Putting $z = t$, (I) becomes

$$\begin{cases} x - y = 3 + t \\ x - 2y = 4 - t \end{cases}$$ 1

$$\therefore \begin{cases} y = 2t - 1 \\ x = 3t + 2 \end{cases}$$

condition for (1), (2) are consistent

Solution set of (I) $= \{(3t+2,\ 2t-1,\ t) : t \in \mathbb{R}\}$ 3

4

(b) Putting $(x, y, z) = (3t+2,\ 2t-1,\ t)$ in 3^{rd} eq. of (II), we have

$(5+p)t + 1 = q$ 1

(i) If $p \neq -5$, then $\forall q \in \mathbb{R}$, $(3t+2,\ 2t-1,\ t)$ is a solution of (II), where $\left(t = \dfrac{q-1}{5+p}\right)$ ← provided $p \neq -5$ 1 2

$\Delta \neq 0$, unique solution

(ii) If $p = -5$, (II) is not solvable unless $q = 1$, $\dfrac{\Delta_i}{\Delta} = 0$

in which case

Solution set

$(3t+2,\ 2t-1,\ t)$ satisfies (II) $\forall t \in \mathbb{R}$. 2

condition for (I) is consistent

$\Delta = 0$ infinite solution $\quad t = \frac{1-1}{5+p} = 0$

6

Alternative solution Case: Case (

(1) $\Delta \neq 0 \Rightarrow$

(2) $\Delta = 0 \Rightarrow p = -5$

(c) (i) By (b), if $p \neq -5$, then, since $q = 1$, $(2, -1, 0)$ is the only solution of (II).

Substituting in 4^{th} eq. of (III), $2^2 + (-1)^2 + 0^2 \neq 11$.

$\therefore$ (III) has no solutions. inconsistent 3

(ii) If $p = -5$, and $q = 1$ substituting $(3t+2,\ 2t-1,\ t)$ in 4^{th} eq. of (III), 1

$7t^2 + 4t - 3 = 0$

$t = -1$ or $\frac{3}{7}$

$\therefore$ $(-1, -3, -1)$ and $\left(\frac{23}{7}, -\frac{1}{7}, \frac{3}{7}\right)$ are solutions of (III). 3

7

2 (a) Consider the sequence $\{-b_n\}$.

Since $-b_n \leqslant -b_{n+1}$ $\forall n$ and

$-b_n \leqslant -M$ $\forall n$,

$\{-b_n\}$ converges.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} -(-b_n)$

$= -\lim_{n\to\infty}(-b_n)$, since the first limit exists. 1

$\{b_n\}$ converges. 1

5

Consider the sequence $\{-b_n\}$

$\{-b\}$ is strictly increasing and bounded

$\Rightarrow \{-b_n\}$ converges $=$ limit exists

$\Rightarrow \lim_{n\to\infty} b_n = -\lim_{n\to\infty}\{-b_n\}$

(b) Since G.M. $\leqslant$ A.M. and $x_1 < y_1$, $(\because a < b)$

$x_n \leqslant y_n$ for $n = 1, 2, \ldots$ 2

It is obvious that $x_n, y_n \geqslant 0$ $\forall n$.

Further, for $n \geqslant 1$

$x_{n+1} = \sqrt{x_n y_n} \geqslant \sqrt{x_n x_n} = x_n$, $\quad \forall n,\ x_{n+1} \geqslant x_n \geqslant a$ 2

$y_{n+1} = \dfrac{x_n + y_n}{2} \leqslant \dfrac{y_n + y_n}{2} = y_n$ $\quad y_{n+1} \leq y_n \leq b$ 2

Thus $\{x_n\}$ is increasing and bounded above by b, and $\{y_n\}$ is decreasing and bounded below by a. Therefore both $\{x_n\}$, $\{y_n\}$ are convergent. 1 1

SOLUTION	Marks	Remarks

(b) Let $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$

Since $y_{n+1} = \dfrac{x_n + y_n}{2}$

$\therefore \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} \dfrac{x_n + y_n}{2}$ 2

$= \dfrac{1}{2}\left(\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n\right)$ 1

i.e. $y = \dfrac{x + y}{2}$

$x = y$ 1

12

SOLUTION	Marks	Remarks

(a) (i) E + E + E = E $C^{15}_3 = 455$ [1] 1

$\underset{\text{odd}}{O} + E = E$ $C^{15}_2 \times C^{15}_1 = 1575$

The number of ways of obtaining an even sum is

2030 [2] 2

3

(ii) The thirty numbers can be divided into 3 groups of ten numbers each as follows :

(a) Those divisible by 3 ,

(b) those that leave a remainder of 1 when divided by 3,

(c) those that leave a remainder of 2 when divided by 3 .

A sum divisible by 3 can be formed iff either

(1) the three numbers are selected from (a) , or 1

(2) the three numbers are selected from (b) , or 1

(3) the three numbers are selected from (c) , or 1

(4) a number is selected from each of (a) , (b) , (c) . 1

$\therefore$ the required number of ways $= 3C^{10}_3 + 10^3$ 2

$= 1360$

6

(b) We shall prove by induction on n .

