香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八二年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION, 1982

Pure Mathematics I

MARKING SCHEME

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Solution

1. (a) For $0 < \lambda < 1$, let $\varphi(t) = \lambda t + (1-\lambda) - t^{\lambda}$.

$$\frac{d\varphi}{dt} = \lambda - \lambda t^{\lambda - 1}$$

$$= \lambda\left(1 - \frac{1}{t^{1-\lambda}}\right) \begin{cases} = 0 & \text{if } t = 1 \\ < 0 & \text{if } 0 < t < 1 \\ > 0 & \text{if } t > 1 \end{cases}$$

$$\therefore \varphi(t) \geq \varphi(1) = 0 \quad \forall t > 0$$

i.e. $\lambda t + (1-\lambda) \geq t^{\lambda} \quad \forall t > 0$.

Next, if either of α and β is zero, the second inequality is obvious.

If $\alpha, \beta > 0$, let $t = \frac{\alpha}{\beta} > 0$.

Substituting in the first inequality

$$\lambda\frac{\alpha}{\beta} + (1-\lambda) \geq \left(\frac{\alpha}{\beta}\right)^{\lambda}$$

or $\lambda\alpha + (1-\lambda)\beta \geq \alpha^{\lambda}\beta^{1-\lambda} \quad \forall \alpha, \beta \geq 0$.

(7 marks)

(b) Let $\lambda = \frac{1}{p}$, $1 - \lambda = \frac{1}{q}$, then $0 < \lambda < 1$.

For each i, let $\alpha = a_i^p \geq 0$

$\beta = b_i^q \geq 0$

By (a)
$$\frac{1}{p}a_i^p + \frac{1}{q}b_i^q \geq (a_i^p)^{\frac{1}{p}}(b_i^q)^{\frac{1}{q}} = a_i b_i.$$

$$\therefore \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n}\left(\frac{1}{p}a_i^p + \frac{1}{q}b_i^q\right)$$

$$= \frac{1}{p}\sum_{i=1}^{n} a_i^p + \frac{1}{q}\sum_{i=1}^{n} b_i^q$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

(5 marks)

...ation

1. (b) (cont'd) If all x_i or all y_i are zero, the inequality is obvious.

Otherwise, let $a_i = \dfrac{x_i}{\left[\sum_{r=1}^{n} x_r^p\right]^{\frac{1}{p}}} \geqslant 0$

$$b_i = \frac{y_i}{\left[\sum_{r=1}^{n} y_r^q\right]^{\frac{1}{q}}} \geqslant 0 .$$

Then a_i , $b_i \geqslant 0$ and $\sum_{i=1}^{n} a_i^p = \sum_{i=1}^{n} b_i^q = 1$.

By (b) $\displaystyle\sum_{i=1}^{n} \frac{x_i}{\left[\sum_{r=1}^{n} x_r^p\right]^{\frac{1}{p}}} \times \frac{y_i}{\left[\sum_{r=1}^{n} y_r^q\right]^{\frac{1}{q}}} \leqslant 1$

i.e. $\displaystyle\sum_{i=1}^{n} x_i y_i \leqslant \left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} y_i^q\right]^{\frac{1}{q}}$.

(5 marks)

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2. (a) Since [illegible] r_3, r_4 are roots of $f(x) = 0$,

Let $f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$

$$\frac{df}{dx} = (x - r_1)\frac{d}{dx}\left[(x - r_2)(x - r_3)(x - r_4)\right] + (x - r_2)(x - r_3)(x - r_4)$$

$$= (x - r_1)\left\{(x - r_2)\frac{d}{dx}\left[(x - r_3)(x - r_4)\right] + (x - r_3)(x - r_4)\right\} + (x - r_2)(x - r_3)(x - r_4)$$

$$= (x - r_1)(x - r_2)(x - r_3) + (x - r_1)(x - r_2)(x - r_4) + (x - r_1)(x - r_3)(x - r_4) + (x - r_2)(x - r_3)(x - r_4)$$

$$= \sum_{i=1}^{4} \frac{f(x)}{x - r_i} \qquad \forall\, x \neq r_i \ (i = 1, 2, 3, 4)$$

(4 marks)

(b) Since $f(r_i) = 0$ for $i = 1, 2, 3, 4$,

$$\frac{f(x)}{x - r_i} = \frac{f(x) - f(r_i)}{x - r_i}$$

$$= \frac{(x^4 - r_i^4) + a_1(x^3 - r_i^3) + a_2(x^2 - r_i^2) + a_3(x - r_i)}{x - r_i}$$

$$= (x^3 + r_i x^2 + r_i^2 x + r_i^3) + a_1(x^2 + r_i x + r_i^2) + a_2(x + r_i) + a_3 \qquad \forall\, x \neq r_i$$

(5 marks)

(c) By (a) and (b), $f'(x) = \sum \frac{f(x)}{x - r_i}$

$$= \left[\sum_{i=1}^{4} x^3 + x^2\sum_{i=1}^{4} r_i + x\sum_{i=1}^{4} r_i^2 + \sum_{i=1}^{4} r_i^3\right] + \left[\sum_{i=1}^{4} a_1 x^2 + a_1 x\sum_{i=1}^{4} r_i + a_1\sum_{i=1}^{4} r_i^2\right] + \left[\sum_{i=1}^{4} a_2 x + a_2\sum_{i=1}^{4} r_i + \sum_{i=1}^{4} a_3\right]$$

$$= 4x^3 + (\sum r_i + \sum a_1)x^2 + (\sum r_i^2 + a_1\sum r_i + \sum a_2)x + (\sum r_i^3 + a_1\sum r_i^2 + a_2\sum r_i + \sum a_3)$$

$$= 4x^3 + (S_1 + 4a_1)x^2 + (S_2 + a_1 S_1 + 4a_2)x + (S_3 + a_1 S_2 + a_2 S_1 + 4a_3)$$

