7. A continuous real-valued function f is said to be convex in an interval (a, b) if for any x, y in (a, b) and for any λ, μ in $[0, 1]$ with $\lambda + \mu = 1$,

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y) .$$

(a) Show that, for any t in the interval (x, y), there exist λ, μ in $(0, 1)$ with $\lambda + \mu = 1$ such that

$$t = \lambda x + \mu y .$$

Draw a diagram to illustrate the inequality in the definition of a convex function.

(b) f is convex. Show that, for any x, t, y in (a, b) with $x < t < y$,

$$\frac{f(t) - f(x)}{t - x} \leqslant \frac{f(y) - f(t)}{y - t} .$$

(c) Let g be a function with a second derivative $g''(x) \geqslant 0$ on (a, b).
For any x, y in (a, b) with $x < y$, consider the function

$$h(t) = \lambda g(t) + \mu g(y) - g(\lambda t + \mu y)$$

for $t \in [x, y]$, where λ and μ are fixed numbers in $[0, 1]$ such that $\lambda + \mu = 1$. Show that h is monotonic decreasing and hence show that g is a convex function.

(d) For $x_1, x_2 > 0$, $p > 1$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$, show that $(\lambda_1 x_1 + \lambda_2 x_2)^p \leqslant \lambda_1 x_1^p + \lambda_2 x_2^p$.

8. Let L_1 and L_2 be two rays from the origin O inclining at angles $\frac{\pi}{4}$ and $-\frac{\pi}{4}$, respectively, to the positive x-axis. P and Q are points on L_1 and L_2, respectively, such that $OP = p$ and $OQ = \frac{1}{p}$.

(a) P' and Q' are points on L_1 and L_2, respectively, such that $OP' = p'$ and $OQ' = \frac{1}{p'}$. If $M(u, v)$ denotes the point at which PQ and $P'Q'$ meet, express u and v in terms of p and p' .

Find $\lim_{p' \to p} u$ and $\lim_{p' \to p} v$.

(b) Let $\xi(p) = \lim_{p' \to p} u$ and $\eta(p) = \lim_{p' \to p} v$.
As p varies, show that the locus of $(\xi(p), \eta(p))$ consists of a branch (H) of a hyperbola.

(c) A and B are points on L_1 and L_2, respectively. Show that AB meets H at no point, one point, or two points according as $OA \cdot OB$ is less than, equal to, or greater than 1 .

END OF PAPER

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HONG KONG EXAMINATIONS AUTHORITY

一 九 八 三 年 香 港 高 級 程 度 會 考

HONG KONG ADVANCED LEVEL EXAMINATION 1983

純數學 試卷一

PURE MATHEMATICS PAPER I

三小時完卷 Three hours

上午九時至正午十二時 9.00 a.m.—12.00 noon

本試卷必須用英文作答 This paper must be answered in English

This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

In this paper, you may use without proof the fact that a monotonic increasing (decreasing) sequence which is bounded above (below) converges.

1. (a) Prove that the following system of linear equations in the unknowns x, y and z has a unique solution if a, b and c are all non-zero and distinct:

$$\begin{cases} ax + by + cz = k \\ a^2x + b^2y + c^2z = k^2 \\ a^3x + b^3y + c^3z = k^3 . \end{cases}$$

In such a case, find the solution (x_0, y_0, z_0) in terms of a, b, c and k, and show that it is impossible for exactly one of x_0, y_0 and z_0 to be zero.

(b) Find all values of d for which the following system is solvable:

$$\begin{cases} -x + 2y - z = d \\ x + 4y + z = d^2 \\ -x + 8y - z = d^3 . \end{cases}$$

Give the solutions for each of the values of d.

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2. Let $M = \begin{pmatrix} p & q & r \\ r & p & q \\ q & r & p \end{pmatrix}$, where p, q and r are non-negative real numbers satisfying $p + q + r = 1$.

(a) Show that $\det(M) = 1 - 3(pq + qr + rp) = \frac{1}{2}[(p - q)^2 + (q - r)^2 + (r - p)^2]$.
Hence deduce that $0 \leq \det(M) \leq 1$.

(b) Using mathematical induction, or otherwise, show that for any positive integer n, M^n is of the form

$$\begin{pmatrix} p_n & q_n & r_n \\ r_n & p_n & q_n \\ q_n & r_n & p_n \end{pmatrix},$$

where p_n, q_n and r_n are non-negative real numbers satisfying $p_n + q_n + r_n = 1$.

(c) Suppose at least two of p, q and r are non-zero. Using (a) and (b), or otherwise, show that

(i) $\lim_{n \to \infty} \det(M^n) = 0$,

(ii) $\lim_{n \to \infty} [3p_n - (p_n + q_n + r_n)] = 0$ and hence
$\lim_{n \to \infty} p_n = \frac{1}{3}$.

