HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION

MATHEMATICS
Extended Part
Module 2 (Algebra and Calculus)

(Sample Paper)

Time allowed: 2 hours 30 minutes

This paper must be answered in English

INSTRUCTIONS

1. This paper consists of Section A and Section B. Each section carries 50 marks.

2. Answer **ALL** questions in this paper.

3. All working must be clearly shown.

4. Unless otherwise specified, numerical answers must be exact.

Not to be taken away before the end of the examination session

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$
$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$	

* *

Section A (50 marks)

1. Find $\frac{d}{dx}(\sqrt{2x})$ from first principles.

(4 marks)

2. A snowball in a shape of sphere is melting with its volume decreasing at a constant rate of $4 \text{ cm}^3\text{s}^{-1}$. When its radius is 5 cm, find the rate of change of its radius.

(4 marks)

3. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = 2x \ln(x^2 + 1)$. It is given that the curve passes through the point $(0,1)$.

   Find the equation of the curve.

(4 marks)

4. Find $\int \left(x^2 - \frac{1}{x}\right)^4 dx$.

(4 marks)

5. By considering $\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$, find the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$.

(4 marks)

6. Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$.

   Find the equation of the tangent to C at the point $(1, 1)$.

(5 marks)

7. Solve the system of linear equations

$$\begin{cases} x+7y-6z=-4 \\ 3x-4y+7z=13 \\ 4x+3y+z=9 \end{cases}.$$

(5 marks)

8. (a) Using integration by parts, find $\int x\cos x \, \mathrm{d}x$.

(b)

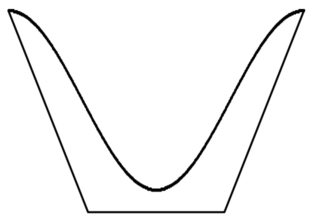

Figure 1

The **inner surface** of a container is formed by revolving the curve $y=-\cos x$ (for $0 \le x \le \pi$) about the y-axis (see Figure 1). Find the capacity of the container.

(6 marks)

9.

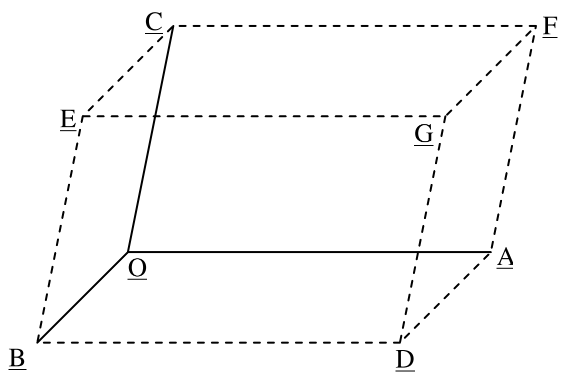

Figure 2

Let $\overrightarrow{OA}=4\mathbf{i}+3\mathbf{j}$, $\overrightarrow{OB}=3\mathbf{j}+\mathbf{k}$ and $\overrightarrow{OC}=3\mathbf{i}+\mathbf{j}+5\mathbf{k}$. Figure 2 shows the parallelepiped $OADBECFG$ formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

(a) Find the area of the parallelogram $OADB$.

(b) Find the volume of the parallelepiped $OADBECFG$.

(c) If C' is a point different from C such that the volume of the parallelepiped formed by $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC'}$ is the same as that of $OADBECFG$, find a possible vector of $\overrightarrow{OC'}$.

(6 marks)

10. Let $0° < \theta < 180°$ and define $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Prove, by mathematical induction, that

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}$$

for all positive integers n .

(b) Solve $\sin 3\theta + \sin 2\theta + \sin\theta = 0$.

(c) It is given that $A^3 + A^2 + A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Find the value(s) of a .

(8 marks)

Section B (50 marks)

11. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Let I and O be the 3×3 identity matrix and zero matrix respectively.

(i) Prove that $P^3 - 2P^2 - P + I = O$.

(ii) Using the result of (i), or otherwise, find P^{-1} .

(5 marks)

(b) (i) Prove that $D = P^{-1}AP$.

(ii) Prove that D and A are non-singular.

(iii) Find $(D^{-1})^{100}$.

Hence, or otherwise, find $(A^{-1})^{100}$.

(7 marks)

12. Let $f(x)=\dfrac{4}{x-1}-\dfrac{4}{x+1}-1$, where $x\neq\pm1$.

(a) (i) Find the x- and y-intercept(s) of the graph of $y=f(x)$.

(ii) Find $f'(x)$ and prove that

$$f''(x)=\frac{16(3x^2+1)}{(x-1)^3(x+1)^3}$$

for $x\neq\pm1$.

(iii) For the graph of $y=f(x)$, find all the extreme points and show that there are no points of inflexion.

(6 marks)

(b) Find all the asymptote(s) of the graph of $y=f(x)$.

(2 marks)

(c) Sketch the graph of $y=f(x)$.

(3 marks)

(d) Let S be the area bounded by the graph of $y=f(x)$, the straight lines $x=3$, $x=a$ $(a>3)$ and $y=-1$.

Find S in terms of a .

Deduce that $S<4\ln 2$.

(3 marks)

13. (a) Let $a > 0$ and $\mathrm{f}(x)$ be a continuous function.

Prove that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$.

Hence, prove that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{2}\int_0^a [\mathrm{f}(x)+\mathrm{f}(a-x)]\,\mathrm{d}x$.

(3 marks)

(b) Show that $\int_0^1 \frac{\mathrm{d}x}{x^2-x+1} = \frac{2\sqrt{3}\pi}{9}$.

(5 marks)

(c) Using (a) and (b), or otherwise, evaluate $\int_0^1 \frac{\mathrm{d}x}{(x^2-x+1)(e^{2x-1}+1)}$.

(6 marks)

14.

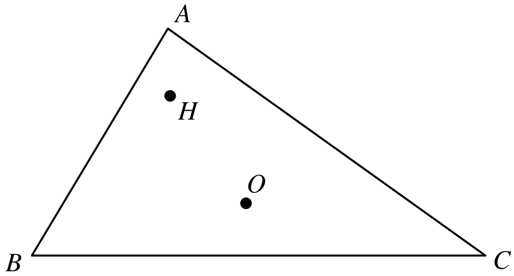

Figure 3

In Figure 3, ΔABC is an acute-angled triangle, where O and H are the circumcentre and orthocentre respectively. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OH} = \mathbf{h}$

(a) Show that $(\mathbf{h}-\mathbf{a}) // (\mathbf{b}+\mathbf{c})$.

(3 marks)

(b) Let $\mathbf{h}-\mathbf{a} = t(\mathbf{b}+\mathbf{c})$, where t is a non-zero constant.
Show that

(i) $t(\mathbf{b}+\mathbf{c})+\mathbf{a}-\mathbf{b} = s(\mathbf{c}+\mathbf{a})$ for some scalar s ,

(ii) $(t-1)(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a}) = 0$.

(5 marks)

(c) Express $\mathbf{h}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(2 marks)

END OF PAPER