1980 A-Level Pure Mathematics Paper I

1. Let V be the set of all 3×1 real matrices. For any $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ in V and any real number λ,

   we define $\underline{x}+\underline{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix}$, $\lambda\underline{x} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix}$. It is known that under this addition and scalar multiplication,

   V forms a real vector space with zero vector $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

   (a) For a given 3×3 real matrix A, let $E = \{\underline{x}\in V : A\underline{x} = \underline{0}\}$.

   (i) Show that E forms a vector subspace of V.

   (ii) For $\underline{b}$ in V, suppose we have $\underline{p}$ in V such that $A\underline{p} = \underline{b}$. Show that, for any $\underline{y}$ in V,

   $A\underline{y} = \underline{b}$ if and only if $\underline{y} = \underline{p} + \underline{x}$ for some $\underline{x}$ in E.

   (b) (i) Find all solutions to $\begin{cases} x-y-z=0 \\ 10x+5y-4z=0 \\ 5x+5y-z=0 \end{cases}$

   (ii) Suppose $x=\dfrac{1}{2}$, $y=\dfrac{4}{3}$, $z=\sqrt{2}$ is a solution to the system of equations

   $\begin{cases} x-y-z=b_1 \\ 10x+5y-4z=b_2 \\ 5x+5y-z=b_3. \end{cases}$.

   Find all solutions to the system. (1980)

2. Let F denote the set of all positive-valued continuous functions on the set R of all real numbers. For any $f, g\in F$, define $f * g$ by $(f*g)(x) = f(x)g(x)$ $\forall x\in R$. It is known that F forms a group under the operation *. The identity I of this group and the inverse g of $f\in F$ are given respectively by

   $I(x)=1$ $\forall x\in R$, $g(x)=\dfrac{1}{f(x)}$ $\forall x\in R$.

   Define a relation $\sim$ in F as follows:

   For $f, g\in F$, $f\sim g$ if there are polynomials p, q in F such that $p * f = q * g$.

   (a) Show that ~ is an equivalence relation on F.

   (b) Let $f/\sim$ be the equivalence class of f with respect to ~, and let F/~ be the quotient set consisting of all these equivalence classes. For any $f/\sim, g/\sim \in F/\sim$, define $f/\sim \otimes g/\sim$ to be $(f*g)/\sim$.

   (i) Show that $\otimes$ is well defined on F/~, i.e., if $f/\sim = f_1/\sim$ and $g/\sim = g_1/\sim$, then $f/\sim\otimes g/\sim = f_1/\sim \otimes g_1/\sim$.

   (ii) Show that F/~ forms a group under $\otimes$. (1980)

3. (a) If $x > 0$ and p is a positive integer, show that $\frac{x^{p+1}-1}{p+1} \ge \frac{x^{p}-1}{p}$, and that the equality holds only if $x = 1$.

(b) Let $x_1, x_2, \ldots, x_n$ be positive numbers and $\sum_{i=1}^{n} x_i \ge n$.

(i) Show that, for any positive integer m, $\sum_{i=1}^{n} x_i^{\,m} \ge n$.

(ii) If $\sum_{i=1}^{n} x_i^{\,m} = n$ for some integer m greater than one, show that $x_1 = x_2 = \ldots = x_n = 1$.

(c) Using (b), or otherwise, show that, for any positive numbers $y_1, y_2, \ldots, y_n$, and positive integer m,

$$\frac{y_1^{\,m} + y_2^{\,m} + \ldots + y_n^{\,m}}{n} \ge \left(\frac{y_1 + y_2 + \ldots + y_n}{n}\right)^m$$

and that the equality holds only when $m = 1$ or $y_1 = y_2 = \ldots = y_n$. (1980)

4. (a) The terms of a sequence $y_1, y_2, y_3, \ldots$ satisfy the relation $y_k = Ay_{k-1} + B$ $(k \ge 2)$ where A, B are constants independent of k and $A \ne 1$. Guess an expression for y_k $(k \ge 2)$ in terms of y_1, A, B and k and prove it.

(b) The terms of a sequence $x_0, x_1, x_2, \ldots$ satisfy the relation $x_k = (a+b)x_{k-1} - abx_{k-2}$ $(k \ge 2)$, where a, b are non-zero constants independent of k and $a \ne b$.

(i) Express $x_k - ax_{k-1}$ $(k \ge 2)$ in terms of $(x_1 - ax_0)$, b and k.

(ii) Using (a) or otherwise, express x_k $(k \ge 2)$ in terms of x_0, x_1, a, b and k.