(4 marks)

2\. (d) Differentiating w.r.t. x, $f'(x) = 4x^3 + 3a_1x^2 + 2a_2x + a_3$

Comparing coefficients with $f'(x)$ in (c)

$$S_1 + 4a_1 = 3a_1$$

$$S_2 + a_1S_1 + 4a_2 = 2a_2$$

$$S_3 + a_1S_2 + a_2S_1 + 4a_3 = a_3$$

i.e. $S_1 + a_1 = 0$

$$S_2 + a_1S_1 + 2a_2 = 0$$

$$S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0 .$$

$\therefore$

Answer:

$$b_1 = a_1$$

$$b_2 = a_1$$

$$b_3 = 2a_2$$

$$b_4 = a_1$$

$$b_5 = a_2$$

$$b_6 = 3a_3$$

(4 marks)

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3\. (a) For $n = 1, 2, \ldots ,$

$$x_{2n+1} = \frac{p^2 + x_{2n}}{1 + x_{2n}}$$

$$= \frac{p^2 + \dfrac{p^2 + x_{2n-1}}{1 + x_{2n-1}}}{1 + \dfrac{p^2 + x_{2n-1}}{1 + x_{2n-1}}}$$

$$= \frac{2p^2 + x_{2n-1} + p^2x_{2n-1}}{1 + p^2 + 2x_{2n-1}}$$

(3 marks)

$$\therefore x_{2n-1} - x_{2n+1} = \frac{x_{2n-1} + p^2x_{2n-1} + 2x_{2n-1}^2 - 2p^2 - x_{2n-1} - p^2x_{2n-1}}{1 + p^2 + 2x_{2n-1}}$$

$$= \frac{2(x_{2n-1}^2 - p^2)}{1 + p^2 + 2x_{2n-1}} \quad \ldots\ldots\ldots\ldots \quad (*) \qquad 1$$

We shall next show inductively that $x_{2n-1} > p$ for $n = 1, 2, \ldots$

Now

$$x_{2k+1} - p = \frac{2p^2 + x_{2k-1} + p^2x_{2k-1} - p - p^3 - 2px_{2k-1}}{1 + p^2 + 2x_{2k-1}}$$

$$= \frac{(x_{2k-1} - p)(p - 1)^2}{1 + p^2 + 2x_{2k-1}} \quad \ldots\ldots\ldots\ldots \quad (**) \qquad 2$$

If $k = 1$, $x_1 > p$. 1

If $x_{2k-1} > p$ for some $k \geq 1$,

$x_{2k+1} > p$ by (**) 1+1

$\therefore$ $x_{2n-1} > p$ for $n = 1, 2, \ldots$.

and by (*) $x_{2n-1} > x_{2n+1}$ for $n = 1, 2, \ldots$.

(7 marks)

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3 (b) Since $y_1 > y_2 > \dots > y_{n-1} > y_n > \dots > p$,

$\{y_n\}$ is monotonic decreasing and bounded.

$\therefore \{y_n\}$ converges.

Let $\lim_{n\to\infty} y_n = y$

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{2p^2 + x_{2n-3} + p^2 x_{2n-3}}{1 + p^2 + 2x_{2n-3}}$$

$$= \frac{2p^2 + \lim_{n\to\infty} x_{2n-3} + p^2 \lim_{n\to\infty} x_{2n-3}}{1 + p^2 + 2 \lim_{n\to\infty} x_{2n-3}}$$

$$2y^2 + (1 + p^2)y = 2p^2 + y + p^2 y.$$

$$y = \pm p.$$

Since $y_n > p > 1 \quad \forall n$,

$\lim_{n\to\infty} y_n = p$.

(7 marks)

$$X_{2n+3} - X_{2n+1} = \frac{2p^2 + (1+p^2)X_{2n+1}}{(1+p^2) + 2X_{2n+1}} - \frac{2p^2 + (1+p^2)X_{2n-1}}{(1+p^2) + 2X_{2n-1}}$$

$$= \frac{(2p^2 + (1+p^2)X_{2n+1})((1+p^2) + 2X_{2n-1}) - ((1+p^2) + 2X_{2n+1})(2p^2 + (1+p^2)X_{2n-1})}{((1+p^2) + 2X_{2n+1})((1+p^2) + 2X_{2n-1})}$$

$$= \frac{(1+p^2)^2 (X_{2n+1} - X_{2n-1})}{((1+p^2) + 2X_{2n+1})(1+p^2 + 2X_{2n-1})}$$

4. (a) The probability that A gets k heads

$$= C_k^m \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{m-k}$$

$$= C_k^m \left(\frac{1}{2}\right)^m$$

$\therefore$ the probability that A and B both get k heads

$$= C_k^m \left(\frac{1}{2}\right)^m C_k^n \left(\frac{1}{2}\right)^n = C_k^m C_k^n \left(\frac{1}{2}\right)^{m+n}, \quad k \le n$$

Required probability $= \sum_{k=0}^{n} C_k^m C_k^n \left(\frac{1}{2}\right)^{m+n}$

But $(1 + x)^{m+n} = \sum_{i=0}^{m+n} C_i^{m+n} x^i$

$$(1 + x)^m (1 + x)^n = \left(\sum_{k=0}^{m} C_k^m x^k\right)\left(\sum_{k=0}^{n} C_k^n x^k\right)$$

Since $(1 + x)^{m+n} = (1 + x)^m (1 + x)^n$, by comparing coefficient of x^n,

$$C_n^{m+n} = \sum_{k=0}^{m} C_k^m C_{n-k}^n$$

$$= \sum_{k=0}^{n} C_k^m C_k^n$$

(8 marks)

(b) Let x_n be the probability that A wins,

y_n be the probability that A and B draw.

$$\frac{y_{n+1}}{y_n} = \frac{C_{n+1}^{2n+2}\left(\frac{1}{2}\right)^{2n+2}}{C_n^{2n}\left(\frac{1}{2}\right)^{2n}} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}\left(\frac{1}{2}\right)^2$$

$$= \frac{2n+1}{2n+2} < 1$$

$\therefore y_{n+1} < y_n$.