For $n = 1$, R.S. $= \sum_{j_1=N} \dfrac{N!}{j_1!} y_1^{j_1}$

$= \dfrac{N!}{N!} y_1^{N} = $ L.S. 1

Assume the equality holds for some $k \geq 1$

i.e. $(y_1 + y_2 + \dots + y_k)^N$

$$= \sum_{j_1+j_2+\dots+j_k=N} \frac{N!}{j_1!j_2!\dots j_k!} y_1^{j_1} y_2^{j_2} \dots y_k^{j_k} \quad \forall N \in \mathbb{N}$$ 1+1

Let $N - r = j_{k+1}$, then

$$(y_1 + y_2 + \dots + y_k + y_{k+1})^N = [(y_1 + y_2 + \dots + y_k) + y_{k+1}]^N$$

$$= \sum_{r=0}^{N} \frac{N!}{r!(N-r)!} y_{k+1}^{N-r} (y_1 + y_2 + \dots + y_k)^r$$ 1

$$= \sum_{r+j_{k+1}=N} \left[\frac{N!}{r!j_{k+1}!} y_{k+1}^{j_{k+1}} \sum_{j_1+j_2+\dots+j_k=r} \frac{r!}{j_1!j_2!\dots j_k!} y_1^{j_1} y_2^{j_2} \dots y_k^{j_k} \right]$$ 1 1.

$$= \sum_{j_1+j_2+\dots+j_k+j_{k+1}=N} \frac{N!}{j_1!j_2!\dots j_k!j_{k+1}!} y_1^{j_1} y_2^{j_2} \dots y_k^{j_k} y_{k+1}^{j_{k+1}}$$

$\therefore$ the equality holds for $n = k+1$ and hence $\forall n \in \mathbb{N}$ 2

8

[2]

[4]

[2]

SOLUTION	Marks	Re.

4. For any complex numbers $z = x + iy$,

$y = \operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$ ....(1)

$x = \operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$

$\therefore x \leq \sqrt{x^2 + y^2} = |z|$ and $x \geq -\sqrt{x^2+y^2} = -|z|$ ....(2)

(a) $|u + v|^2 = (u + v)(\overline{u + v})$ 1

$= u\bar{u} + v\bar{v} + (u\bar{v} + \bar{u}v)$

$= |u|^2 + |v|^2 + (u\bar{v} + \overline{u\bar{v}})$ 1

$= |u|^2 + |v|^2 + 2\operatorname{Re}(u\bar{v})$ 1

$\leq |u|^2 + |v|^2 + 2|u\bar{v}|$ 1

$= |u|^2 + |v|^2 + 2|u||v|$

$= (|u| + |v|)^2$ ....(3)

$|u + v| \leq |u| + |v|$ 4

(b) We shall prove that $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_1$ 1

(i) "$S_1 \Rightarrow S_2$"

If $|u + v| = |u| + |v|$ or $|u - v| = |u| + |v|$, 3

from (3) either $\operatorname{Re}(u\bar{v}) = |u\bar{v}|$, or, replacing v by $-v$, $-\operatorname{Re}(u\bar{v}) = |u\bar{v}|$.

From (2) $\operatorname{Im}(u\bar{v}) = 0$.

MARKS

5. (c) "Existence"

For any $x \in A$, let $y = f(x) \in f[A]$.

We define $g(y) = x$.

Since f is injective, x is uniquely determined by y.

Further, by definition of $f[A]$, g is defined for all $y \in f[A]$. Hence g is a mapping from $f[A]$ to A such that

$$g(f(x)) = x \quad \forall x \in A.$$ 3

"Surjective"

This is trivial by definition of g. 1

"Injective"

$\forall y_1, y_2 \in f[A]$, let $y_1 = f(x_1)$, $y_2 = f(x_2)$.

Then $g(y_1) = g(y_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow x_1 = x_2$

$\Rightarrow y_1 = y_2$ since f is a mapping 2

$\therefore$ g is injective and hence bijective.

"Uniqueness"

Given a bijective mapping $h : f[A] \to A$ such that

$h(f(x)) = x \quad \forall x \in A$.

$\forall y \in f[A]$, let $g(y) = x_1$, $h(y) = x_2$.

Then $f(x_1) = y$, by definition of g.

$\therefore x_1 = h(f(x_1)) = h(y) = x_2$

i.e. $h(y) = g(y) \quad \forall y \in f[A]$. 2

8

(a) (i) First $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ and the associative law can be assumed. identity associative 1

(ii) For any $A = \begin{bmatrix} x_1 & y_1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ 0 & 1 \end{bmatrix} \in G$,

since $x_1, x_2 \neq 0$, $x_1 x_2 \neq 0$ and

$$AB = \begin{bmatrix} x_1 x_2 & x_1 y_2 + y_1 \\ 0 & 1 \end{bmatrix} \in G$$

$\therefore$ multiplication is closed in G. closure 1

(iii) For any $A = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \in G$, let $B = \begin{bmatrix} \frac{1}{x} & -\frac{y}{x} \\ 0 & 1 \end{bmatrix}$,

Since $x \neq 0$, then $AB = BA = I$.

$\therefore A^{-1} = B$ exists in G. Inverse 2

Hence G is a group under the usual multiplication.

4

(b) i) For any $A \in S$, since $I \in G$ and $AI = A$.

$\therefore A \sim A$ and $\sim$ is reflexive. 1

(ii) For any $A, B \in S$, if $A \sim B$, let $AD = B$, where $D \in G$. $D^{-1}D = DD^{-1} = I$

Then $D^{-1} \in G$ and $BD^{-1} = A$.

$AD = B$

$ADD^{-1} = BD^{-1}$

$A = BD^{-1}$

$\therefore B \sim A$ and $\sim$ is symmetric. 1

(iii) For any $A, B, C \in S$, if $A \sim B$ and $B \sim C$,

let $AD_1 = B$ and $BD_2 = C$, where $D_1, D_2 \in G$.