(c) If the terms of the sequence $x_0, x_1, x_2, \ldots$ satisfy the relation $x_k = \frac{1}{3}x_{k-1} + \frac{2}{3}x_{k-2}$ $(k \ge 2)$, express $\lim_{k \to \infty} x_k$ in terms of x_0 and x_1. (1980)

5. (a) (i) Let $\omega^3 = 1$ and $\omega \ne 1$. Show that the expression $x^3 - 3uvx - (u^3 + v^3) = 0$ can be factorized as $(x-u-v)(x-\omega u-\omega^2 v)(x-\omega^2 u-\omega v)$

(ii) Find a solution to the following system of equations

$$\begin{cases} u^3 + v^3 = 6 \\ uv = 2 \end{cases}$$

Hence, or otherwise, find the roots of the equations $x^3 - 6x - 6 = 0$

(b) Given an equation $x^3 + px + q = 0$............ (*)

(i) Show that, if (*) has a multiple root, then $27q^2 + 4p^3 = 0$

(ii) Using the method indicated in (a) (ii), or otherwise, show that, if $27q^2 + 4p^3 = 0$, then (*) has a multiple root. (1980)

6. Let a, b be real numbers such that $a < b$ and let m, n be positive integers.

(a) If for all real numbers x, u, $[(1+u)x - (au+b)]^{m+n} = \sum_{k=0}^{m+n} A_k(x)u^k$ .... (*)

show that $A_k(x) = C_k^{m+n}(x-a)^k(x-b)^{m+n-k}$ for $k = 0, 1, \ldots, m+n$, where C_k^{m+n} is the coefficient of t^k in the expansion of $(1+t)^{m+n}$.

(b) By integrating both sides of (*) with respect to x, or otherwise, calculate $\int_a^b (x-a)^m (x-b)^n \, dx$.

(c) By differentiating both sides of (*) with respect to x, or otherwise, find $\frac{d^r}{dx^r}\{(x-a)^m (x-b)^n\}$ at $x = a$, where r is a positive integer. (1980)

7. Let C be the set of complex numbers. A function $f : C \to C$ is said to be an isometry if it preserves distance, that is, if $|f(z_1) - f(z_2)| = |z_1 - z_2|$ for all $z_1, z_2 \in C$.

(a) If f is an isometry, show that $g(z) = \frac{f(z) - f(0)}{f(1) - f(0)}$ is an isometry satisfying $g(1) = 1$ and $g(0) = 0$.

(b) If g is an isometry satisfying $g(1) = 1$, $g(0) = 0$, show that
(i) the real parts of $g(z)$ and z are equal for all $z \in C$,
(ii) $g(i) = i$ or $-i$.

(c) If g is an isometry satisfying $g(1) = 1$, $g(0) = 0$ and $g(i) = i$ (respectively $-i$), show that $g(z)=z$ (respectively $\bar{z}$) for all $z \in C$.

(d) Show that any isometry f has the form $f(z) = az + b$ or $f(z) = a\bar{z} + b$ with a and b constant and $|a| = 1$. (1980)

8. N balls are distributed randomly among n cells. Each of the n^N possible distributions has probability n^{-N}.

(a) (i) Calculate the probability P_k that a given cell contains exactly k balls.

(ii) Show that the most probable number k_0 satisfies the inequality $\frac{N-n+1}{n} \le k_0 \le \frac{N+1}{n}$.

(iii) Compute the mean number $\sum_{k=0}^{N} kP_k$ of balls in a given cell and show that it can differ from k_0 by at most one.

(b) Let $A(N, n)$ be the number of distributions leaving none of the cells empty. Show that

$A(N, n+1) = \sum_{k=1}^{N} C_k^N A(N-k, n)$, where C_k^N is the coefficient of t^k in the expansion of $(1+t)^N$. Hence show by mathematical induction (on n), or otherwise, that

$$A(N, n) = \sum_{j=0}^{n} (-1)^i C_j^n (n-j)^N.$$ (1980)

1980 A-Level Pure Mathematics Paper II

1. Let $P(t) = (x(t), y(t))$ be a point on the unit circle with parametric equations $x(t) = \frac{1-t^2}{1+t^2}$, $y(t) = \frac{2t}{1+t^2}$, Q be the point (a, 0), $0 < a < 1$. An arbitrary line of slope m passing through Q cuts the circle at the points $R = P(t_1)$ and $S = P(t_2)$. Let T be the point where RO meets the line through Q parallel to SO, where O is the origin.

   (a) Show that $t_1 t_2 = \frac{a-1}{a+1}$, $t_1 + t_2 = \frac{-2}{m(a+1)}$.