By symmetry, the probability that B wins = probability that A wins.

$$2x_n + y_n = 1.$$

Since $y_{n+1} < y_n$

$x_{n+1} > x_n$

For $n = 3$, $y_3 = \frac{5}{16}$

$x_3 = \frac{11}{32} > y_3$

5. (a) For any $A, B \subset X$,

$A, B \subset A \cup B \Rightarrow f[A], f[B] \subset f[A \cup B]$

and $f[A] \cup f[B] \subset f[A \cup B]$.

Next, for any y, $y \in f[A \cup B] \Rightarrow \exists x \in A \cup B$ s.t: $f(x) = y$
$\Rightarrow$ ($\exists x \in A$ s.t. $f(x) = y$) or ($\exists x \in B$ s.t. $f(x) = y$)

$\therefore y \in f[A] \cup f[B]$

and $f[A \cup B] \subset f[A] \cup f[B]$

Hence $f[A \cup B] = f[A] \cup f[B] \quad \forall A, B \subset X$.

(5 marks)

(b) Suppose $f[A \cap B] = f[A] \cap f[B] \quad \forall A, B \subset X$.

Let $A = \{x_1\}$ and $B = \{x_2\}$ be any two singletons in X.

Since $f[A] \cap f[B] = f[A \cap B]$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore f$ is injective.

(5 marks)

(c) Suppose f is bijective.

For any $A \subset X$, $y \in f[X \setminus A] \Rightarrow \exists x \in X \setminus A$ s.t. $y = f(x)$
$\Rightarrow y \notin f[A]$ (since f is injective)
$\Rightarrow y \in Y \setminus f[A]$.

$\therefore f[X \setminus A] \subset Y \setminus f[A]$.

For any y, $y \in Y \setminus f[A] \Rightarrow \exists x \in X$ s.t. $f(x) = y$ (since f is surjective)

Obviously $x \notin A$, $\therefore x \in X \setminus A$

i.e. $y \in f[X \setminus A]$

and $Y \setminus f[A] \subset f[X \setminus A]$.

Hence $f[X \setminus A] = Y \setminus f[A] \quad \forall A \subset X$.

(7 marks)

6. (a) Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} t & t \\ t & t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

$$\Leftrightarrow \begin{cases} t(x+y) = 2tx \\ t(x+y) = 2ty \\ t(z+w) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = y \\ z = -w \end{cases} \quad \forall t \neq 0$$

$\therefore X = \begin{pmatrix} r & r \\ s & -s \end{pmatrix}$, $r, s \in \mathbb{R}$.

Let $r = s = 1$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then $|Q| = -2 \neq 0$.

Q is non-singular and $\begin{pmatrix} t & t \\ t & t \end{pmatrix} = Q^{-1}\begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix}Q$.

(4 marks)

(b) Consider the mapping $f : G \to H$ such that

$$f\left(\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}\right) = Q^{-1}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}Q.$$

$$= \begin{pmatrix} \frac{t}{2} & \frac{t}{2} \\ \frac{t}{2} & \frac{t}{2} \end{pmatrix} \quad \text{for all } \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in G.$$

Clearly f is bijective.

Further, for all $A, B \in G$, $f(A)f(B) = (Q^{-1}AQ)(Q^{-1}BQ)$
$= Q^{-1}ABQ$
$= f(AB)$

Let E be a multiplicative identity in G, i.e.

$EA = AE = A \quad \forall A \in G$.

Since f is bijective, every matrix in H can be written as $f(A)$ for exactly one A in G. Then for any matrix $f(A)$ in H,

$f(E)f(A) = f(EA) = f(A)$
$f(A)f(E) = f(AE) = f(A)$

$\therefore f(E)$ is a multiplicative identity in H.

(7 marks)

6. (c) Multiplication is closed in G since $\begin{pmatrix} t_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} t_1t_2 & 0 \\ 0 & 0 \end{pmatrix} \in G$.

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity and $\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & 0 \end{pmatrix}$ is the inverse of $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}$,

$t \neq 0$. $\therefore$ G is a group.

Further, for any $f(A)$, $f(B)$ in H, $f(A)f(B) = f(AB) \in H$ since $AB \in G$

$\therefore$ multiplication is closed in H.

For any $f(A)$ in H, $A^{-1} \in G$ and $f(A)f(A^{-1}) = f(AA^{-1}) = f(E)$

$$f(A^{-1})f(A) = f(A^{-1}A) = f(E)$$

$\therefore$ $f(A)$ has an inverse.

Together with (b), H is a group.

(6 marks)

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7. (a) $A\underline{x} = \lambda \underline{x} \iff A\underline{x} = \lambda I \underline{x}$

$$\iff (A - \lambda I)\underline{x} = \underline{0}$$

This equation has a non-zero solution

iff $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda = 1 \text{ or } 2.$$

Let $\lambda_1 = 1$, $\lambda_2 = 2$.

(3 marks)

(b) Let $A\underline{x}_1 = \underline{x}_1$

$A\underline{x}_2 = 2\underline{x}_2$.

Then $AX = \begin{pmatrix} x_{11} & 2x_{12} \\ x_{21} & 2x_{22} \end{pmatrix}$.

But $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 3x_{11} - x_{21} & 3x_{12} - x_{22} \\ 2x_{11} & 2x_{12} \end{pmatrix}$.