3. (a) Let w be a complex number. Show that
$|w - i| = |w + i|$ if and only if w is real.

(b) The complex number u satisfies the equation
$|2u - i| = 1$. .... (*)
Sketch the locus of u in the Argand plane.

(c) Show that the complex number $u\ (\neq i)$ satisfies equation (*) if and only if
$v = \dfrac{iu}{i - u}$ is real.
In this case, show that the points representing u, v and i in the Argand plane are collinear.

4. Given a sequence $\{a_n\}$ such that

(1) $a_1 > a_2 > 0$,

(2) $a_{n+2} = \frac{1}{2}(a_{n+1} + a_n)$ for $n = 1, 2, \ldots$.

(a) Show that for $n \geq 1$,
$$a_{n+2} - a_n = \frac{(-1)^n}{2^n}(a_1 - a_2),$$
and hence show that the sequence $\{a_1, a_3, a_5, \ldots\}$ is strictly decreasing and that the sequence $\{a_2, a_4, a_6, \ldots\}$ is strictly increasing.

(b) For any positive integers m and n, show that
$a_{2m} < a_{2n-1}$.

(c) Show that the two sequences $\{a_1, a_3, a_5, \ldots\}$ and $\{a_2, a_4, a_6, \ldots\}$ converge to the same limit.

5. On a rainy day, each man arriving at a dinner party leaves his umbrella and takes one when he departs. Suppose that each man's choice of an umbrella at the end of the dinner is completely random.
Let $P_{n,k}$ be the probability that, in a party of n men, exactly k men take back their own umbrellas.

(a) Show that $P_{n,k} = (k+1)P_{n+1,k+1}$.

(b) Let $F_n(x) = P_{n,0} + P_{n,1}x + \ldots + P_{n,k}x^k + \ldots + P_{n,n}x^n$. Show that

(i) $F_n(x) = \frac{d}{dx}F_{n+1}(x)$,

(ii) $F_n^{(k)}(1) = 1$ for $0 \leq k \leq n$.

[$F_n^{(k)}(x)$ denotes the k-th derivative of $F_n(x)$ for $k > 0$. $F_n^{(0)}(x) = F_n(x)$.]

(c) Use the expansion
$$F_n(x) = F_n(a) + \frac{(x-a)}{1!}F_n^{(1)}(a) + \ldots + \frac{(x-a)^k}{k!}F_n^{(k)}(a) + \ldots + \frac{(x-a)^n}{n!}F_n^{(n)}(a)$$
of $F_n(x)$ at a for $a = 0$ and $a = 1$ to show that
$$P_{n,k} = \frac{1}{k!}\sum_{j=k}^{n}\frac{(-1)^{j-k}}{(j-k)!} \quad \text{for } 0 \leq k \leq n.$$

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6. Let A and B be two non-empty sets and let $f : A \to B$ and $g : B \to A$ be two mappings. The power set of A is denoted by $\mathscr{P}(A)$ and the direct image of a subset X of A under f is denoted by $f[X]$.

(a) A function $\Phi : \mathscr{P}(A) \to \mathscr{P}(A)$ is defined by

$$\Phi(X) = g[B \setminus f[X]] \quad \text{for all} \quad X \in \mathscr{P}(A).$$

Show that if $X_1 \subset X_2 \subset A$, then $\Phi(X_1) \supset \Phi(X_2)$.

(b) A function $\Psi : \mathscr{P}(A) \to \mathscr{P}(A)$ is defined by

$$\Psi(X) = A \setminus \Phi(X) \quad \text{for all} \quad X \in \mathscr{P}(A).$$

Let $\mathscr{F} = \{X \in \mathscr{P}(A) : \Psi(X) \subset X\}$. Denote by S the intersection of all members of $\mathscr{F}$.

Show that if $X_1 \subset X_2 \subset A$, then $\Psi(X_1) \subset \Psi(X_2)$. Hence verify that $S \in \mathscr{F}$.

(c) Prove that $\Psi(S) \in \mathscr{F}$ and $A \setminus S = \Phi(S)$.

7. (a) Prove that for any positive integer n,

$$\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{r=1}^{n} \left\{ \frac{1}{r!} \prod_{k=0}^{r-1} \left(1 - \frac{k}{n}\right) \right\}.$$

Hence, or otherwise, show that for $n \geqslant 2$,

$$2 < \left(1 + \frac{1}{n}\right)^n < 3,$$

and that $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is a convergent sequence.

