(k-1)(4k-5)}{6} .$$

[You may use the result $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$]

(c) Use the above results to solve the following problem :

Three different numbers are taken at random from the first $2k$ positive integers. Find the probability $p(2k)$ that they form the sides of a triangle (non-degenerate).

Evaluate the limit $\lim_{k \to \infty} p(2k)$.

4. Let $z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, where n is a positive integer.

(a) Prove that $z^m = 1$ if and only if m is divisible by n .

Hence evaluate $\sum_{r=0}^{n-1} z^{mr}$ for the cases :

(i) m is divisible by n ,

(ii) m is not divisible by n .

(b) Let $f(x) = \sum_{k=0}^{n-1} a_k x^k$. Use the result of (a) to show that,

for any given j $(0 \leq j \leq n-1)$,

$$\sum_{r=0}^{n-1} f(z^r)\, z^{(n-j)r} = na_j \,.$$

(c) Given a polynomial $g(x)$. Let $h(x)$ and $f(x)$ be polynomials with $\deg f \leq n-1$ such that $g(x) = (x^n - 1)h(x) + f(x)$.

Show that

$$f(x) = \frac{1}{n}\sum_{j=0}^{n-1}\left\{\sum_{r=0}^{n-1} g(z^r)\, z^{(n-j)r}\right\} x^j \,.$$

5. Let A and B be two non-empty sets and let f and g be two mappings from A to B . Define relations R and S in A as follows :

xRy if $f(x) = f(y)$ and $g(x) = g(y)$,

xSy if $f(x) = f(y)$ or $g(x) = g(y)$.

(a) (i) Show that R is an equivalence relation.

(ii) For $A = B = \{1, 2, 3\}$, find mappings f and g such that S is not an equivalence relation.

(b) Let u be the natural surjection from A onto the quotient set A/R taking each $a \in A$ to the equivalence class a/R .

Show that there exists a unique mapping $h : A/R \longrightarrow B$ such that

$$f = h \circ u \,.$$

Furthermore, suppose that g is a constant mapping. Show that if f is surjective then h is bijective.

6. (a) Let α and β be two complex numbers with $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$. Show that

(i) if $\bar{\alpha}\beta = 1$, then $\alpha = \beta$,

(ii) if $|\alpha| < 1$, then

$$\frac{|\alpha - \beta|}{|1 - \bar{\alpha}\beta|} \leqslant 1 ,$$

where the equality holds if and only if $|\beta| = 1$.

(b) Let a and b be two complex numbers with $b \neq 0$. Consider the function

$$f(z) = \frac{z - a}{bz - 1}$$

defined on the set $D = \mathbb{C} \setminus \left\{\frac{1}{b}\right\}$. Suppose $1, -1, i \in D$ and $|f(1)| = |f(-1)| = |f(i)| = 1$.

(i) Show that $b = \bar{a}$ and $|f(z)| = 1$ for all $z \in D$ with $|z| = 1$.

(ii) Show that $f(z)$ is a constant function if $|a| = 1$.

7. Let $\mathscr{D}$ be the set of all 3×3 real matrices, the sum of whose elements in any one row or any one column is 1.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be in $\mathscr{D}$.

Define $S(A) = \left\{ X \in \mathscr{D} : AX = J \right\}$, where $J = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

(a) Show that if $(x_1\ x_2\ x_3)A = (x'_1\ x'_2\ x'_3)$ and

$$A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} \text{, then } x_1 + x_2 + x_3 = x'_1 + x'_2 + x'_3$$

and $y_1 + y_2 + y_3 = y'_1 + y'_2 + y'_3$.

(b) Show that

(i) $JB = J = BJ$ for all $B \in \mathscr{D}$,

(ii) $S(B) \neq \phi$ for all $B \in \mathscr{D}$,

(iii) $S(J) = \mathscr{D}$.

(c) If A is invertible, use the above results to show that

$$A^{-1} \in \mathscr{D} \text{ and } S(A) = \left\{ J \right\} .$$

(d) Show that if A is not invertible, then there exists a non-zero matrix C such that the row sums and column sums of C are zero and that AC is the zero matrix.

Hence show that $S(A) \neq \left\{ J \right\}$.

8. Let $f(x)$ and $g(x)$ be non-zero polynomials with integral coefficients. Suppose for any positive integer n, there exists an integer a_n such that $g(n) = a_n f(n)$, i.e. $g(n)$ is divisible by $f(n)$.

(a) Show that $a_n = 0$ for only a finite number of n.

Hence deduce that it is impossible for $\deg f(x) > \deg g(x)$.