$\therefore \begin{cases} 2x_{11} - x_{21} = 0 \\ x_{12} - x_{22} = 0 \end{cases}$

$|X| = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}$

$= x_{11}x_{22} - x_{12}x_{21}$

$= x_{11}x_{22} - 2x_{11}x_{22}$

$= -x_{11}x_{22}$

Now $|X| = 0 \Rightarrow x_{11} = 0$ or $x_{22} = 0$

$\Rightarrow x_{21} = 0$ or $x_{12} = 0$

$\Rightarrow \underline{x}_1 = \underline{0}$ or $\underline{x}_2 = \underline{0}$, which is not true.

$\therefore$ X is non-singular

7 marks

7. (c) (i)

$$A\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix}=\lambda_1\begin{pmatrix}x_{11}\\x_{21}\end{pmatrix}=\begin{pmatrix}\lambda_1x_{11}\\\lambda_1x_{21}\end{pmatrix}$$

$$A\begin{pmatrix}x_{12}\\x_{22}\end{pmatrix}=\begin{pmatrix}\lambda_2x_{12}\\\lambda_2x_{22}\end{pmatrix}$$

$$\therefore\ A\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}=\begin{pmatrix}\lambda_1x_{11}&\lambda_2x_{12}\\\lambda_1x_{21}&\lambda_2x_{22}\end{pmatrix}$$

$$=\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix}\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$$

$$A=X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}X^{-1}$$

$$A^n=\left[X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}X^{-1}\right]\left[X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}X^{-1}\right]\cdots\left[X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}X^{-1}\right]\ n \text{ times}$$

$$=X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}^nX^{-1}$$

$$=X\begin{pmatrix}\lambda_1^n&0\\0&\lambda_2^n\end{pmatrix}X^{-1}$$

$Ax=\begin{pmatrix}3&-1\\2&0\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\2x_{11}&x_{12}\end{pmatrix}=\begin{pmatrix}x_{11}&2x_{12}\\2x_{11}&2x_{12}\end{pmatrix}$; $X\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}=\begin{pmatrix}x_{11}&x_{12}\\2x_{11}&x_{12}\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}=\begin{pmatrix}x_{11}&2x_{12}\\2x_{11}&2x_{12}\end{pmatrix}$

(ii) By (a), let $\lambda_1=1$, $\begin{cases}2x_1-x_2=0\\2x_1-x_2=0\end{cases}$

$\begin{pmatrix}1\\2\end{pmatrix}$ is a solution of $A\underline{x}=1\underline{x}$.

Let $\lambda_2=2$, $\begin{cases}x_1-x_2=0\\2x_1-2x_2=0\end{cases}$

$\begin{pmatrix}1\\1\end{pmatrix}$ is a solution of $A\underline{x}=2\underline{x}$.

$$\therefore\ \begin{pmatrix}3&-1\\2&0\end{pmatrix}^n=\begin{pmatrix}1&1\\2&1\end{pmatrix}\begin{pmatrix}1^n&0\\0&2^n\end{pmatrix}\begin{pmatrix}1&1\\2&1\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}2^{n+1}-1&1-2^n\\2^{n+1}-2&2-2^n\end{pmatrix}$$

(7 marks)

$u=\bar{u}$, ... $\bar{v}$ ... $\Rightarrow\ \frac{1}{u}+\frac{1}{v}+1=0$

$u+v+1=0\Rightarrow\ \overline{u+v+1}=0$

$\Rightarrow\ \bar{u}+\bar{v}+1=0$

$\therefore\ \frac{1}{u}+\frac{1}{v}+1=\bar{u}+\bar{v}+1=0.$

If $\frac{1}{u}+\frac{1}{v}+1=0$, $\frac{u+v+uv}{uv}=0$ $u+v+1=0$ and $\frac{1}{u}+\frac{1}{v}+1=0$ $\Rightarrow |u|=|v|=1$. (3 marks)

$\Rightarrow\ u+v+uv=0$

$\Rightarrow\ u=-v(1+u)$

$\Rightarrow\ u=-v(-v)$

$=v^2$ (since $1+u+v=0$)

$\therefore\ v^3=uv$

$=-(u+v)$

$=+1$

$\Rightarrow\ |v|=1$

and hence $|u|=1$

$\therefore\ |u|=|v|=1$ iff $\frac{1}{u}+\frac{1}{v}+1=0$ (3 marks)

Next, $u+v+1=0\ \Rightarrow\ u^2+v^2+1+2(u+v+uv)=0$.

If $\frac{1}{u}+\frac{1}{v}+1=0$, $\frac{u+v+uv}{uv}=0\Rightarrow u^2+v^2+1=0$

If $u^2+v^2+1=0$, $u+v+uv=0\Rightarrow\frac{1}{u}+\frac{1}{v}+1=0$.

$\therefore\ u^2+v^2+1=0\Leftrightarrow\frac{1}{u}+\frac{1}{v}+1=0\Leftrightarrow|u|=|v|=1$. (4 marks)

(b) The three sides AB, BC and CA are, respectively,

$|z_2-z_1|$, $|z_3-z_2|$ and $|z_1-z_3|$.

