2023-DSE
MATH EP
M2

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HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2023

Candidate Number									

MATHEMATICS Extended Part

Module 2 (Algebra and Calculus)

Question-Answer Book

8:30 am – 11:00 am (2½ hours)

This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) The diagrams in this paper are not necessarily drawn to scale.

(8) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

* A 0 3 2 E 0 0 1 *

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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SECTION A (50 marks)

1. Let a be a constant. If the coefficient of x in the expansion of $(2-3x)^5\left(x+\frac{a}{x}\right)^2$ is $\frac{160}{3}$, find a and the coefficient of x^2 in the expansion. (5 marks)

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2. Let $f(x) = -x\sin x$.

(a) Prove that $f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)=\pi\sin^2\left(\frac{h}{2}\right)-h\cos h$.

(b) Using (a), find $f'\left(\frac{\pi}{2}\right)$ from first principles.

(5 marks)

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3. (a) Find a pair of constants p and q such that $11\sin x + 7\cos x \equiv p(3\sin x + \cos x) + q(3\cos x - \sin x)$.

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{11\sin x + 7\cos x}{3\sin x + \cos x}\,dx$.

(6 marks)

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4. (a) Prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

(b) Using (a), solve the equation $\sec^3 x - 6\sec^2 x + 8 = 0$, where $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

(5 marks)

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5. Let A be a 2×2 real matrix such that $A^2 + A + I = 0$, where I is the 2×2 identity matrix.

(a) Prove that $A^3 = I$.

(b) Prove that A is non-singular.

(c) Someone claims that $\left(A^{1000} + (A^{-1})^{2000}\right)^{-1}$ can be expressed in the form of $\alpha I + \beta A$, where α and β are real numbers. Is the claim correct? Explain your answer.

(7 marks)

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6. (a) Let r be a positive constant. Figure 1 shows the shaded region which is inside the circle $x^2+(y-r)^2=r^2$ and below the horizontal line $y=h$, where $0 \le h \le 2r$. Prove that the volume of the solid of revolution generated by revolving the shaded region about the y-axis is $\frac{\pi}{3}h^2(3r-h)$.

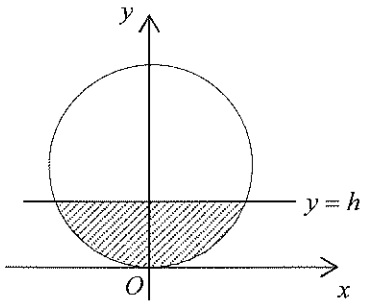

Figure 1

(b) A solid metal sphere of radius 10 cm is put into an empty right cylindrical container of radius 11 cm and height 20 cm, with longitudinal section as shown in Figure 2. Starting from time $t=0$, water is added to the container at a constant rate of $1\,\text{cm}^3$ per second. Let h cm be the depth of water after t seconds. By expressing $\frac{\mathrm{d}h}{\mathrm{d}t}$ in terms of h, find the greatest value of $\frac{\mathrm{d}h}{\mathrm{d}t}$.

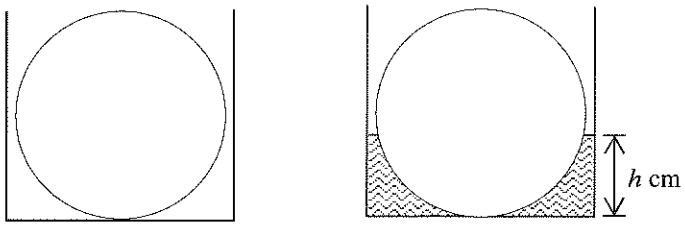

Figure 2

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7. Let $f(x)$ be a function defined on the interval $(-2, 2)$. Denote the curve $y = f(x)$ by Γ. At any point (x, y) on Γ, the slope of the tangent to Γ is $\frac{k - 3x}{\sqrt{4 - x^2}}$, where k is a constant. It is given that Γ passes through the origin.

