香港考試局
HONG KONG EXAMINATIONS AUTHORITY

一九八一年香港高級程度會考
HONG KONG ADVANCED LEVEL EXAMINATION 1981

# 純數學 試卷一 | PURE MATHEMATICS PAPER I

三小時完卷 | Three hours

上午九時至正午十二時 | 9.00 a.m.—12.00 noon

本試卷必須用英文作答 | This paper must be answered in English

**This paper consists of eight questions all carrying equal marks.**
**Answer any SIX questions.**

(a) Solve the following system of equations :

(I) $\begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \end{cases}$ .

(b) Find all possible values of $p$ and $q$ such that the following system of equations is solvable :

(II) $\begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \\ x + y + pz = q \end{cases}$

(c) Find the solutions, if possible, of the system of equations

(III) $\begin{cases} x - y - z = 3 \\ x - 2y + z = 4 \\ x + y + pz = 1 \\ x^2 + y^2 + z^2 = 11 \end{cases}$

2. Let $\{a_n\}$ be a sequence of real numbers. It is known that, if

(i) $\{a_n\}$ satisfies $a_n \leq a_{n+1}$ for all $n$, and

(ii) there exists a real number $K$ such that $a_n \leq K$ for all $n$,

then $\{a_n\}$ converges.

(a) Show that a sequence $\{b_n\}$ of real numbers is convergent if

(i) $b_n \geq b_{n+1}$ for all $n$, and

(ii) there exists a real number $M$ such that $b_n \geq M$ for all $n$.

(b) Given two positive real numbers $a$ and $b$ such that $a < b$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences satisfying

$$x_1 = a, \quad y_1 = b, \quad \text{and}$$

$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}$$

for all positive integers $n$.

Show that both $\{x_n\}$ and $\{y_n\}$ converge and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n .$$

3. (a) In how many ways can three different numbers be selected from the thirty numbers 1, 2, ... , 30 such that their sum is

(i) divisible by 2,

(ii) divisible by 3 ?

(b) Using the binomial theorem

$$(x_1 + x_2)^N = \sum_{r=0}^{N} C_r^N x_1^{N-r} x_2^r$$

for all positive integers $N$, prove that

$$(y_1 + y_2 + \ldots + y_n)^N = \sum_{j_1+j_2+\ldots+j_n=N} \left[\frac{N!}{j_1!\,j_2!\ldots j_n!} y_1^{j_1} y_2^{j_2} \ldots y_n^{j_n}\right]$$

for all positive integers $n$ and $N$.

4. Given any two complex numbers $u$ and $v$.

(a) Prove the inequality $|u + v| \leq |u| + |v|$.

(b) Prove that the following three statements are equivalent :

$S_1$ : $|u + v| = |u| + |v|$ or $|u - v| = |u| + |v|$,

$S_2$ : $\mathrm{Im}(u\bar{v}) = 0$,

$S_3$ : $u\bar{v} = \bar{u}v$,

where $\mathrm{Im}(u\bar{v})$ denotes the imaginary part of $u\bar{v}$.

(c) If both $|u + v| \neq |u| + |v|$ and $|u - v| \neq |u| + |v|$, prove that, for any complex number $z$, there exist real numbers $a$ and $b$ such that $z = au + bv$.



5. Let $A$ be a set. Suppose $f$ is a mapping from $A$ into itself such that $f$ is injective but not surjective. Denote the set $\{f(a) : a \in A\}$ by $f[A]$.

(a) Show that

(i) $A \neq \phi$,

(ii) $A$ consists of more than two elements.

(b) If $\varphi : A \rightarrow A$ is a mapping such that

$$\varphi(f(x)) = x$$

for all $x$ in $A$, show that $\varphi$ is not bijective.

(c) Prove that there exists a unique mapping $g : f[A] \rightarrow A$ such that

(i) $g(f(x)) = x$ for all $x$ in $A$, and

(ii) $g$ is bijective.

6. For any $2 \times 2$ real matrix $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, let $|A| = xw - yz$.

Let

$$S = \{A : |A| \neq 0\} \quad \text{and}$$

$$G = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} : x \neq 0 \right\}.$$

(a) Does $G$ form a group under the usual multiplication of matrices ? Prove your assertion.

(b) A relation $\sim$ is defined on $S$ as follows :

For $A, B \in S$, $A \sim B$ if there exists $D \in G$ such that $AD = B$.

Show that $\sim$ is an equivalence relation on $S$.

(c) Let $\mathbb{C}$ be the set of complex numbers. A mapping $\Phi : S \rightarrow \mathbb{C} \setminus \{0\}$ is defined by

$$\Phi(A) = \frac{x}{|A|} + i\frac{z}{|A|},$$

where $A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Show that

(i) $\Phi$ is surjective,

(ii) $\Phi(A) = \Phi(B)$ if and only if $A \sim B$ as defined in (b).

7. (a) Let $f(x) = x - 1 - \log_e x$ for all $x > 0$. Find the minimum value of $f(x)$ and show that $\log_e x \leq x - 1$ for all $x > 0$. For what value of $x$ will the equality hold?

