7. Suppose that the function $f(x)$ is continuous for all $x \geq 0$ and $\lim_{x \to \infty} f(x) = \ell$ exists.

(a) Show that
$$\int_a^b \frac{f(rx) - f(sx)}{x}\,dx = \int_{ra}^{sa} \frac{f(x)}{x}\,dx - \int_{rb}^{sb} \frac{f(x)}{x}\,dx ,$$
where $0 < a < b$ and $0 < r < s$.

(b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers such that $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = +\infty$. Show that
$$\lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \left(f(0) - \ell\right) \log_e \left(\frac{s}{r}\right) .$$

[Hint : you may assume without proof that the following theorem holds :

If $g(x)$ and $h(x)$ are continuous on the closed interval $[c, d]$ and $h(x) > 0$ for all x in $[c, d]$, then
$$\int_c^d g(x)h(x)\,dx = g(x_0)\int_c^d h(x)\,dx$$
for some x_0 in $[c, d]$.]

(c) Furthermore, if $f(x) \geq c$ for all $x \geq 0$, where c is a positive constant, is it then true that
$$\lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx)}{x}\,dx - \lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(sx)}{x}\,dx \ ?$$
Prove your assertion.

8. Suppose that $f(x) = \int_1^x \frac{1}{\sqrt{1 + t^5}}\,dt$ for $x \geq 1$.

(a) (i) Show that $f(x) < f(y)$ whenever $1 \leq x < y$.

(ii) Show that $f(x) < \frac{2}{3}$ for all $x \geq 1$.

(iii) Find an x_0 such that $f(x_0) > \frac{1}{3}$.

(b) Let $g(u)$, where $0 \leq u < f(x_0)$, be a function such that $f\left(g(u)\right) = u$.

(i) Show that $g'(u) = [1 + g^5(u)]^{\frac{1}{2}}$ and $g''(u) = \frac{5}{2}\left(g(u)\right)^4$.

(ii) Let $h(u) = e^u - g(u)$, where $0 \leq u < f(x_0)$. Prove that $h''(u) < h'(u)$.

Hence prove that $h(u)$ does not have a minimum.

END OF PAPER

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八二年香港高級程度會考

HONG KONG ADVANCED LEVEL EXAMINATION 1982

純數學 PURE MATHEMATICS

試卷一 PAPER I

三小時完卷 Three hours

上午九時至正午十二時 9.00 a.m.—12.00 noon

本試卷必須用英文作答 This paper must be answered in English

This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

1. (a) Let $0 < \lambda < 1$. Show that
$$\lambda t + (1 - \lambda) \geq t^{\lambda} \quad \text{for all} \quad t > 0 .$$
Deduce that
$$\lambda\alpha + (1 - \lambda)\beta \geq \alpha^{\lambda}\beta^{1-\lambda} \quad \text{for all} \quad \alpha, \beta > 0 .$$

(b) Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of non-negative real numbers such that $\sum_{i=1}^{n} a_i^{\,p} = \sum_{i=1}^{n} b_i^{\,q} = 1$. Using the result in (a), show that $\sum_{i=1}^{n} a_i b_i \leq 1$.

Hence show that, for any two sets of non-negative real numbers $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$,
$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^{\,p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^{\,q}\right)^{\frac{1}{q}} .$$

2. Let $f(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ be a real polynomial and $f'(x)$ be its derivative. Suppose r_1, r_2, r_3 and r_4 are the four real roots of $f(x) = 0$. For $p = 1, 2$ and 3, S_p is defined by $S_p = \sum_{i=1}^{4} r_i^p$.

(a) Show that $f'(x) = \sum_{i=1}^{4} \frac{f(x)}{x - r_i}$ for all $x \neq r_1, r_2, r_3, r_4$.

(b) Show that, for $i = 1, 2, 3$, or 4,
$$\frac{f(x)}{x - r_i} = (x^3 + r_ix^2 + r_i^2x + r_i^3) + a_1(x^2 + r_ix + r_i^2) + a_2(x + r_i) + a_3$$
for all $x \neq r_i$.

(c) Using (a) and (b), show that
$$f'(x) = 4x^3 + (S_1 + 4a_1)x^2 + (S_2 + S_1a_1 + 4a_2)x + (S_3 + S_2a_1 + S_1a_2 + 4a_3).$$

(d) Find $b_j\ (j = 1, 2, \ldots, 6)$ in terms of $a_i\ (i = 1, 2, 3, 4)$ such that
$$S_1 + b_1 = 0$$
$$S_2 + b_2S_1 + b_3 = 0$$
$$S_3 + b_4S_2 + b_5S_1 + b_6 = 0.$$

3. Let p be a real number greater than 1. $\{x_n\}$ is a given sequence such that

(1) $x_1 > p$,

and (2) $x_{n+1} = \dfrac{p^2 + x_n}{1 + x_n}$ for $n = 1, 2, \ldots$.