Consider the two complex numbers $u=\frac{z_2-z_1}{z_1-z_3}$, $v=\frac{z_3-z_2}{z_1-z_3}$.

$$u+v+1=\frac{z_2-z_1}{z_1-z_3}+\frac{z_3-z_2}{z_1-z_3}+1$$

$=0$

$$\therefore\ \left|\frac{z_2-z_1}{z_1-z_3}\right|=\left|\frac{z_3-z_2}{z_1-z_3}\right|=1$$

$$\Longleftrightarrow\ \left(\frac{z_2-z_1}{z_1-z_3}\right)^2+\left(\frac{z_3-z_2}{z_1-z_3}\right)^2+1=0$$

$$\Longleftrightarrow\ (z_2-z_1)^2+(z_3-z_2)^2+(z_1-z_3)^2=0$$

$$\Longleftrightarrow\ z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2$$

i.e. $|z_2-z_1|=|z_3-z_2|=|z_1-z_3|$

iff $z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2$

(7 marks)

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Pure Mathematics II

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1. (a) Putting $y = mx + c$ in (E),

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$$

$$\frac{a^2m^2 + b^2}{a^2b^2}x^2 + \frac{2mc}{b^2}x + \frac{c^2 - b^2}{b^2} = 0$$

Condition for tangency is

$$\frac{4m^2c^2}{b^4} - 4\left(\frac{a^2m^2 + b^2}{a^2b^2}\right)\left(\frac{c^2 - b^2}{b^2}\right) = 0$$

i.e. $a^2m^2 + b^2 - c^2 = 0$.

Since $P(h, k)$ lies on the tangent,

$$c = k - mh.$$

$$a^2m^2 + b^2 - (k - mh)^2 = 0$$

or $(a^2 - h^2)m^2 + 2hkm + (b^2 - k^2) = 0$ .... (1)

(6 marks)

(b) The tangents of the angles between tangents from P to the line $y = nx$ are given by

$$\frac{m_1 - n}{1 + m_1n}$$

$$\frac{m_2 - n}{1 + m_2n}.$$

If these angles are equal $\dfrac{m_1 - n}{1 + m_1n} = \dfrac{n - m_2}{1 + m_2n}$.

Expanding $(1 - n^2)(m_1 + m_2) + 2n(m_1m_2 - 1) = 0$ .... (2)

(5 marks)

(c) From (1), $m_1 + m_2 = \dfrac{-2hk}{a^2 - h^2}$

$$m_1m_2 = \frac{b^2 - k^2}{a^2 - h^2}$$

Sub. in (2), $(1 - n^2)\left(\dfrac{-2hk}{a^2 - h^2}\right) + 2n\left(\dfrac{b^2 - k^2}{a^2 - h^2} - 1\right) = 0$

$$-2hk(1 - n^2) + 2n(b^2 - a^2 + h^2 - k^2) = 0$$

$$2nh^2 - 2nk^2 - 2(1 - n^2)hk + 2n(b^2 - a^2) = 0$$

$\therefore$ equation of locus of P is

$$nx^2 - ny^2 + (n^2 - 1)xy - n(a^2 - b^2) = 0.$$

(6 marks)

$$\cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta$$

or $\cos[(n+1)\cos^{-1}x] + \cos[(n-1)\cos^{-1}x] = 2\cos(n\cos^{-1}x)\cos(\cos^{-1}x) = 2x\cos(n\cos^{-1}x)$

$$= \cos n\theta\cos\theta - [\sin(n-1)\theta\cos\theta + \cos(n-1)\theta\sin\theta]\sin\theta$$

$$= \cos n\theta\cos\theta - \sin(n-1)\theta\cos\theta\sin\theta - \cos(n-1)\theta(1-\cos^2\theta)$$

$$= \cos n\theta\cos\theta - \cos(n-1)\theta - [\sin(n-1)\theta\sin\theta - \cos(n-1)\theta\cos\theta]\cos\theta$$

$$= \cos n\theta\cos\theta - \cos(n-1)\theta + \cos n\theta\cos\theta$$

$$= 2\cos n\theta\cos\theta - \cos(n-1)\theta\,.$$

$\therefore\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ (4 marks)

from def $\rightarrow T_0(x) = 1$ and $T_1(x) = x \leftarrow$ from def

$\therefore\ T_2(x) = 2x^2 - 1$ is a polynomial of degree 2 with leading coefficient 2^1. (from above)

Since $T_{k+2}(x) = 2x\,T_{k+1}(x) - T_k(x)$, if, for $1 \le n \le k$, $T_{n+1}(x)$ is a polynomial of degree $k+1$ with leading coefficient 2^k, then $T_{k+2}(x)$ is a polynomial of degree $k+2$ with leading coefficient 2^{k+1}.

$\therefore\ T_n(x)$ is a polynomial of degree n with leading coefficient $2^{n-1}\ \forall\, n \ge 1$.

(4 marks)

(b) $\cos\theta = \frac{1}{2}\left[(\cos\theta + i\sin\theta) + (\cos-\theta + i\sin-\theta)\right]$

$$\therefore\ \cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n} C_k^n (\mathrm{cis}\,\theta)^k(\mathrm{cis}-\theta)^{n-k}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} C_k^n (\mathrm{cis}\,k\theta)(\mathrm{cis}(k-n)\theta)$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} C_k^n\,\mathrm{cis}(2k-n)\theta$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} C_k^n \cos(2k-n)\theta \quad (\text{since imaginary part} = 0)$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} C_k^n \cos(n-2k)\theta$$

$$\therefore\ a_k = \frac{1}{2^n}C_k^n\,.$$

(6 marks)

2. (c) Since $C_k^n\cos(n-2k) = C_{n-k}^n\cos(2k-n)$, (by putting $\theta = \cos^{-1}x$)

by (b), $x^n = \dfrac{1}{2^n}\displaystyle\sum_{k=0}^{n} C_k^n \cos[(n-2k)\cos^{-1}x]$

$$= \frac{1}{2^n}\sum_{k=0}^{\frac{n-1}{2}} 2C_k^n\cos[(n-2k)\cos^{-1}x]$$

$$= \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}} C_k^n\,T_{n-2k}(x)\,,\quad n = 1, 3, 5, \ldots$$

3 marks

put $j = n-k$, $k = n-j$.