Then $D_1 D_2 \in G$ and

$AD_1 D_2 = BD_2 = C$.

$\therefore A \sim C$ and $\sim$ is transitive. 2

(i), (ii), (iii) $\Rightarrow$ $\sim$ is an equivalence relation on S

4

6.(c) (i) For any $u \in \mathbb{C} \setminus \{0\}$, let $u = x + iz$.

Then either $x \neq 0$ or $z \neq 0$.

Suppose $x \neq 0$, consider the matrix $A = \begin{bmatrix} x & 0 \\ z & \frac{1}{x} \end{bmatrix}$.

Since $|A| = 1$, $A \in S$ and $\Phi(A) = \frac{x}{1} + i\frac{z}{1}$.

The case $z \neq 0$ is similar.

$\therefore \Phi$ is surjective.

$\forall u = x + iz \in \mathbb{C} \setminus \{0\}$
$\exists A = \begin{bmatrix} x & 0 \\ z & \frac{1}{x} \end{bmatrix} \in S$
s.t. $\Phi(A) = u$

3

(ii) Let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} \in S$ and

$D = \begin{bmatrix} x_0 & y_0 \\ 0 & 1 \end{bmatrix} \in G$.

"If" part If $AD = B$,

i.e. if $\underset{A}{\begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}} \underset{D}{\begin{bmatrix} x_0 & y_0 \\ 0 & 1 \end{bmatrix}} = \overset{(AD)}{\begin{bmatrix} x_0 x_1 & x_1 y_0 + y_1 \\ x_0 z_1 & y_0 z_1 + w_1 \end{bmatrix}}$

$= \underset{B}{\begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix}}$, ... (1)

then $|B| = |A||D| = |A| x_0$ .... (2)

Now $\Phi(B) = \frac{x_2}{|B|} + i\frac{z_2}{|B|}$ 1

$= \frac{x_0 x_1}{|B|} + i\frac{x_0 z_1}{|B|}$, by (1), 1

$= \frac{x_1}{|A|} + i\frac{z_1}{|A|}$, by (2),

$= \Phi(A)$

"Only if" part If $\Phi(A) = \Phi(B)$, then

$\frac{x_1}{|A|} + i\frac{z_1}{|A|} = \frac{x_2}{|B|} + i\frac{z_2}{|B|}$

i.e. $\left.\begin{aligned} x_1 &= \frac{|A|}{|B|} x_2 \\ z_1 &= \frac{|A|}{|B|} z_2 \end{aligned}\right\}$ .... (3) 1

SOLUTION	Marks	Remarks
6 (c) (ii) Since $\lvert A\rvert \neq 0$, $A^{-1} = \begin{bmatrix} \frac{w_1}{\lvert A\rvert} & -\frac{y_1}{\lvert A\rvert} \\ -\frac{z_1}{\lvert A\rvert} & \frac{x_1}{\lvert A\rvert} \end{bmatrix}$ exists. $A^{-1} = \frac{[A_{ij}]^T}{\lvert A\rvert}$		
Consider the matrix $D' = A^{-1}B$.	2	
$\begin{bmatrix} \frac{w_1}{\lvert A\rvert} & -\frac{y_1}{\lvert A\rvert} \\ -\frac{z_1}{\lvert A\rvert} & \frac{x_1}{\lvert A\rvert} \end{bmatrix} \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2 w_1 - y_1 z_2}{\lvert A\rvert} & \frac{w_1 y_2 - w_2 y_1}{\lvert A\rvert} \\ \frac{x_1 z_2 - x_2 z_1}{\lvert A\rvert} & \frac{x_1 w_2 - y_2 z_1}{\lvert A\rvert} \end{bmatrix}$		
by (3) $= \begin{bmatrix} \frac{w_1 x_1 - y_1 z_1}{\lvert A\rvert^2}\lvert B\rvert & \frac{w_1 y_2 - w_2 y_1}{\lvert A\rvert} \\ \frac{x_1 z_1 - x_1 z_1}{\lvert A\rvert^2}\lvert B\rvert & \frac{x_2 w_2 - y_2 z_2}{\lvert B\rvert} \end{bmatrix}$		
$= \begin{bmatrix} \frac{\lvert B\rvert}{\lvert A\rvert} & \frac{w_1 y_2 - y_1 w_2}{\lvert A\rvert} \\ 0 & 1 \end{bmatrix} \in G$	1	
$\therefore AD' = B$ and $A \sim B$. $AD' = B$, $A^{-1}AD' = A^{-1}B$, $D' = A^{-1}B$	6	
( )		

RESTRICTED 內部文件 (13)

7

SOLUTION	Marks	Remarks

$\frac{df}{dx} = 1 - \frac{1}{x}$

$= 0$ iff $x = 1$.

$f'(x) < 0 \ \forall x \in (0,1)$ $\quad f'(x) > 0 \ \forall x \in (0, \infty)$

f is strictly decreasing in $(0, 1)$ and strictly increasing in $(1, \infty)$. — 2

f is minimum at $x = 1$.

$x - 1 - \log x \geq f(1) = 0 \quad \forall x > 0$

i.e. $\log x \leq x - 1 \quad \forall x > 0$.

The equality holds iff $x = 1$. — 4

Since $x_i > 0$ for each $i = 1, 2, \ldots, n$, by (a)

$\log x_i \leq x_i - 1$.

$\log x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} = \lambda_1 \log x_1 + \lambda_2 \log x_2 + \cdots + \lambda_n \log x_n$ — 2

$\leq \lambda_1(x_1 - 1) + \lambda_2(x_2 - 1) + \cdots + \lambda_n(x_n - 1)$

$= (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

$= 0$

$\because$ log is strictly increasing,

$x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \leq \log^{-1} 0 = 1$.