(a) Express x_{2n+1} in terms of x_{2n-1} and show that $x_{2n+1} < x_{2n-1}$ for $n = 1, 2, \ldots$.

(b) Let $y_n = x_{2n-1}$, $n = 1, 2, \ldots$.
Show that $\{y_n\}$ converges and find its limit.

[Hint: you may use the fact that every monotonic decreasing sequence which is bounded below is convergent.]

4. (a) A tosses m fair coins followed by B with $n < m$ coins. Show that the probability that they get the same number of heads is $C_n^{m+n}\left(\frac{1}{2}\right)^{m+n}$.

(b) A and B play a game in which they both toss n coins. The one who gets more heads wins. If they get the same number of heads, they draw. Show that the more coins they toss, the less likely it is that they will draw.

Hence, or otherwise, show that if $n > 3$, it is more likely for A to win than for A and B to draw.

5. Let $f: X \to Y$ be a mapping from a set X into a set Y. For any subset S of X, the direct image of S under f is defined by
$$f[S] = \{y : y = f(x) \text{ for some } x \in S\}.$$

(a) Show that $f[A \cup B] = f[A] \cup f[B]$ for all subsets A and B of X.

(b) Show that if $f[A \cap B] = f[A] \cap f[B]$ for all subsets A and B of X, then f is injective.

(c) Show that $f[X \setminus A] = Y \setminus f[A]$ for all subsets A of X if f is bijective.

6. Given two sets of 2×2 matrices
$$G = \left\{\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} : t \in \mathbf{R} \setminus \{0\}\right\}$$
and
$$H = \left\{\begin{pmatrix} t & t \\ t & t \end{pmatrix} : t \in \mathbf{R} \setminus \{0\}\right\}.$$

(a) Find all the 2×2 matrices X such that
$$X\begin{pmatrix} t & t \\ t & t \end{pmatrix} = \begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix}X \quad \text{for all } t \in \mathbf{R} \setminus \{0\}.$$
Hence find a non-singular matrix Q such that
$$\begin{pmatrix} t & t \\ t & t \end{pmatrix} = Q^{-1}\begin{pmatrix} 2t & 0 \\ 0 & 0 \end{pmatrix}Q.$$

(b) Using (a), find a bijective mapping $f : G \to H$ such that
$f(AB) = f(A)f(B)$ for all A and B in G.
Show that if E is a multiplicative identity in G, $f(E)$ is a multiplicative identity in H.

(c) Show that G is a group under matrix multiplication.
Hence, using (b), show that H is also a group under matrix multiplication.

7. Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and let $\underset{\sim}{x}$ denote a 2×1 matrix.

(a) Find the two real values λ_1 and λ_2 of λ with $\lambda_1 < \lambda_2$ such that the matrix equation

(1) $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$

has non-zero solutions.

(b) Let $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ be non-zero solutions of (1) corresponding to λ_1 and λ_2, respectively. Show that if $\underset{\sim}{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ and $\underset{\sim}{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$, then the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is non-singular.

(c) Using (a) and (b), show that

$$AX = X\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and hence

$$A^n = X\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}X^{-1} ,$$

where n is a positive integer.

Evaluate $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}^n$.

8. (a) Suppose u and v are two non-zero complex numbers such that

$$u + v + 1 = 0 .$$

Show that

$$|u| = |v| = 1$$

if and only if

$$\frac{1}{u} + \frac{1}{v} + 1 = 0 .$$

Hence, or otherwise, show that

$$|u| = |v| = 1$$

if and only if

$$u^2 + v^2 + 1 = 0 .$$

(b) Let A, B and C be three distinct points on the complex plane representing the complex numbers z_1, z_2 and z_3, respectively.

Using the second result of (a), show that ABC is an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2 .$$

END OF PAPER

香 港 考 試 局

HONG KONG EXAMINATIONS AUTHORITY

一 九 八 二 年 香 港 高 級 程 度 會 考

HONG KONG ADVANCED LEVEL EXAMINATION 1982

純數學 PURE MATHEMATICS
試卷二 PAPER II

三小時完卷 Three hours

下午二時至下午五時 2.00 p.m.—5.00 p.m.