$k = \frac{n+1}{2}$, $j = \frac{n-1}{2}$

$k = n$, $j = 0$

$$\sum_{k=\frac{n+1}{2}}^{n} {}^nC_k\cos[(n-2k)\cos^{-1}x]$$

$$= \sum_{j=\frac{n-1}{2}}^{0} {}^nC_{n-j}\cos[(n-2(n-j))\cos^{-1}x]$$

$$= \sum_{j=0}^{\frac{n-1}{2}} {}^nC_j\cos[(-n+2j)\cos^{-1}x]$$

$$= \sum_{k=0}^{\frac{n-1}{2}} {}^nC_k\cos[(n-2k)\cos^{-1}x]$$

$$x^n = \frac{1}{2^n}\sum_{k=0}^{n} {}^nC_k\cos[(n-2k)\cos^{-1}x]$$

$$= \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}} {}^nC_k\cos[(n-2k)\cos^{-1}x]$$

$$\therefore\ x^n = \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}} {}^nC_k\,T_{n-2k}(x).$$

Since $\lim_{h\to 0} f(x_0) = f(x_0)$ and $f(x_0 + h) = f(x_0) + f(h)$,

$$\lim_{h\to 0} f(h) = \lim_{h\to 0} f(x_0 + h) - \lim_{h\to 0} f(x_0)$$
$$= 0 .$$

$$\therefore \quad f(x) = \lim_{h\to 0} f(x) + \lim_{h\to 0} f(h)$$
$$= \lim_{h\to 0}\left[f(x) + f(h)\right]$$
$$= \lim_{h\to 0} f(x + h) \qquad \forall\ x \in \mathbb{R}.$$

i.e. f is cont. at every $x \in \mathbb{R}$.

(5 marks)

(b) We shall first induce on n for $n \geqslant 0$.

For $n = 0$, $f(0) = f(0 + 0)$
$= f(0) + f(0)$
$f(0) = 0$.

Assume $f(kx) = kf(x)$ for some $k \geqslant 0$.

Then $f((k + 1)x) = f(kx) + f(x)$
$= kf(x) + f(x)$
$= (k + 1)f(x)$.

Hence $f(nx) = nf(x) \qquad \forall\ n \geqslant 0.$

Next $f(x) + f(-x) = f(x - x) = 0 \qquad \forall\ x$
$\Rightarrow f(-x) = -f(x)$.

$\therefore$ if $n < 0$, $f(nx) = -nf(-x) = nf(x)$.

Let $r = \frac{p}{q}$, where $p, q \in Z$, $q \neq 0$.

By (b), $qf\left(\frac{p}{q}\right) = f\left(q\,\frac{p}{q}\right)$
$= f(p)$
$= pf(1)$
$\therefore \quad f(r) = f(1) \times r$

(8 marks)

(c) For any $x \in \mathbb{R}$, let $\{a_n\}$ be a sequence of rational numbers such that $\lim_{n\to\infty} a_n = x$.

As f is continuous,

$$f(x) = \lim_{n\to\infty} f(a_n)$$
$$= \lim_{n\to\infty} f(1) \times a_n$$
$$= f(1) \lim_{n\to\infty} a_n$$
$$= f(1) \cdot x .$$

The answer follows where $k = f(1)$.

(4 marks)

4. (a) For $n = 0$,

$$\text{R.S.} = I_0 = \int_0^1 e^t\,dt$$
$$= e^t\Big|_0^1 = e - 1$$

L.S. $= (-1) + e =$ R.S.

Assume that $I_k = (-1)^{k+1} k! + e \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!}$ for some $k \geqslant 0$,

$$I_{k+1} = \int_0^1 e^t t^{k+1}\,dt$$

Put $u = t^{k+1}$, $dv = e^t dt$
$du = (k+1)t^k$, $v = e^t$

$$I_{k+1} = e^t t^{k+1}\Big|_0^1 - (k+1)\int_0^1 t^k e^t dt$$
$$= e - (k+1)I_k$$
$$= e - \left[(-1)^{k+1} k! - e \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!}\right](k+1)$$
$$= e + (-1)^{k+2}(k+1)! - e(k+1) \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!}$$
$$= (-1)^{k+2}(k+1)! - e\sum_{i=0}^{k} (-1)^i \frac{(k+1)!}{(k-i)!} + e$$
$$= (-1)^{k+2}(k+1)! + e\sum_{j=1}^{k+1} (-1)^j \frac{(k+1)!}{(k+1-j)!} + e , \quad \text{where } i = j-1$$
$$= (-1)^{k+2}(k+1)!\ e\sum_{j=0}^{k+1} (-1)^j \frac{(k+1)!}{(k+1-j)!}$$

$$\therefore \quad I_n = (-1)^{n+1} n! + e \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} \qquad \forall\ n \geqslant 0 .$$

6 marks

(b) For $0 \leqslant t \leqslant 1$, $n \geqslant 1$, $t^n \leqslant e^t t^n \leqslant et^n$

$$\therefore \quad I_n = \int_0^1 e^t t^n dt \leqslant \int_0^1 et^n dt$$
$$= e\frac{t^{n+1}}{n+1}\Bigg|_0^1$$
$$= \frac{e}{n+1}$$
$$< \frac{e}{n}$$

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4. (b) Also $I_n \geq \int_0^1 t^n dt$

$$= \left.\frac{t^{n+1}}{n+1}\right|_0^1$$

$$= \frac{1}{n+1}$$

i.e. $\frac{1}{n+1} \leq I_n \leq \int_0^1 e t^n dt < \frac{e}{n}$

4 marks

(c) By (a) and (b) $\frac{1}{n+1} \leq (-1)^{n+1} n! + e \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} < \frac{e}{n}$

Assume, for contradiction, that $e = \frac{p}{q}$, where $p, q \geq 1$.