The equality holds iff $x_1 = x_2 = \cdots = x_n = 1$. — 6

For each i, let $\lambda_i = \frac{p_i}{p_1 + p_2 + \cdots + p_n}$, — 1

$x_i = \frac{a_i}{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}$, — 1

then $\lambda_i, x_i > 0$.

(14)

SOLUTION	Marks	Remarks

By (b), $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \leq 1$

$\left(\frac{a_1}{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}\right)^{\lambda_1} \left(\frac{a_2}{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}\right)^{\lambda_2} \cdots \left(\frac{a_n}{\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n}\right)^{\lambda_n} \leq 1$

$\therefore (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})$ $\quad \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$

$\leq (\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n)^{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$

i.e. $(a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n})^{\frac{1}{p_1 + p_2 + \cdots + p_n}}$

$\leq \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}$ — 2

The equality holds iff $x_i = 1 \ \forall i$,

i.e. $a_i = \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}$ — 1

or $a_1 = a_2 = \cdots = a_n$ — 7

$a_i = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = 1$

(a) By De Moivre's theorem,

$$\cos(2n+1)\theta + i\sin(2n+1)\theta = (\cos\theta + i\sin\theta)^{2n+1}$$

$$= \sum_{j=0}^{2n+1} C_j^{2n+1} (\cos\theta)^{2n+1-j} (i\sin\theta)^j$$

Considering imaginary parts of both sides,

$$i\sin(2n+1)\theta = C_1^{2n+1}(\cos\theta)^{2n}(i\sin\theta) + C_3^{2n+1}(\cos\theta)^{2n-2}(i\sin\theta)^3 + \dots + C_{2n+1}^{2n+1}(i\sin\theta)^{2n+1}$$ 1

$$\therefore \sin(2n+1)\theta = \sum_{r=0}^{n}(-1)^r C_{2r+1}^{2n+1}(\cos\theta)^{2n-2r}(\sin\theta)^{2r+1}$$

$$= (\sin\theta)^{2n+1}\sum_{r=0}^{n}(-1)^r C_{2r+1}^{2n+1}(\cot^2\theta)^{n-r},$$

$(\sin\theta \neq 0)$ 1 | 5

(b) Putting $\theta = \frac{k\pi}{2n+1}$ in (a), $k = 1, 2, \dots, n$, 1

$$\left(\sin\frac{k\pi}{2n+1}\right)^{2n+1}\sum_{r=0}^{n}(-1)^r C_{2r+1}^{2n+1}\left(\cot^2\frac{k\pi}{2n+1}\right)^{n-r} = \sin\left[(2n+1)\frac{k\pi}{2n+1}\right] = 0$$ 1

Since $\left(\sin\frac{k\pi}{2n+1}\right)^{2n+1} \neq 0$, $\cot^2\frac{k\pi}{2n+1}$, $k = 1, 2, \dots, n$, are n roots of the equation $\sum_{r=0}^{n}(-1)^r C_{2r+1}^{2n+1} x^{n-r} = 0$. 1

Further, these roots are distinct as $0 < \frac{k\pi}{2n+1} < \frac{\pi}{2}$. 1

$\therefore$ sum of roots $= -\frac{\text{coeff of } x^{n-1}}{\text{coeff of } x^n}$

8II Q7 SOLUTION 81.7 | Marks | Remarks

7. (a) (i) $I = \int \sin(\log x)\,dx$

$$= x\sin(\log x) - \int x\cdot\frac{1}{x}\cos(\log x)\,dx$$

$$= x\sin(\log x) - \left[x\cos(\log x) - \int x\cdot\frac{1}{x}(-\sin(\log x))\,dx\right]$$ 1

$$\therefore 2I = x\left[\sin(\log x) - \cos(\log x)\right] + c$$ 1

$$I = \frac{x}{2}\left[\sin(\log x) - \cos(\log x)\right] + C$$ 3 1

(ii) Let $t = x + \sqrt{x^2+1}$, then $x = \frac{t^2-1}{2t}$

x in terms of t

$$dx = \left(\frac{1}{2} + \frac{1}{2t^2}\right)dt$$

dx in terms of dt, t

$$\therefore \int\frac{dx}{x+\sqrt{x^2+1}} = \int\frac{1}{t}\left(\frac{1}{2}+\frac{1}{2t^2}\right)dt$$ 2 1

$$= \frac{1}{2}\log t - \frac{1}{4t^2} + C$$ 1

$$= \frac{1}{2}\log(x+\sqrt{x^2+1}) - \frac{1}{4(x+\sqrt{x^2+1})^2} + C$$ 1 | 3

6

$$= \cancel{uF(u^2)}\Big|_0^x - \int_0^x \cancel{u\frac{d}{du}F(u^2)} = uF(u^2)\Big|_0^x - \int_0^x u\,d[F(u^2)]$$