本試卷必須用英文作答 This paper must be answered in English

This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

1. Given an ellipse (E) : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(a) P is a point (h, k). If $y = mx + c$ is the equation of a tangent drawn from P to (E), show that

$$(a^2 - h^2)m^2 + 2hkm + (b^2 - k^2) = 0 .$$

(b) Two non-parallel lines of slopes m_1 and m_2 are equally inclined to the line $y = nx$. Show that

$$(1 - n^2)(m_1 + m_2) + 2n(m_1 m_2 - 1) = 0 .$$

(c) Tangents of (E) equally inclined to the line $y = nx$ intersect at the point P. Using the results of (a) and (b), or otherwise, find the equation of the locus of P.

2. Let $T_n(x) = \cos(n \arc\cos x)$, where $-1 < x < 1$, $n = 0, 1, 2, \ldots$.

(a) Prove that $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$, and hence show that $T_n(x)$ is a polynomial in x of degree n with leading coefficient 2^{n-1}, where $n = 1, 2, \ldots$.

(b) Using De Moivre's Theorem, or otherwise, determine a_k $(k = 0, 1, 2, \ldots, n)$ in

$$\cos^n\theta = \sum_{k=0}^{n} a_k \cos(n - 2k)\theta \ .$$

(c) Show that $x^n = \dfrac{1}{2^{n-1}} \displaystyle\sum_{k=0}^{\frac{n-1}{2}} C_k^n\, T_{n-2k}(x)$ for $n = 1, 3, 5, \ldots$.

3. Let f be a real-valued function defined on the set $\mathbf{R}$ of real numbers such that

(1) f is continuous at $x_0 \in \mathbf{R}$, and
(2) $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

(a) Evaluate $\lim\limits_{h \to 0} f(h)$.
Hence prove that f is continuous at every $x \in \mathbf{R}$.

(b) Prove that $f(nx) = n f(x)$ for all integers n (zero, positive and negative).
Hence show that $f(r) = r f(1)$ for any rational number, r.

(c) Using the above results, prove that there exists a constant k such that $f(x) = kx$ for all $x \in \mathbf{R}$.
[Hint: you may use the fact that for all $x \in \mathbf{R}$, there exists a sequence $\{a_n\}$ of rational numbers such that $\lim\limits_{n \to \infty} a_n = x$.]

4. Let $I_n = \displaystyle\int_0^1 e^t t^n\, dt$, where n is any non-negative integer.

(a) Prove, by induction, that

$$I_n = (-1)^{n+1} n! + e \sum_{i=0}^{n} (-1)^i \frac{n!}{(n-i)!} \ .$$

(b) Show that $\dfrac{1}{n+1} < I_n < \displaystyle\int_0^1 e\,t^n\, dt < \dfrac{e}{n}$ for all $n > 1$.

(c) Using the above results, show that e must be an irrational number.

5. Let f be a function defined on (a, b); f is said to be <u>S-differentiable</u> at $x \in (a, b)$ if

$$f_s(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}$$

exists. $f_s(x)$ is called the <u>symmetric derivative</u> of f at x .

(a) Show that if f is differentiable at x, then it is S-differentiable at x .

(b) By considering the function F defined by

$$F(x) = \begin{cases} x \sin\dfrac{1}{x} & \text{if } x \neq 0 , \\ 0 & \text{if } x = 0 , \end{cases}$$

show that the converse of the statement in (a) is not true.

(c) Let f and g be continuous and S-differentiable at x. Show that both the sum $f + g$ and the product fg are S-differentiable at x.

6. For a positive integer n, the polynomial $P_n(x)$ is defined by

$$P_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n \ .$$

(a) (i) By applying Leibniz' rule* to $\dfrac{d^k}{dx^k}\{(x + 1)^n (x - 1)^n\}$,
show that, for any positive integer $k < n$, the value of $\dfrac{d^k}{dx^k}(x^2 - 1)^n$ is 0 at $x = \pm 1$.

(ii) Let f be a function which has continuous derivatives up to the n-th order in the interval $[-1, 1]$. Prove that

$$\int_{-1}^{1}\left[\frac{d^r}{dx^r}(x^2 - 1)^r\right] f(x)\,dx = (-1)^r \int_{-1}^{1} (x^2 - 1)^r \frac{d^r}{dx^r} f(x)\,dx \quad \text{for } r = 1, 2, \ldots, n.$$

(b) (i) Prove that $\displaystyle\int_{-1}^{1} P_n(x)\,x^k\,dx = 0$ for $k = 0, 1, 2, \ldots, (n - 1)$.