Then $\frac{1}{n+1} \leq (-1)^{n+1} n! + \frac{p}{q} \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} < \frac{p}{qn} \quad \forall n \geq 1.$

$$\frac{q}{n+1} \leq (-1)^{n+1} n!\, q + p \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} < \frac{p}{n} \quad \forall n \geq 1.$$

Since $(-1)^{n+1} n!\, q + p \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!}$ is an integer

and $\frac{q}{n+1} > 0$,

taking n_0 to be greater than p, $\because n_0 > p \quad \frac{p}{n_0} < 1$

than $0 < \frac{q}{n_0+1} \leq (-1)^{n_0+1} n_0!\, q + p \sum_{i=0}^{n_0} (-1)^i \frac{n_0!}{(n_0-i)!} < \frac{p}{n_0} < 1$,

which is a contradiction.

$\therefore$ e cannot be a rational number.

7 marks

5. (a) Suppose $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{1}{2}\left[\frac{f(x+h) - f(x)}{h} + \frac{f(x-h) - f(x)}{-h}\right].$$

$$\lim_{h \to 0} \frac{1}{2}\left[\frac{f(x+h) - f(x)}{h} + \frac{f(x-h) - f(x)}{-h}\right]$$

$$= \frac{1}{2}\left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}\right]$$

$$= f'(x)$$

$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$ exists.

(4 marks)

(b) Put $x = 0$.

$$\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \lim_{h \to 0} \frac{h \sin\frac{1}{h} + h \sin\frac{1}{-h}}{2h}$$

$$= \lim_{h \to 0} \frac{0}{2h} = 0$$

Hence $F_s(0)$ exists.

$$\frac{F(0+h) - F(0)}{h} = \frac{h \sin\frac{1}{h} - 0}{h}$$

$$= \sin\frac{1}{h}.$$

$$= \begin{cases} 1 & \text{if } \frac{1}{h} = \frac{(4n+1)\pi}{2} \\ -1 & \text{if } \frac{1}{h} = \frac{(4n+3)\pi}{2} \end{cases}$$

$\therefore \lim_{h \to 0} \sin\frac{1}{h}$ does not exist.

and F is not differentiable at $x = 0$.

(7 marks)

(c) $\frac{(f+g)(x+h) - (f+g)(x-h)}{2h}$

$$= \frac{f(x+h) - f(x-h) + g(x+h) - g(x-h)}{2h}$$

Since $f_s(x)$ and $g_s(x)$ exist,

$\lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x-h)}{2h}$ exists.

5. (c) and is equal to $\lim_{h\to 0}\dfrac{f(x+h)-f(x-h)}{2h}+\lim_{h\to 0}\dfrac{g(x+h)-g(x-h)}{2h}$

i.e. $(f+g)_s(x) = f_s(x) + g_s(x)$

$$\frac{(fg)(x+h)-(fg)(x-h)}{2h} = \frac{f(x+h)\,g(x+h)-f(x-h)\,g(x-h)}{2h}$$

$$= g(x+h)\left[\frac{f(x+h)-f(x-h)}{2h}\right] + f(x-h)\left[\frac{g(x+h)-g(x-h)}{2h}\right]$$

Since $f_s(x)$ and $g_s(x)$ exist and f, g are continuous at x,

$(fg)_s(x) = g(x)f_s(x) + f(x)g_s(x)$.

(6 marks)

6. (b) (i) For $k = 0, 1, 2, \ldots (n-1)$, let $f(x) = x^k$.

By (a)(ii)

$$\int_{-1}^{1} P_n(x)\, x^k\, dx = (-1)^k \int_{-1}^{1}\left[\frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n\right]\frac{d^k}{dx^k} f(x)dx$$

$$= (-1)^k (k!) \int_{-1}^{1} \frac{d^{n-k}}{dx^{n-k}}(x^2-1)^n\, dx$$

$= 0$ by (a)(i)

(3 marks)

(ii) $P_m(x)$ is a polynomial of degree m.

Let $P_m(x) = a_0x^m + a_1x^{m-1} + \ldots + a_m$.

$$\therefore I = \int_{-1}^{1} P_n(x)P_m(x)dx = \int_{-1}^{1} P_n(x)\left[a_0x^m + a_1x^{m-1} + \ldots + a_m\right]dx$$

$$= \sum_{j=0}^{m} \int_{-1}^{1} P_n(x)a_j x^{m-j}\, dx$$

If $m < n$, this is equal to zero by (b)(ii).

The case for $m > n$ is similar.

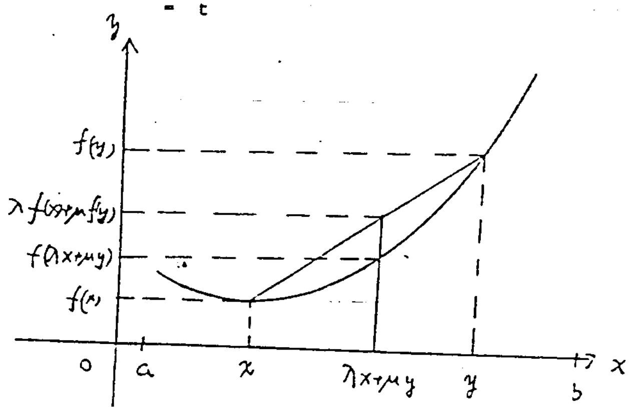
(6 marks)

7. (a) For $x < t < y$, let

$$\lambda = \frac{y-t}{y-x}, \quad \mu = \frac{t-x}{y-x}.$$

Then $\lambda + \mu = 1$ and $\lambda, \mu \in (0, 1)$.

$$\lambda x + \mu y = \frac{y-t}{y-x}x + \frac{t-x}{y-x}y$$
$$= t$$



4 marks

(b) Let x, t, y be in (a, b) such that $x < t < y$.