(b) $$\int_0^x F(u^2)\,du = uF(u^2)\Big|_0^x - \int_0^x u f(u^2)2u\,du$$ 2

$$= xF(x^2) - \int_0^x 2u^2 f(u^2)\,du$$ ~~2u f(t) dt~~

$\because F(u) = \int_0^u f(t)\,dt$

$F(x^2) = \int_0^{x^2} f(t)\,dt$

$$= xF(x^2) - \int_0^{x^2}\sqrt{t}\,f(t)\,dt, \quad \text{where } t = u^2,$$ 2

$u = \sqrt{t}$

$du = \frac{dt}{2\sqrt{t}}$

$$= x\int_0^{x^2} f(u)\,du - \int_0^{x^2}\sqrt{u}\,f(u)\,du$$ 1

$$= \int_0^{x^2}(x-\sqrt{u})\,f(u)\,du.$$ 1 | 6

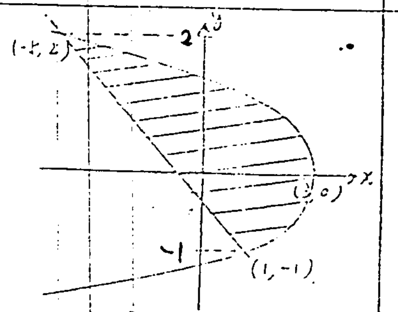
SOLUTION | Marks | Remarks

(c) The two curves intersect at

$$\begin{cases} x = 3 - 2y^2 \\ x = -1 - 2y \end{cases},$$

i.e. at $y = -1$ or 2 — 1

$x = +1 \quad -5$



Area bounded $= \int_{-1}^{2} \left[(3 - 2y^2) - (-1 - 2y)\right] dy$ — 2

$= \int_{-1}^{2} \left[-2y^2 + 2y + 4\right] dy$

$= -\frac{2}{3}y^3 + y^2 + 4y \Big|_{-1}^{2}$ — 1

$= 9$ — 1

Total: 5

Solution | Marks | Remarks

(a) Let $F = (x_0, y_0, z_0)$. Since PF is perpendicular to the plane $\Pi_1 : x + y - z - 1 = 0$,

$2 - x_0 : 2 - y_0 : -1 - z_0 = 1 : 1 : -1$ — 2

$\therefore y_0 = x_0$ $\qquad \frac{2 - x_0}{1} = \frac{2 - y_0}{1} = \frac{-1 - z_0}{-1}$ symmetric form

$z_0 = 1 - x_0$.

As F lies in Π_1, $x_0 + y_0 - z_0 - 1 = 0$ — 2

Thus $x_0 + x_0 - (1 - x_0) - 1 = 0$

$x = 2 + t$
$y = 2 + t$
$z = -1 - t$

$(2+t) + (2+t) + (1+t) - 1 = 0$

$3t = 4$

$t = \frac{4}{3}$

$\therefore x_0 = \frac{2}{3}$

$y_0 = \frac{2}{3}$

$z_0 = \frac{1}{3}$

$F = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ — 1

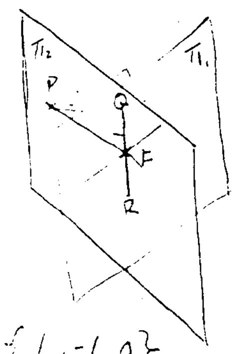
Total: 5

(b) Substituting $P(2, 2, -1)$ and $F\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ in $x - y = 0$, it is seen that they both lie in Π_2. — 2

Let the line of intersection of Π_1 and Π_2 be

$$x = \frac{2}{3} + at$$
$$y = \frac{2}{3} + bt$$
$$z = \frac{1}{3} + ct.$$

direction ratios



Substituting in Π_1 and Π_2, we have

$at + bt - ct = 0$ and

$at - bt = 0$.

$\therefore a = b$ $\qquad$ or $\{1, 1, -1\} \times \{1, -1, 0\}$

$c = 2a$

The line of intersection is given by = the equation of QR — 2

$x = \frac{2}{3} + t \qquad y = \frac{2}{3} + t \qquad z = \frac{1}{3} + 2t$ — 2

Solution	marks	Remarks
$\lvert PF\rvert = \sqrt{\left(2-\frac{2}{3}\right)^2 + \left(2-\frac{2}{3}\right)^2 + \left(-1-\frac{1}{3}\right)^2} = \frac{4\sqrt{3}}{3}$	1	
$\lvert QF\rvert = \lvert RF\rvert = \frac{1}{\sqrt{3}}(PF) = \frac{4}{3}$	1	
$= \sqrt{t^2 + t^2 + 4t^2}$		
$\therefore t = \pm\frac{2}{9}\sqrt{6}$	2	
$Q = \left(\frac{2}{3}+\frac{2}{9}\sqrt{6},\ \frac{2}{3}+\frac{2}{9}\sqrt{6},\ \frac{1}{3}+\frac{4}{9}\sqrt{6}\right)$		
$R = \left(\frac{2}{3}-\frac{2}{9}\sqrt{6},\ \frac{2}{3}-\frac{2}{9}\sqrt{6},\ \frac{1}{3}-\frac{4}{9}\sqrt{6}\right)$		
	12	

$6t^2 = \frac{16}{9}$

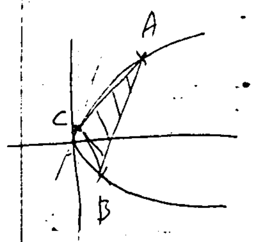
$t^2 = \frac{8}{27}$

$t = \pm\frac{2\sqrt{2}}{3\sqrt{3}} = \pm\frac{2\sqrt{6}}{9}$

Let $x_1 \neq x_2$.

The equation of the tangent ... point $C(x_3, y_3)$ is given by $8y_3 y = \frac{1}{2}(x_3 + x)$, whose slope is $\frac{1}{16y_3}$. — 1 — Slope can also be found by diff.