(ii) Evaluate $\displaystyle\int_{-1}^{1} P_m(x) P_n(x)\,dx$ for any positive integers m and n with $m \neq n$.

* Leibniz' rule: For any functions f and g whose derivatives up to the k-th order exist,

$$(fg)^{(k)} = \sum_{r=0}^{k} C_r^k f^{(k-r)} g^{(r)} \ ,$$

where C_r^k denote the binomial coefficients, $f^{(0)} = f$, $g^{(0)} = g$ and $f^{(r)}$, $g^{(r)}$ are the r-th derivatives of f and g, respectively, for $r = 1, 2, \ldots, k$.

7. A continuous real-valued function f is said to be convex in an interval (a, b) if for any x, y in (a, b) and for any λ, μ in $[0, 1]$ with $\lambda + \mu = 1$,

$$f(\lambda x + \mu y) \leqslant \lambda f(x) + \mu f(y) .$$

(a) Show that, for any t in the interval (x, y), there exist λ, μ in $(0, 1)$ with $\lambda + \mu = 1$ such that

$$t = \lambda x + \mu y .$$

Draw a diagram to illustrate the inequality in the definition of a convex function.

(b) f is convex. Show that, for any x, t, y in (a, b) with $x < t < y$,

$$\frac{f(t) - f(x)}{t - x} \leqslant \frac{f(y) - f(t)}{y - t} .$$

(c) Let g be a function with a second derivative $g''(x) \geqslant 0$ on (a, b).
For any x, y in (a, b) with $x < y$, consider the function

$$h(t) = \lambda g(t) + \mu g(y) - g(\lambda t + \mu y)$$

for $t \in [x, y]$, where λ and μ are fixed numbers in $[0, 1]$ such that $\lambda + \mu = 1$. Show that h is monotonic decreasing and hence show that g is a convex function.

(d) For $x_1, x_2 > 0$, $p > 1$ and $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$, show that $(\lambda_1 x_1 + \lambda_2 x_2)^p \leqslant \lambda_1 x_1^p + \lambda_2 x_2^p$.

8. Let L_1 and L_2 be two rays from the origin O inclining at angles $\frac{\pi}{4}$ and $-\frac{\pi}{4}$, respectively, to the positive x-axis. P and Q are points on L_1 and L_2, respectively, such that $OP = p$ and $OQ = \frac{1}{p}$.

(a) P' and Q' are points on L_1 and L_2, respectively, such that $OP' = p'$ and $OQ' = \frac{1}{p'}$. If $M(u, v)$ denotes the point at which PQ and $P'Q'$ meet, express u and v in terms of p and p'.

Find $\lim_{p' \to p} u$ and $\lim_{p' \to p} v$.

(b) Let $\xi(p) = \lim_{p' \to p} u$ and $\eta(p) = \lim_{p' \to p} v$.
As p varies, show that the locus of $(\xi(p), \eta(p))$ consists of a branch (H) of a hyperbola.

(c) A and B are points on L_1 and L_2, respectively. Show that AB meets H at no point, one point, or two points according as $OA \cdot OB$ is less than, equal to, or greater than 1.

END OF PAPER

香港考試局
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一九八三年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 1983

純數學 試卷一
PURE MATHEMATICS PAPER I

三小時完卷 Three hours
上午九時至正午十二時 9.00 a.m.—12.00 noon
本試卷必須用英文作答 This paper must be answered in English

This paper consists of eight questions all carrying equal marks.
Answer any SIX questions.

In this paper, you may use without proof the fact that a monotonic increasing (decreasing) sequence which is bounded above (below) converges.

1. (a) Prove that the following system of linear equations in the unknowns x, y and z has a unique solution if a, b and c are all non-zero and distinct:

$$\begin{cases} ax + by + cz = k \\ a^2x + b^2y + c^2z = k^2 \\ a^3x + b^3y + c^3z = k^3 . \end{cases}$$

In such a case, find the solution (x_0, y_0, z_0) in terms of a, b, c and k, and show that it is impossible for exactly one of x_0, y_0 and z_0 to be zero.

(b) Find all values of d for which the following system is solvable:

$$\begin{cases} -x + 2y - z = d \\ x + 4y + z = d^2 \\ -x + 8y - z = d^3 . \end{cases}$$

Give the solutions for each of the values of d.