(b) Let $x_1, x_2, \ldots, x_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be positive numbers such that

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \lambda_1 + \lambda_2 + \ldots + \lambda_n = 1 .$$

Show that

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \leq 1 .$$

For what values of $x_1, x_2, \ldots, x_n$ will the equality hold?

(c) Prove that, if $a_1, a_2, \ldots, a_n$ and $p_1, p_2, \ldots, p_n$ are positive numbers, then

$$\left(a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n}\right)^{\frac{1}{p_1 + p_2 + \ldots + p_n}} \leq \frac{p_1 a_1 + p_2 a_2 + \ldots + p_n a_n}{p_1 + p_2 + \ldots + p_n} .$$

When will the equality hold?

8. (a) Using DeMoivre's theorem, show that

$$\sin(2n + 1)\theta = \sin^{2n+1}\theta \sum_{r=0}^{n} (-1)^r C^{2n+1}_{2r+1} (\cot^2\theta)^{n-r} ,$$

where $n$ is a positive integer, $0 < \theta < \frac{\pi}{2}$ and $C^{2n+1}_{2r+1}$ are binomial coefficients.

(b) Using (a) and the relations between the coefficients and roots of the equation $\sum_{r=0}^{n} (-1)^r C^{2n+1}_{2r+1} x^{n-r} = 0$, show that

$$\sum_{k=1}^{n} \cot^2 \frac{k\pi}{2n + 1} = \frac{n(2n - 1)}{3} , \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

and deduce that

$$\sum_{k=1}^{n} \operatorname{cosec}^2 \frac{k\pi}{2n + 1} = \frac{n(2n + 2)}{3} . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

(c) Let $A_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Using equations (i) and (ii) of (b), show that

$$\frac{\pi^2}{6}\left(\frac{2n}{2n + 1}\right)\left(\frac{2n - 1}{2n + 1}\right) < A_n < \frac{\pi^2}{6}\left(\frac{2n}{2n + 1}\right)\left(\frac{2n + 2}{2n + 1}\right).$$

Hence evaluate $\lim_{n \to \infty} A_n$.

(It is known that for $0 < \theta < \frac{\pi}{2}$, $\tan\theta > \theta > \sin\theta$ .)

END OF PAPER



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|---|---|
| 純數學 | **PURE MATHEMATICS** |
| 試卷二 | **PAPER II** |
| 三小時完卷 | Three hours |
| 下午二時至下午五時 | 2.00 p.m.—5.00 p.m. |
| 本試卷必須用英文作答 | This paper must be answered in English |

**This paper consists of eight questions each carrying equal marks.**
**Answer any SIX questions.**

1. (a) Find the integrals

(i) $\int \sin(\log_e x)\,dx$ by parts,

(ii) $\int \frac{dx}{x + \sqrt{x^2+1}}$ by using the substitution $t = x + \sqrt{x^2+1}$ .

(b) Let $F(u) = \int_0^u f(t)\,dt$ , where $f(t)$ is a continuous function.

Show that

$$\int_0^x F(u^2)\,du = \int_0^{x^2} (x - \sqrt{u})\,f(u)\,du$$

for all $x > 0$ .

(c) Find the area bounded by the curve $2y^2 + x - 3 = 0$ and the straight line $2y + x + 1 = 0$ .

2. Given two planes $\pi_1 : x + y - z - 1 = 0$ ,
$\pi_2 : x - y = 0$ .

(a) Find the foot $F$ of the perpendicular from the point $P(2, 2, -1)$ to the plane $\pi_1$ .

(b) Find two points $Q$ and $R$ in the plane $\pi_2$ such that $PQR$ is an equilateral triangle with the line segment $PF$ as a median.

3. Suppose that $A(x_1, y_1)$ and $B(x_2, y_2)$ are distinct points lying on the parabola $8y^2 = x$ .

(a) Show that there is a point $C(x_3, y_3)$ lying on the parabolic arc $AB$ such that the tangent at $C$ is parallel to the chord joining $A$ and $B$ .

Express $x_3$ and $y_3$ in terms of $y_1$ and $y_2$ .

(b) Let $h = y_2 - y_1$ . Find the area of $\Delta ABC$ in terms of $h$ .

(c) Show that the circle passing through the points $A$ , $B$ and $C$ will intersect the parabola again at a point $D(x_4, y_4)$ .

Express $x_4$ and $y_4$ in terms of $y_1$ and $y_2$ .



4. (a) Resolve $\frac{1}{(1+x)(1+2x)\ldots(1+nx)}$ into partial fractions.

(b) Use the result in (a) to prove the identity

$$\sum_{r=0}^{n} (-1)^{n-r}\, C_r^n\, r^n = n! \ ,$$

where $C_r^n$ are binomial coefficients.

(c) Prove that the $n$th derivative with respect to $t$ of $(e^t - 1)^n$ takes the value $n!$ when $t$ is zero.

5. (a) Suppose that a function $f(x)$ is increasing on the interval $\{x : x \geq 1\}$ .