By (a),
$$f(t) = f(\lambda x + \mu y)$$
$$\leqslant \lambda f(x) + \mu f(y)$$
$$= \frac{y-t}{y-x}f(x) + \frac{t-x}{y-x}f(y)$$

$(y-x)f(t) \leqslant (y-t)f(x) + (t-x)f(y)$ $\quad(\because y - x > 0)$

$(y-t)f(t) - (y-t)f(x) \leqslant (t-x)f(y) - (x-t)f(t)$

$\therefore \dfrac{f(t)-f(x)}{t-x} \leqslant \dfrac{f(y)-f(t)}{y-t}$ $\quad$ (As $t - x, y - t > 0$)

4 marks

(c) $h(t) = \lambda g(t) + \mu g(y) - g(\lambda t + \mu y)$

$h'(t) = \lambda g'(t) - \lambda g'(\lambda t + \mu y)$

Since $g'' \geqslant 0$ on (a, b), g' is increasing

$\therefore \lambda t + \mu y \geqslant t \Rightarrow g'(\lambda t + \mu y) \geqslant g'(t)$

$\therefore h'(t) \leqslant 0$

i.e. h is monotonic decreasing.

Now
$$h(y) = \lambda g(y) + \mu g(y) - g(\lambda y + \mu y)$$
$$= (\lambda + \mu)g(y) - g((\lambda + \mu)y)$$
$$= 0 \quad \text{as } \lambda + \mu = 1.$$

$\therefore \lambda g(t) + \mu g(y) - g(\lambda t + \mu y) \leqslant 0$

i.e. g is convex.

6 marks

(d) Let $g(x) = x^p$ in (c), where $p > 1$, $x > 0$.

$g'(x) = px^{p-1}$

$g''(x) = p(p-1)x^{p-2} \geqslant 0$

$\therefore g$ is convex.

By (a) $\quad g(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 g(x_1) + \lambda_2 g(x_2)$

i.e. $\quad (\lambda_1 x_1 + \lambda_2 x_2)^p \leqslant \lambda_1 x_1^p + \lambda_2 x_2^p$.

3 marks

(a) Let $P = \left(\frac{p}{\sqrt{2}}, \frac{p}{\sqrt{2}}\right)$

$Q = \left(\frac{1}{p\sqrt{2}}, -\frac{1}{p\sqrt{2}}\right)$

$P' = \left(\frac{p'}{\sqrt{2}}, \frac{p'}{\sqrt{2}}\right)$

$Q' = \left(\frac{1}{p'\sqrt{2}}, -\frac{1}{p'\sqrt{2}}\right)$

Equation of PQ is $y - \frac{p}{\sqrt{2}} = \frac{p^2+1}{p^2-1}\left(x - \frac{p}{\sqrt{2}}\right)$

Similarly, equation of P'Q' is $y - \frac{p'}{\sqrt{2}} = \frac{p'^2+1}{p'^2-1}\left(x - \frac{p'}{\sqrt{2}}\right)$

Solving the above

$$u = \frac{pp'+1}{\sqrt{2}(p+p')}$$

$$v = \frac{pp'-1}{\sqrt{2}(p+p')}$$

$$\therefore \lim_{p' \to p} u = \frac{p^2+1}{2\sqrt{2}\,p}$$

$$\lim_{p' \to p} v = \frac{p^2-1}{2\sqrt{2}\,p}$$

7 marks

(b) Eliminating p from $\begin{cases} x = \frac{p^2+1}{2\sqrt{2}\,p} \\ y = \frac{p^2-1}{2\sqrt{2}\,p} \end{cases}$

$$x^2 - y^2 = \frac{1}{8p^2}[(p^2+1)^2 - (p^2-1)^2]$$

$$= \frac{1}{2}$$

Since $(\xi(p), \eta(p))$ lies on the first and fourth quadrants, its locus is a brance of the hyperbola.

$$(H) : \frac{x^2}{\frac{1}{2}} - \frac{y^2}{\frac{1}{2}} = 1$$

3 marks

8. (c) Let $OA = a$, $OB = b$.

Equation of AB is $y - \frac{a}{\sqrt{2}} = \frac{a+b}{a-b}\left(x - \frac{a}{\sqrt{2}}\right)$.

Substituting $x = \left(y - \frac{a}{\sqrt{2}}\right)\left(\frac{a-b}{a+b}\right) + \frac{a}{\sqrt{2}}$ in H

$$\left[\left(y - \frac{a}{\sqrt{2}}\right)\left(\frac{a-b}{a+b}\right) + \frac{a}{\sqrt{2}}\right]^2 - y^2 = \frac{1}{2}$$

$$\frac{-8ab}{(a+b)^2}y^2 + \frac{(a-b)8ab}{(a+b)^2}y + \frac{4a^2b^2 - (a+b)^2}{(a+b)^2} = 0$$

The discriminant $= \frac{(a-b)^2 64a^2b^2}{(a+b)^4} + \frac{32ab}{(a+b)^2}\cdot\frac{4a^2b^2 - (a+b)^2}{(a+b)^2}$

$$= \frac{32ab}{(a+b)^4}[(a-b)^2ab + 4a^2b^2 - (a+b)^2]$$

$$= \frac{32ab}{(a+b)^4}(ab-1)(a+b)^2$$

Since the discriminant $\begin{smallmatrix}<\\=\\>\end{smallmatrix} 0$ according as $ab \begin{smallmatrix}<\\=\\>\end{smallmatrix} 1$, the line AB meets (H) at no point, one point, or 2 points according as $ab \begin{smallmatrix}<\\=\\>\end{smallmatrix} 1$.

7 marks