Slope of AB $= \frac{y_2 - y_1}{x_2 - x_1}$. — 1



Equating slopes, $\boxed{\frac{1}{16y_3} = \frac{y_2 - y_1}{x_2 - x_1}}$ — 1

iff $y_3 = \frac{1}{16}\left(\frac{x_2 - x_1}{y_2 - y_1}\right) = \frac{y_1 + y_2}{2}$ — 1

Hence $x_3 = 8y_3^2 = 2(y_1 + y_2)^2$ — 1

Since y_3 lies between y_1 and y_2, C lies on the parabolic arc AB. — 1 — Can also be proved by Mean Value Theorem

The result still holds if $x_1 = x_2$.

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Area of $\triangle ABC = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$ — 2 — mark for ...

$= \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & \frac{y_1+y_2}{2} & 1 \end{vmatrix}\right|$

$= \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 - \left(\frac{x_1+x_2}{2}\right) & 0 & 0 \end{vmatrix}\right|$

$= \frac{1}{2}\left|(y_1 - y_2) \times 2\left[(y_1+y_2)^2 - 2(y_1^2 + y_2^2)\right]\right|$

$= |h|^3$ — 2

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SOLUTION | Marks | Remarks

(c) Let the equation of the circle be

$$x^2 + y^2 + Lx + My + N = 0 .$$

Substituting $x = 8y^2$,

$$64y^4 + (8L + 1)y^2 + My + N = 0 .$$ 3

Since y_1, y_2, y_3 are real roots of the equation, its fourth real root, y_4, satisfies

$$y_1 + y_2 + y_3 + y_4 = \frac{-\text{coeff of } y^3}{64} = 0$$ 3

$$y_4 = -(y_1 + y_2 + y_3) = -\frac{3}{2}(y_1 + y_2)$$

and $x_4 = 8y_4^2 = 18(y_1 + y_2)^2$.

Let $\frac{1}{(1+x)(1+2x)\dots(1+nx)} \equiv \frac{A_1}{1+x} + \frac{A_2}{1+2x} + \dots + \frac{A_n}{1+nx}$

$$1 \equiv \sum_{r=1}^{n} \frac{A_r(1+x)(1+2x)\dots(1+nx)}{(1+rx)}$$ 1

Putting $x = -\frac{1}{r}$, $r = 1, 2, \dots, n$, we have

$$1 = A_r(1 - \frac{1}{r})(1 - \frac{2}{r})\dots(1 - \frac{r-1}{r})(1 - \frac{r+1}{r})\dots(1 - \frac{n}{r})$$ 2

$$= A_r\frac{(r-1)(r-2)\dots1(-1)(-2)\dots(r-n)}{r^{n-1}}$$

$$\therefore A_r = \frac{(-1)^{n-r}r^{n-1}}{(r-1)!(n-r)!}$$ 2

5

$$\frac{1}{(1+x)(1+2x)\dots(1+nx)} \equiv \sum_{r=1}^{n} \frac{(-1)^{n-r}r^{n-1}}{(r-1)!(n-r)!(1+rx)}$$

( ) Putting $x = 0$ in the partial fractions in (a), 2

$$1 = \sum_{r=1}^{n} A_r$$

$$= \sum_{r=1}^{n} \frac{(-1)^{n-r}r^{n-1}}{(r-1)!(n-r)!}$$

$$= \sum_{r=1}^{n} \frac{(-1)^{n-r}r^{n}}{r!(n-r)!}$$ 1

$$= \sum_{r=0}^{n} \frac{(-1)^{n-r}r^{n}}{r!(n-r)!}$$ 2

( )

$$= \sum_{r=0}^{n} \frac{(-1)^{n-r}C_r^n r^n}{n!}$$ 1

$$\therefore \sum_{r=0}^{n} (-1)^{n-r}C_r^n r^n = n!$$

5

SOLUTION	Marks	Remarks

$$(e^t - 1)^n = \sum_{r=0}^{n} (-1)^{n-r} C_r^n \, e^{rt}$$ 2

Since $\frac{d^n}{dt^n} e^{rt} = r^n e^{rt}$ 1

$$\frac{d^n}{dt^n}(e^t - 1)^n = \frac{d^n}{dt^n} \sum_{r=0}^{n} (-1)^{n-r} C_r^n \, e^{rt}$$

$$= \sum_{r=0}^{n} (-1)^{n-r} C_r^n \, r^n e^{rt}$$ 1

$$= \sum_{r=0}^{n} (-1)^{n-r} C_r^n \, r^n, \quad \text{at } t = 0.$$ 1

$= n!$, by (b). 1

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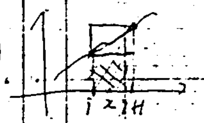
$$\boxed{\begin{array}{l}(a-b)^n \\ r+1^{th} \text{ term} = C_r^n a^{n-r}(-b)^r\end{array}}$$

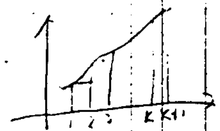
$$(e^t - 1)^n = \sum_{r=0}^{n} (e^t)^r (-1)^{n-r} C_r^n$$

8(a) $f(i) \leqslant f(x) \leqslant f(i+1)$ for $i \leqslant x \leqslant i+1$, $i = 1, 2, \ldots, k$. 1

Hence $1 \times f(i) \leqslant \int_i^{i+1} f(x)\,dx \leqslant f(i+1) \times 1$.