   (b) Express the coordinates of T in terms of a, t_1, t_2

   (c) Verify that the locus of T is an ellipse with equation $(1-a^2)(x-\frac{a}{2})^2 + y^2 = C$, where C is a constant. What is C?

(1980)

2. In a 3-dimensional space with a Cartesian coordinate system, two lines l_1 and l_2 are given by the pairs of equations :

$$l_1: \begin{cases} x+2y+3z-3=0 \\ x+2y+2z-4=0 \end{cases}, \qquad l_2: \begin{cases} x+y+z-1=0 \\ 2x+3y+5z-2=0 \end{cases}.$$

   (a) Let P_α be the plane $\alpha(x + 2y + 3z - 3) + (x + 2y + 2z - 4) = 0$ and Q_β be the plane
   $\beta(x + y + z - 1) + (2x + 3y + 5z - 2) = 0$.
   Show that P_α is parallel to Q_β if and only if there exists $m \neq 0$ such that

$$\begin{pmatrix} 1 & -m & 1-2m \\ 2 & -m & 2-3m \\ 3 & -m & 2-5m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{..........................} \quad (*)$$

   (b) Find the value of m for which there are numbers α and β satisfying (*) in (a). Hence find the equations of the two parallel planes M_1 and M_2 containing l_1 and l_2 respectively.

   (c) Find the equation of the plane N containing l_1 and perpendicular to M_2.

   (d) Let l_1' be the projection of l_1 on M_2 (i.e. l_1' is the intersection of N and M_2). Find its equation.

(1980)

3. (a) Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be an n^{th} degree polynomial in the complex variable z with real coefficients. Show that

   (i) $|f(z)|^2 = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \cos(k-j)\theta$, where $z = r(\cos\theta + i\sin\theta)$,

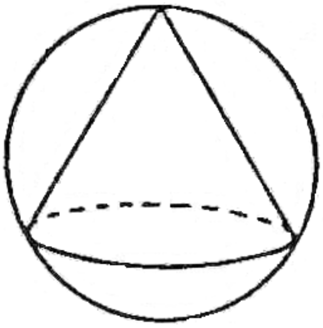
   (ii) $\frac{1}{2\pi}\int_0^{2\pi} |f(\cos\theta + i\sin\theta)|^2 \, d\theta = \sum_{k=0}^{n} a_k{}^2$.

   (b) If C_k^n is the coefficient of t^k in the binomial expansion of $(1+t)^n$, show that

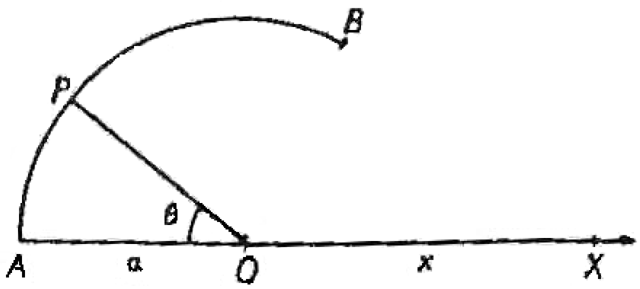
$$\sum_{k=0}^{n}\left(C_k^n\right)^2 = \frac{2^n}{\pi}\int_0^{\pi}(1+\cos\theta)^n\,d\theta = \frac{2^{2n+1}}{\pi}\int_0^{\frac{\pi}{2}}(\cos t)^{2n}\,dt.$$

(1980)

4. (a) A right circular cone is inscribed in a sphere of radius a as shown in the figure. Determine the height of the cone if it is to have maximum volume.



(b) Two points A and B lie on the circumference of a circle with centre O and radius a such that $\angle AOB = \frac{2}{3}\pi$. X is a point on AO produced with $OX = x$; P is a point on arc AB with $\angle AOP = \theta$. For each $x > 0$, let $g(x) = \int_0^{\frac{2\pi}{3}} \frac{a\sin\theta}{r(x,\theta)}\,d\theta$, where $r(x, \theta)$ is the distance between P and X.



(i) Show that $g(x) = \dfrac{3a}{x + a + (a^2 + x^2 - ax)^{\frac{1}{2}}}$.

(ii) Prove that on $(0, \infty)$, $0 < g(x) < \frac{3}{2}$.

(1980)

5. (a) Let f and g be two continuous functions defined on the real line R and let $x_0 \in R$, show that if $f(x) = g(x)$ for all $x \in R \backslash \{x_0\}$, then $f(x_0) = g(x_0)$.

(b) If a real polynomial $p(x)$ can be written as $p(x) = (x - x_0)^m q(x)$ for some positive integer m and polynomial $q(x)$ with $q(x_0) \neq 0$, show that the expression is unique, that is, if k is a positive integer and $h(x)$ is a polynomial with $h(x_0) \neq 0$ such that $p(x) = (x - x_0)^k h(x)$, then $m = k$ and $q(x) = h(x)$ for all $x \in R$.