(a) Find the equation of Γ in terms of k.

(b) Suppose that Γ has a turning point.

(i) Find the range of values of k.

(ii) Does Γ have a point of inflexion? Explain your answer.

(7 marks)

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8. (a) Let $\theta \in \mathbf{R}$. Using mathematical induction, prove that $\sin\theta \sum_{k=1}^{n} \sin 2k\theta = \sin n\theta \sin(n+1)\theta$ for all positive integers n.

(b) Using (a), find a pair of rational numbers a and b such that $\sum_{k=1}^{111} \sin\frac{k\pi}{11}\cos\frac{k\pi}{11} = a\sin b\pi$, where $0 < b < \frac{1}{2}$.

(8 marks)

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SECTION B (50 marks)

9. Define $f(x) = xe^{-x^2}$ for all $x \in \mathbf{R}$. Denote the graph of $y = f(x)$ by G.

(a) Find $f'(x)$ and $f''(x)$. (3 marks)

(b) Find the maximum point(s) and the minimum point(s) of G. (3 marks)

(c) Let L be the tangent to G at the point $\left(1, \frac{1}{e}\right)$.

(i) Find the equation of L.

(ii) By considering $f''(x)$, explain why G lies below L in the interval $(0, 1)$.

(iii) Find the area of the region bounded by G, L and the y-axis.

(6 marks)

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10. Let O be the origin. The position vectors of P and Q are $-2\mathbf{i} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ respectively. Denote the circle passing through O, P and Q by C. Let R be a point lying on PQ such that OR is perpendicular to OQ.

(a) By considering the ratio of PR to RQ, find $\overrightarrow{OR}$. (3 marks)

(b) OR produced meets C at another point S. Find $\overrightarrow{OS}$. (3 marks)

(c) Let Π be the plane which contains C.

(i) Find a non-zero vector which is perpendicular to Π.

(ii) Let G be the centre of C. Denote the projection of point $A(-6, -22, 2)$ on Π by B. Describe the geometric relationship between O, B and G. Explain your answer. (6 marks)

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11. (a) Consider the system of linear equations in real variables x, y, z

$$(E): \begin{cases} x + ay + (a+1)z = 2 \\ x + (a+4)y + (2a+4)z = b+1 \\ 2x + 3y + 5z = b \end{cases}, \text{ where } a, b \in \mathbf{R}.$$

(i) Assume that (E) has a unique solution. Find the range of values of a.

(ii) Assume that $a = 1$. If (E) is consistent, find b.

(iii) Assume that $a \neq 1$ and (E) is inconsistent. Find the range of values of b.

(7 marks)

(b) Consider the system of linear equations in real variables x, y, z

$$(F): \begin{cases} x + 2y + 3z = 2 \\ x + 6y + 8z = s+1 \\ 2x + 3y + 5z = s \end{cases}, \text{ where } s \in \mathbf{R}.$$

Does there exist a pair of real constants m and n (independent of s) such that for every $s \in \mathbf{R}$, (F) has a real solution (x, y, z) satisfying $mx + ny + z = -2$? Explain your answer. (6 marks)

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12. (a) Let a be a non-zero constant. Prove that $\int_0^1 x^2 e^{ax}\,dx = \frac{(a^2-2a+2)e^a-2}{a^3}$. (3 marks)

(b) Using (a) and integration by substitution, evaluate $\int_0^{e-1} x\left(\ln(1+x)\right)^2\,dx$. (4 marks)

(c) Evaluate $\int_0^{\frac{\pi}{2}} \left(\ln(1+(e-1)\cos x)\right)^2 \sin 2x\,dx$. (3 marks)

(d) Evaluate $\int_{\frac{\pi}{2}}^{\pi} \left(\ln(1+(e-1)\sin x)\right)^2 \sin 2x\,dx$. (3 marks)

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