(b) Prove the identity

$$\sum_{k=1}^{n} C_k^n (-1)^{k-1} x^{k-1} = \sum_{i=0}^{n-1} (1 - x)^i.$$

Using integration, or otherwise, show that

$$C_1^n - \frac{1}{2} C_2^n + \ldots + (-1)^{n-1} \frac{1}{n} C_n^n = \sum_{i=0}^{n-1} \frac{1}{i+1}.$$

8. Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of real numbers, $S = \sum_{i=1}^{n} a_i b_i$ and $B_i = \sum_{j=1}^{i} b_j$.

(a) Show that

$$S = a_n B_n + \sum_{i=1}^{n-1} (a_i - a_{i+1}) B_i.$$

(b) If $\{a_1, a_2, \ldots, a_n\}$ is monotonic (decreasing or increasing) and $|B_i| \leqslant K$ for all i, show that

$$|S| \leqslant K(|a_1| + 2|a_n|).$$

(c) Using (b), or otherwise, show that if $n \geqslant 3$,

$$\left| \sum_{k=n}^{n+p} \frac{(-1)^k}{\sqrt[k]{k}} \right| \leqslant 3.$$

for any positive integer p.

END OF PAPER

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八三年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION 1983

純數學 PURE MATHEMATICS

試卷二 PAPER II

三小時完卷 Three hours

下午二時至下午五時 2.00 p.m.—5.00 p.m.

本試卷必須用英文作答 This paper must be answered in English

This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

Provided by dse.life

In this paper, you may use without proof the fact that a monotonic increasing (decreasing) sequence which is bounded above (below) converges.

1. Evaluate

(a) $\displaystyle\int \frac{dx}{\sqrt{(x+a)(x+b)}}$,

(b) $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$,

[Hint : Put $u = \frac{\pi}{4} - x$.]

(c) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left\{\cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \ldots + \cos\frac{(n-1)\pi}{n}\right\}$.

2. Let $f(x) = x^2(x-2)e^{-x}$.

(a) Sketch the graph of the function $f(x)$ by first finding its stationary point(s), intercept(s) and asymptote(s).

(b) Evaluate $\displaystyle\lim_{k\to\infty} A_k$, where A_k is the area bounded by the curve, the positive x-axis and the line $x = k$ $(k > 0)$.

3. (a) Given a line in space

$$L : \begin{cases} x = \ell t + x_0 \\ y = mt + y_0 \\ z = nt + z_0 . \end{cases}$$

Show that the plane $Ax + By + Cz + D = 0$ contains L if and only if

$A\ell + Bm + Cn = 0$

and $Ax_0 + By_0 + Cz_0 + D = 0$.

(b) Given two distinct lines

$$L_1 : \begin{cases} x = \ell_1 t + x_1 \\ y = m_1 t + y_1 \\ z = n_1 t + z_1 , \end{cases}$$

$$L_2 : \begin{cases} x = \ell_2 t + x_2 \\ y = m_2 t + y_2 \\ z = n_2 t + z_2 . \end{cases}$$

(i) Suppose L_1 and L_2 intersect at a point. Show that the equation of the plane passing through L_1 and L_2 is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

(ii) Suppose L_1 and L_2 are parallel. Find the equation of the plane passing through L_1 and L_2.

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4. Let $f(x)$ and $g(x)$ be two differentiable functions defined on the interval $I = (-\frac{\pi}{2}, \frac{\pi}{2})$ and with the following properties:

(1) $g(x) > 0$,

(2) $\frac{d}{dx} f(x) = \frac{1}{g^2(x)}$,

(3) $\frac{d}{dx} g(x) = -f(x) g(x)$,

(4) $f(0) = 0$ and $g(0) = 1$.

(a) Find, in terms of $f(x)$ and $g(x)$,

(i) $\frac{d}{dx}\left(\frac{1}{g(x)}\right)$,

(ii) $\frac{d}{dx}\left(\frac{1}{g^2(x)}\right)$.

(b) Show that $1 + f^2(x) = \frac{1}{g^2(x)}$.

(c) Show that

$$f(a) g(a) = f(x) g(x) g(a - x) + f(a - x) g(a - x) g(x)$$

for any $a, x \in I$ such that $(a - x) \in I$.

(d) Deduce from (c) that

(i) $f(x + y) g(x + y) = g(x) g(y) [f(x) + f(y)]$ for any $x, y \in I$ such that $x + y \in I$.

(ii) $f(-x) = -f(x)$ for any $x \in I$.

5. Let f be a non-constant real-valued function defined on the set $\mathbf{R}$ of real numbers such that

(1) $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{R}$,

(2) there exists $x_0 \in \mathbf{R}$ at which f is differentiable.