(b) Show that there exists a non-zero polynomial $h(x)$ with rational coefficients such that

$$g(x) = f(x)\,h(x) .$$

(c) If $\deg f(x) = \deg g(x)$, show that $h(x)$ is identically equal to an integer.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1984

純數學 試卷二

PURE MATHEMATICS PAPER II

2.00 pm–5.00 pm (3 hours)

This paper must be answered in English

This paper consists of eight questions all carrying equal marks.

Answer any SIX questions.

1. (a) For any non-negative integer k, let

$$u_k = \int_0^{\pi} \frac{\sin kx}{\sin x}\, dx .$$

Express u_{k+2} in terms of u_k.

Hence, or otherwise, evaluate u_k.

(b) For any non-negative integers m and n, let

$$I(m, n) = \int_0^{\frac{\pi}{2}} \cos^m \theta \sin^n \theta \, d\theta .$$

(i) Show that if $m \geqslant 2$, then

$$I(m, n) = \left(\frac{m-1}{n+1}\right) I(m-2, n+2) .$$

(ii) Evaluate $I(1, n)$ for $n \geqslant 0$.

(iii) Show that if $n \geqslant 2$, then

$$I(0, n) = \left(\frac{n-1}{n}\right) I(0, n-2) .$$

(iv) Evaluate $I(6, 4)$.

2. Let f be a real-valued function defined on the interval $I = (-1, 1)$ and with n th order continuous derivative $f^{(n)}$. For any $0 < h < 1$, let R_m be defined by

$$R_m = \frac{1}{(m-1)!} \int_0^h (h-t)^{m-1} f^{(m)}(t)\, dt ,$$

where $1 \leqslant m \leqslant n$.

(a) Show that

$$R_m = R_{m-1} - \frac{h^{m-1}}{(m-1)!} f^{(m-1)}(0) \quad (2 \leqslant m \leqslant n) .$$

(b) Evaluate R_1 and R_2.

Hence show that

$$f(h) = f(0) + hf'(0) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n .$$

(c) Using (b), or otherwise, show that

$$0 < \ln(1+h) - h + \frac{1}{2}h^2 - \frac{1}{3}h^3 + \frac{1}{4}h^4 < \frac{h^5}{5}$$

for all $0 < h < 1$.

3. A hypocycloid is a curve generated by the motion of a point P on the circumference of a circle which rolls internally without slipping on a larger circle [see Figures 1 and 2] .

Let the radius of the larger circle be a and that of the smaller circle be b , where $2b < a$. Suppose the initial position of P is at $(a, 0)$ [see Figure 1] .

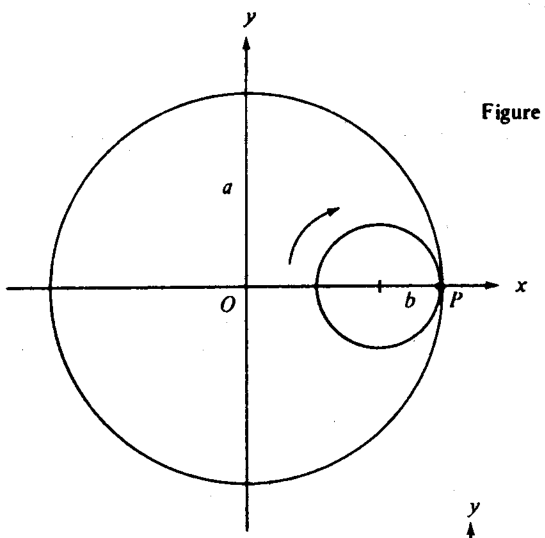


Figure 1

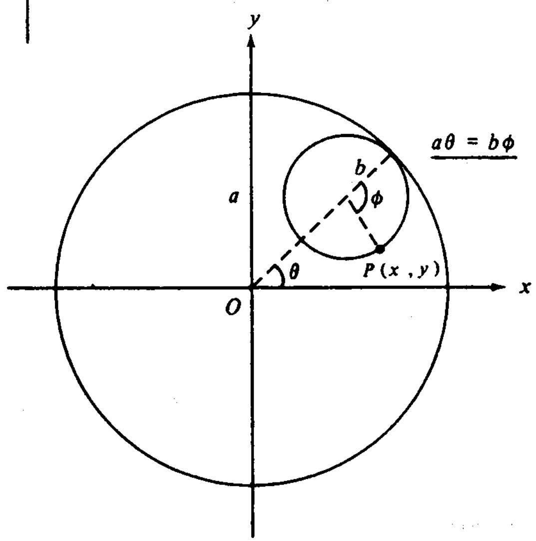


Figure 2

(a) Referring to Figure 2 , show that the parametric equations of the hypocycloid are given by

$$\begin{cases} x = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\right)\theta \\ y = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\right)\theta \end{cases} .$$

(b) Suppose that $b = \frac{a}{4}$. Show that the equations of the hypocycloid can be written as

$$\begin{cases} x = a\cos^3\theta \\ y = a\sin^3\theta \qquad (0 \leqslant \theta \leqslant 2\pi) . \end{cases}$$

By eliminating θ , show that x and y satisfy the equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} .$$

(c) Compute the length of the curve in (b).