(c) Let $p(x)$ be a real polynomial . Show that for any positive integer k , x_0 is a root of multiplicity $k+1$ of the equation $p(x)=0$ if and only if $p(x_0)=0$ and x_0 is a root of multiplicity k of $p'(x)=0$, where $p'(x)$ denotes the derivative of $p(x)$. (1980)

6. (a) Let f and g be real-valued functions defined on the real line **R** and possess the following properties :

(1) $f(x+y)=f(x)g(y)+f(y)g(x)$ for all $x, y \in \mathbf{R}$,

(2) $f(0)=0$, $f'(0)=1$, $g(0)=1$, $g'(0)=0$

Show that $f'(x)=g(x)$ for all $x \in \mathbf{R}$.

(b) Let $f(x)$ be a function with continuous first and second derivatives on $[0,1]$ and $f(0)=f(1)=0$.

(i) Show that $\int_0^1 f(x)f''(x)dx \le 0$, where the equality sign holds only if $f(x)=0$ for all x in $[0,1]$.

(ii) Suppose , in addition , $\int_0^1 [f(x)]^2 dx = 1$. Show that $\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$

(1980)

7. (a) It is known that , for any integer k , $\int_{-\pi}^{\pi} \sin kx dx = 0$ and $\int_{-\pi}^{\pi} \cos kx\, dx = \begin{cases} 2\pi & \text{if } k=0 \\ 0 & \text{if } k \ne 0 \end{cases}$

Using the above results , show that if m , n are positive integers ,

(i) $\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0,$ (ii)

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} \pi & \text{if } m=n, \\ 0 & \text{if } m \ne n, \end{cases}$$

(iii) $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \begin{cases} \pi & \text{if } m=n, \\ 0 & \text{if } m \ne n. \end{cases}$

(b) Define $\phi_i(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} & \text{if } i=0, \\ \frac{\cos mx}{\sqrt{\pi}} & \text{if } i=2m-1 \\ \frac{\sin mx}{\sqrt{\pi}} & \text{if } i=2m, \end{cases}$,

where $m = 1, 2, 3, \ldots$.

Let f be a continuous real-valued function defined on $[-\pi, \pi]$ and let α_i be real constants.

(i) Prove that , for each integer $N \ge 0$,

$$\int_{-\pi}^{\pi} [f(x) - \sum_{i=0}^{N} \alpha_i \phi_i(x)]^2 dx = \int_{-\pi}^{\pi} [f(x)]^2 dx + \sum_{i=0}^{N} {\alpha_i}^2 - 2\sum_{i=0}^{N} \alpha_i p_i ,$$

where $p_i = \int_{-\pi}^{\pi} f(x)\phi_i(x)dx.$ Hence prove that $\int_{-\pi}^{\pi} [f(x) - \sum_{i=o}^{N} \alpha_i \phi_i(x)]^2 dx$ attains its least value for varying α_i when $\alpha_i = p_i$ for each i .

(ii) Show that, for any integer $M \geq 1$, $\sum_{i=0}^{2M} p_i^2 \leq \int_{-\pi}^{\pi} [f(x)]^2 dx$.

(1980)

8. Let Γ and Γ' be two Cartesian coordinate systems on a plane and with the same origin, where Γ' is obtained from Γ by a rotation through an angle θ. If (x, y) and (x', y') are the coordinates of an arbitrary point P with respect to Γ and Γ' respectively, then it is known that $x' = kx + hy$, $y' = -hx + ky$, where $k = \cos\theta$ and $h = \sin\theta$.

(a) The general equation of a conic section in the coordinate system Γ is given by $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

(i) Show that the same conic section is represented in the coordinate system Γ' by

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where
$$A' = Ak^2 + Bkh + Ch^2,$$
$$B' = 2kh(C - A) + B(k^2 - h^2),$$
$$C' = Ah^2 - Bhk + Ck^2,$$
$$D' = Dk + Eh,$$
$$E' = Ek - Dh, \quad F' = F.$$

(ii) Show that $4A'C' - B'^2 = 4AC - B^2$.

(iii) Show that, by choosing a suitable angle θ of rotation, the coefficient B' can be made to vanish.

(b) By a suitable rotation followed by a translation if necessary, bring the equation of the conic section $x^2 - 2xy + y^2 + \frac{7}{2}x - \frac{1}{2}y + \frac{11}{2} = 0$ into the standard form. Write down the equation of its line of symmetry in the original coordinate system.

(1980)