Show that

$$\sum_{i=1}^{k} f(i) \leq \int_1^{k+1} f(x)\,dx \leq \sum_{i=2}^{k+1} f(i) \ ,$$

where $k$ is a positive integer.

Hence show that

$$\log_e[(n-1)!] \leq \int_1^n \log_e x\,dx \leq \log_e(n!) \ ,$$

and that

$$(n-1)! \leq n^n e^{-n+1} \leq n! \ ,$$

where $n$ is a positive integer.

(b) Prove that $\lim_{n\to\infty} \left\{ (an+b)^{\frac{1}{n}} - 1 \right\} = 0$ , where $a$ and $b$ are positive numbers.

(c) Using the results in (a) and (b), or otherwise, find $\lim_{n\to\infty} \frac{(n!)^{\frac{1}{n}}}{n}$ .

6. Given two non-constant functions $\varphi(x)$ and $\lambda(t)$ satisfying the relation

$$\varphi(x+t) + \varphi(x-t) - 2\varphi(x) = \lambda(t)$$

for all real numbers $x$ and $t$ .

(a) Prove that the function $\varphi(x)$ cannot have both an absolute maximum and an absolute minimum.

(b) If $\varphi(x)$ is differentiable for all real $x$ , prove that

$$\varphi'(x+y) - \varphi'(x) = \varphi'(y) - \varphi'(0)$$

for all $x$ and $y$ .

(c) Suppose, further, that $\varphi''(0)$ exists. Prove that $\varphi''(x)$ exists for all $x$ and that $\varphi''(x) = \varphi''(0)$ .

Hence show that $\varphi(x)$ is a polynomial of degree less than or equal to two.

7. Suppose that the function $f(x)$ is continuous for all $x > 0$ and $\lim_{x \to \infty} f(x) = \ell$ exists.

(a) Show that
$$\int_a^b \frac{f(rx) - f(sx)}{x}\,dx = \int_{ra}^{sa} \frac{f(x)}{x}\,dx - \int_{rb}^{sb} \frac{f(x)}{x}\,dx ,$$
where $0 < a < b$ and $0 < r < s$.

(b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers such that $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = +\infty$. Show that
$$\lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \left(f(0) - \ell\right) \log_e \left(\frac{s}{r}\right) .$$

[ Hint : you may assume without proof that the following theorem holds :

If $g(x)$ and $h(x)$ are continuous on the closed interval $[c, d]$ and $h(x) > 0$ for all $x$ in $[c, d]$, then
$$\int_c^d g(x)h(x)\,dx = g(x_0)\int_c^d h(x)\,dx$$
for some $x_0$ in $[c, d]$. ]

(c) Furthermore, if $f(x) \geq c$ for all $x \geq 0$, where $c$ is a positive constant, is it then true that
$$\lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx) - f(sx)}{x}\,dx = \lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(rx)}{x}\,dx - \lim_{n \to \infty} \int_{a_n}^{b_n} \frac{f(sx)}{x}\,dx \; ?$$
Prove your assertion.

8. Suppose that $f(x) = \int_1^x \frac{1}{\sqrt{1 + t^5}}\,dt$ for $x \geq 1$.

(a) (i) Show that $f(x) < f(y)$ whenever $1 \leq x < y$.

(ii) Show that $f(x) < \frac{2}{3}$ for all $x \geq 1$.

(iii) Find an $x_0$ such that $f(x_0) > \frac{1}{3}$.

(b) Let $g(u)$, where $0 < u < f(x_0)$, be a function such that $f\left(g(u)\right) = u$.

(i) Show that $g'(u) = [1 + g^5(u)]^{\frac{1}{2}}$ and $g''(u) = \frac{5}{2}\left(g(u)\right)^4$.

(ii) Let $h(u) = e^u - g(u)$, where $0 < u < f(x_0)$. Prove that $h''(u) < h'(u)$.

Hence prove that $h(u)$ does not have a minimum.

END OF PAPER





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# 純數學 PURE MATHEMATICS
## 試卷一 PAPER I

三小時完卷 Three hours
上午九時至正午十二時 9.00 a.m.—12.00 noon
本試卷必須用英文作答 This paper must be answered in English

**This paper consists of eight questions all carrying equal marks.**
**Answer any SIX questions.**

1. (a) Let $0 < \lambda < 1$. Show that
$$\lambda t + (1 - \lambda) \geq t^\lambda \quad \text{for all} \quad t > 0 .$$
Deduce that
$$\lambda\alpha + (1 - \lambda)\beta \geq \alpha^\lambda \beta^{1-\lambda} \quad \text{for all} \quad \alpha, \beta > 0 .$$

(b) Let $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be two sets of non-negative real numbers such that $\sum_{i=1}^n a_i^p = \sum_{i=1}^n b_i^q = 1$. Using the result in (a), show that $\sum_{i=1}^n a_i b_i \leq 1$.

Hence show that, for any two sets of non-negative real numbers $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$,
$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q\right)^{\frac{1}{q}} .$$