$$\sum_{i=1}^{k} f(i) \leqslant \int_1^{k+1} f(x)\,dx \leqslant \sum_{i=1}^{k} f(i+1) = \sum_{i=2}^{k+1} f(i).$$ 1



write the given condition

Putting $f(x) = \log_e x$ which is strictly increasing for $x > 0$,

$$\sum_{i=1}^{n-1} \log i \leqslant \int_1^n \log x \, dx \leqslant \sum_{i=2}^{n} \log i$$

$$\log[(n-1)!] \leqslant \int_1^n \log x \, dx \leqslant \log n!$$

$$\int \log x \, dx = x\log x - \int x \cdot \tfrac{1}{x} dx = x\log x - x$$

But $\int_1^n \log x \, dx = (x\log x - x)\Big|_1^n$ 1

$= n\log n - n + 1$

$$\log[(n-1)!] \leqslant n\log n - n + 1 \leqslant \log n!$$ 1

$$(n-1)! \leqslant e^{n\log n - n + 1} = n^n e^{-n+1} \leqslant n!$$

$$= e^{\log n^n} \cdot e^{-n+1}$$

6

Sequence $h_n = n^n \cdot e^{-n+1}$

(b) Let $h_n = \sqrt[n]{an+b} - 1$, then

Binomial Expansion

$$an + b = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{2} h_n^2 + \ldots$$

It is easily seen that if n is sufficiently large,

$h_n > 0$ and $an + b > 1 + \frac{n(n-1)}{2} h_n^2$

$$\therefore \frac{2(an+b) - 2}{n(n-1)} > h_n^2 > 0, \quad \sqrt{\frac{2(an+b)-2}{n(n-1)}} > h_n > 0$$ 2

Since $\lim_{n\to\infty} \sqrt{\frac{2(an+b) - 2}{n(n-1)}} = 0$, $\lim_{n\to\infty} h_n = 0$,

5

SOLUTION Marks

By (a)

$$\frac{(n-1)!}{n^n} \leq e^{-n+1} \leq \frac{n!}{n^n} \qquad \frac{n!}{n^{n+1}} \leq e^{-n+1} \leq \frac{n!}{n^n}$$

$$e^{\frac{1}{n}-1} \leq \frac{\sqrt[n]{n!}}{n} \leq \sqrt[n]{n}\, e^{\frac{1}{n}-1} \qquad \frac{n!}{n^n} \leq n e^{1-n}$$ 2

$$e^{1-n} \leq \frac{n!}{n^n} \leq n e^{1-n} \qquad e^{-n+1} \leq \frac{n!}{n^n}$$

By (b)

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 .$$

$$e^{-1} = \lim_{n\to\infty} e^{\frac{1}{n}-1} \leq \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \leq \lim_{n\to\infty} \sqrt[n]{n}\, e^{\frac{1}{n}-1}$$ 4

$$= e^{-1}$$

$$\therefore \quad \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}$$ 6

Assume, for contradiction, that $\varphi(x)$ attains an absolute maximum at x_0 and an absolute minimum at x_1. Then,

$$\forall t, \quad \lambda(t) = \varphi(x_0+t) + \varphi(x_0-t) - 2\varphi(x_0)$$

$$\leq \varphi(x_0) + \varphi(x_0) - 2\varphi(x_0)$$ 2

$$= 0, \quad \text{and}$$

$$\lambda(t) = \varphi(x_1+t) + \varphi(x_1-t) - 2\varphi(x_1)$$

$$\geq \varphi(x_1) + \varphi(x_1) - 2\varphi(x_1)$$ 2

$$= 0 .$$

We thus have $0 \leq \lambda(t) \leq 0 \quad \forall t$. 2

i.e. $\lambda(t) \equiv 0$, a contradiction. 6

for given non-constant functions

b) Differentiating w.r.t. t

??? $\varphi'(x+t) - \varphi'(x-t) = \lambda'(t)$ 2

$\varphi'(x+2t) - \varphi'(x) = \lambda'(t)$ 2

Letting $t = \frac{y}{2}$, $\varphi'(x+y) - \varphi'(x) = \lambda'\left(\frac{y}{2}\right)$ 1

???? $= \varphi'(y) - \varphi'(0)$ 5

(c) $$\varphi''(x) = \lim_{y\to 0} \frac{\varphi'(x+y) - \varphi'(x)}{y}$$

$$= \lim_{y\to 0} \frac{\varphi'(y) - \varphi'(0)}{y} = \varphi''(0)$$ 3

$\therefore \varphi''(x) \equiv$ constant

$\varphi'(x) = ax + b$

$\varphi(x) = \frac{a}{2}x^2 + bx + c$, a polynomial of degree 2 6

4(a) $\int_a^b \frac{f(rx) - f(sx)}{x}\,dx = \int_a^b \frac{f(rx)}{x}\,dx - \int_a^b \frac{f(sx)}{x}\,dx$

$= \int_{ra}^{rb} \frac{f(u)}{u}\,du - \int_{sa}^{sb} \frac{f(v)}{v}\,dv$, where $u = rx$, $v = sx$. 2

$rb < sa$

$= \int_{ra}^{rb} \frac{f(x)}{x}\,dx - \int_{sa}^{sb} \frac{f(x)}{x}\,dx$

$= \int_{ra}^{\boxed{sa}} \frac{f(x)}{x}\,dx + \int_{\boxed{sa}}^{rb} \frac{f(x)}{x}\,dx - \int_{sa}^{\boxed{rb}} \frac{f(x)}{x}\,dx - \int_{\boxed{rb}}^{sb} \frac{f(x)}{x}\,dx$ 2

$= \int_{ra}^{sa} \frac{f(x)}{x}\,dx - \int_{rb}^{sb} \frac{f(x)}{x}\,dx$

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(b) Putting $g(x) = f(x)$, $h(x) = \frac{1}{x} > 0$, since $0 < a < b$, $0 < r < s$,