4. Given two ellipses

$$(E) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

$$(F) : \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 ,$$

where $0 < b < a$.

(a) Show that the line $\ell x + my = 1$ is tangent to (E) if and only if $a^2\ell^2 + b^2m^2 = 1$.

(b) Find the equations of the common tangents to (E) and (F) .

(c) $R(h, k)$ is a point outside (E) . The two tangents drawn from R to (E) touch (E) at two points S and T . Find the equation of the straight line through S and T .

(d) It is furthermore given that the line through S and T in (c) is tangent to (F) . Find and sketch the locus of R .

5. (a) Prove that $\lim_{n \to \infty} \frac{a^n}{n!} = 0$ for any real number a .

(b) Define $f(x) = x^n (1 - x)^n$, where n is a positive integer.

(i) Verify that $f(x) = f(1 - x)$ for every $x \in \mathbf{R}$.

Hence, or otherwise, show that the k th derivative $f^{(k)}$ satisfies

$$f^{(k)}(1 - x) = (-1)^k f^{(k)}(x) .$$

(ii) Show that $f^{(k)}(0)$ and $f^{(k)}(1)$ are integers divisible by $n!$.

6. Let $f(x) = e^{-x^2}$, $x \in \mathbf{R}$ and let $I_n = \left\{ \int_{-1}^{1} [f(x)]^n \, dx \right\}^{\frac{1}{n}}$, where n is a positive integer.

(a) Show that $0 < f(x) \leq 1$ for all x .

Hence deduce that $I_n \leq 2^{\frac{1}{n}}$.

(b) Given any $0 < r < 1$, find the range of x such that $r \leq f(x)$.

Hence deduce that

$$I_n \geq r \left\{ 2\sqrt{\ln\left(\frac{1}{r}\right)} \right\}^{\frac{1}{n}}$$

for any r in the interval $(\frac{1}{e}, 1)$.

(c) Assume that $\lim_{n \to \infty} I_n$ exists. Using (a) and (b), or otherwise, prove that

$$\lim_{n \to \infty} I_n = 1 .$$

7. Define $f : \mathbf{R} \longrightarrow \mathbf{R}$ by $f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}}$.

(a) (i) Show that $f(x)$ is even and that $f(x) > 0$ for all x .

(ii) Evaluate $\lim_{x \to \infty} f(x)$.

(b) (i) Evaluate $f'(x)$ at $x \neq 0, 1$ or -1 .

(ii) Find the sets $\{ x : x > 0 \text{ and } f'(x) = 0 \}$, $\{ x : x > 0 \text{ and } f'(x) > 0 \}$ and $\{ x : x > 0 \text{ and } f'(x) < 0 \}$.

(iii) Find the relative maxima and minima of f .

(c) Sketch the graph of f .

8. Let f be a real-valued function which is continuously differentiable and strictly increasing on the interval $I = [0, \infty)$. Suppose $f(0) = 0$. Let $a \in I$ and $b \in f[I]$.

(a) For any $t \in I$, define $g(t) = bt - \int_0^t f(x)\,dx$.

Prove that g attains its greatest value at $f^{-1}(b)$.

(b) (i) Show that $\int_0^{f^{-1}(b)} x f'(x)\,dx = g(f^{-1}(b))$.

(ii) By a change of variable, show that

$$\int_0^{f^{-1}(b)} x f'(x)\,dx = \int_0^b f^{-1}(x)\,dx .$$

(c) Use (a) and (b) to prove that $\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geqslant ab$.

Referring to Figure 3, what is the geometric meaning of the above inequality if the integrals are interpreted as areas ?

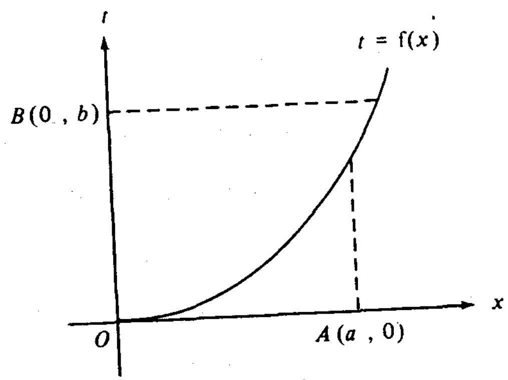


Figure 3

(d) Using (c), show that

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geqslant ab ,$$

where $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$.

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1985

純數學 試卷一

PURE MATHEMATICS PAPER I

9.00 am–12.00 noon (3 hours)

This paper must be answered in English

This paper consists of nine questions all carrying equal marks.

Answer any SEVEN questions.