(a) Show that $f(0) = 1$ and that $f(x) \neq 0$ for all $x \in \mathbf{R}$.

(b) Show that f is differentiable at $x = 0$ and find $f'(0)$ in terms of $f'(x_0)$ and $f(x_0)$.
Hence show that f is differentiable at every $x \in \mathbf{R}$ and that

$$f'(x) = \frac{f'(x_0)}{f(x_0)} f(x).$$

(c) Use the derivative of the function $e^{-\alpha x} f(x)$ to show that $f(x) = e^{\alpha x}$ for some non-zero constant α.

6. (a) For any non-negative integers p and q, the function $F_{p,q}(x)$ is defined by

$$F_{p,q}(x) = \int_0^x \cos^p t \sin qt \, dt.$$

By differentiation, prove that for p and $q \geq 1$,

$$(p + q) F_{p,q}(x) - p F_{p-1,q-1}(x) = -\cos^p x \cos qx + C,$$

where C is a constant.

Determine the value of C.

(b) When p and q are both even or both odd, show that

$$\int_0^{\pi} \cos^p x \sin qx \, dx = 0.$$

(c) Evaluate the integral $\int_0^{\frac{\pi}{2}} \sin^2 x \sin 3x \, dx$.

7. Let k be any positive integer. The k-th harmonic number H_k is defined by

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}.$$

The graph of the function $y = \frac{1}{x}$ is shown in Figure 1.

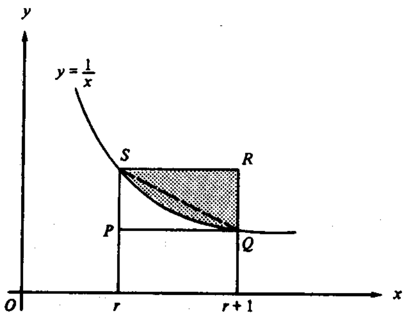

Figure 1

(a) By considering an integral of y, show that

$$\ln k < H_k < 1 + \ln k.$$

Hence show that $\lim_{k \to \infty} \frac{H_k}{\ln k} = 1$.

(b) Let $\gamma_k = H_k - \ln k$. Prove that $\lim_{k \to \infty} \gamma_k$ exists by showing that $\{\gamma_k\}$ is a monotonic sequence.

(c) The area of the shaded region in Figure 1 is denoted by A_r. Show that, for $1 < r < k$,

$$\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) < A_r < \frac{1}{r} - \frac{1}{r+1}.$$

Hence show that $\frac{1}{2} < \lim_{k \to \infty} [H_k - \ln k] < 1$.

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8. Let $L_i : a_i x + b_i y + c_i = 0$ $(i = 1, 2, 3)$ be three distinct straight lines which meet pairwise as shown in Figure 2. Suppose the three points of intersection P_1, P_2 and P_3 are non-collinear.

For any non-zero real constants λ_1, λ_2 and λ_3, consider the equations

$$C(\lambda_1, \lambda_2, \lambda_3) : \lambda_3(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \lambda_1(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + \lambda_2(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) = 0$$

and $T_k : \lambda_j(a_i x + b_i y + c_i) + \lambda_i(a_j x + b_j y + c_j) = 0$,

where (i, j, k) is any permutation of the indices 1, 2 and 3.

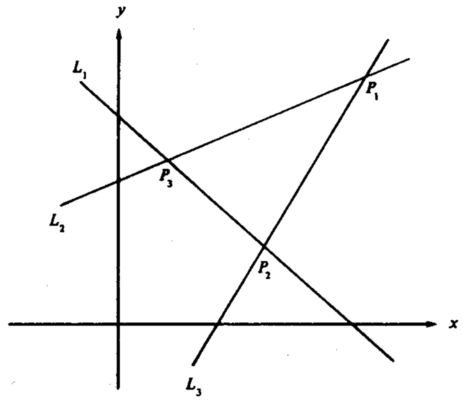

Figure 2

(a) Show that $C(\lambda_1, \lambda_2, \lambda_3)$ represents a conic passing through the points P_1, P_2 and P_3 and that T_k is a tangent to $C(\lambda_1, \lambda_2, \lambda_3)$ at P_k $(k = 1, 2, 3)$.

(b) Let the three lines L_i now be given by

$L_1 : x + y - 2 = 0$
$L_2 : x - y + 2 = 0$
$L_3 : 2x - y = 0$.

Consider all the conics which are of the form $C(\lambda_1, \lambda_2, \lambda_3)$ and whose axes are parallel to the coordinate axes. Find the equation of the locus of the point of intersection of the tangents T_1 and T_2.

END OF PAPER

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