$\int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \int_{ra_n}^{sa_n} \frac{f(x)}{x}\,dx - \int_{rb_n}^{sb_n} \frac{f(x)}{x}\,dx$

$= f(x_n) \int_{ra_n}^{sa_n} \frac{1}{x}\,dx - f(\tilde{x}_n) \int_{rb_n}^{sb_n} \frac{1}{x}\,dx$, where 3

$ra_n \leq x_n \leq sa_n$, $rb_n \leq \tilde{x}_n \leq sb_n$,

$= f(x_n) \log\left(\frac{s}{r}\right) - f(\tilde{x}_n) \log\left(\frac{s}{r}\right)$

$= \left(f(x_n) - f(\tilde{x}_n)\right) \log\left(\frac{s}{r}\right)$ 1

Since as $n \to \infty$, $x_n \to 0$, $\tilde{x}_n \to \infty$, $f(x_n) \to f(0)$ and $f(\tilde{x}_n) \to \ell$, 2

SOLUTION | Marks | Remarks

$\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \lim_{n\to\infty} \left[f(x_n) - f(\tilde{x}_n)\right] \log\left(\frac{s}{r}\right)$ 1

$= \left[f(0) - \ell\right] \log\left(\frac{s}{r}\right)$.

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(c) The equality does not hold since neither

$\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(rx)}{x}\,dx$ nor $\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(sx)}{x}\,dx$ exist.

In fact,

$\int_{a_n}^{b_n} \frac{f(rx)}{x}\,dx \geq c \int_{a_n}^{b_n} \frac{1}{x}\,dx$

$= c \log\left(\frac{b_n}{a_n}\right)$.

Since $c > 0$ and $\lim_{n\to\infty} \log\left(\frac{b_n}{a_n}\right) = \infty$,

$\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(rx)}{x}\,dx$ does not exist.

Similarly $\lim_{n\to\infty} \int_{a_n}^{b_n} \frac{f(sx)}{x}\,dx$ does not exist. 1

5

	Solution	Marks	Remarks

(a) (i) $\frac{df}{dx} = \frac{d}{dx}\int_1^x \frac{1}{\sqrt{1+t^5}}\,dt$ $f(y) - f(x) = \int_x^y \frac{1}{\sqrt{1+t^5}}\,du$

$= \frac{1}{\sqrt{1+x^5}}$

> 0 for all $x \geq 1$.   1

∴ f is strictly increasing there and $f(x) < f(y)$ whenever $1 \leq x < y$.

(ii) For $x \geq 1$, $\frac{1}{\sqrt{1+x^5}} < \frac{1}{\sqrt{x^5}}$

∴ $f(x) = \int_1^x \frac{1}{\sqrt{1+t^5}}\,dt < \int_1^x \frac{dt}{\sqrt{t^5}}$

$= -\frac{2}{3}t^{-\frac{3}{2}}\Big|_1^x$

$= \frac{2}{3} - \frac{2}{3}x^{-\frac{3}{2}}$

$< \frac{2}{3}$ ($\because x \neq 0$)

(iii) Similarly, $\frac{1}{\sqrt{1+x^5}} \geq \frac{1}{\sqrt{2x^5}}$ and

$f(x) = \int_1^x \frac{1}{\sqrt{1+t^5}}\,dt \geq \int_1^x \frac{dt}{\sqrt{2t^5}}$

$= \frac{\sqrt{2}}{3}(1 - x^{-\frac{3}{2}})$ 1

∴ $f(4) \geq \frac{\sqrt{2}}{3}(1 - \frac{1}{8})$

$> \frac{1}{3}$

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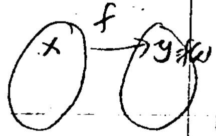
(b) (i) Since $f'(g(u))\,g'(u) = 1$ $g'(u) = \frac{1}{f'(g(u))}$

$g'(u) = \sqrt{1 + (g(u))^5}$ 1

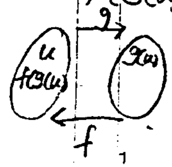
$g''(u) = \frac{1}{2}(1 + (g(u))^5)^{-\frac{1}{2}} \cdot 5(g(u))^4\, g'(u)$

$= \frac{5}{2}(g(u))^4$ 1

Note that g is the inverse of f in the interval defined, $g'(u) = \frac{1}{f'(g(u))}$



$\frac{df(x)}{dx} = \frac{1}{\frac{dx}{df(x)}}$



$\frac{dg(u)}{du} = \frac{1}{\frac{df(g(u))}{d[g(u)]}}$

(ii) $h'(u) = e^u - (1 + g^5(u))^{\frac{1}{2}} = e^u - g'(u)$

$h''(u) = e^u - \frac{5}{2}g^4(u) = e^u - g''(u)$

Since f is increasing in $[1, \infty)$, g is increasing and $g(u) \geq 1 \ \forall u$.

We shall show that $h''(u) \neq h'(u)$. Now $h''(u) \geq h'(u)$

iff $\frac{5}{2}g^4(u_0) \leq (1 + g^5(u_0))^{\frac{1}{2}}$

$\Rightarrow \frac{25}{4}g^8(u_0) \leq 1 + g^5(u_0)$

$\leq 1 + g^8(u_0)$ (since $g(u) \geq 1$)   2

$\Rightarrow g(u_0) < 1$, which is a contradiction.

∴ $h'(u) < h''(u)$.

Further $h(u)$ has a minimum in the open interval only if $h'(u) = 0$ and $h''(u) > 0$ 2

i.e. $h''(u) > h'(u)$, which is false.

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