2022-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2022

Please stick the barcode label here.

Candidate Number									

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)
Question-Answer Book

8:30 am – 11:00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

(1) After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11 and 13.

(2) This paper consists of TWO sections, A and B.

(3) Attempt ALL questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

(4) Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string INSIDE this book.

(5) Unless otherwise specified, all working must be clearly shown.

(6) Unless otherwise specified, numerical answers must be exact.

(7) No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

A032E001

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

****

Answers written in the margins will not be marked.

SECTION A (50 marks)

1. Let $g(x) = \frac{1}{\sqrt{5x+4}}$, where $x > 0$. Prove that $g(1+h) - g(1) = \frac{-5h}{3\sqrt{5h+9}\,(3+\sqrt{5h+9})}$. Hence, find $g'(1)$ from first principles. (4 marks)

Answers written in the margins will not be marked.

Please stick the barcode label here.

2. Let $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

(a) Prove that $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 1 + \sec\theta\csc\theta$.

(b) Solve the equation $\frac{\tan\theta}{1-\cot\theta} + \frac{\cot\theta}{1-\tan\theta} = 5$.

(5 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

3. (a) Using mathematical induction, prove that $\sum_{k=1}^{2n}(-1)^k k^2 = n(2n+1)$ for all positive integers n.

(b) Using (a), evaluate $\sum_{k=11}^{100}(-1)^k k^2$.

(7 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Please stick the barcode label here.

4. Let $y=(7x-2x^2)e^{-x}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Someone claims that there are two points of inflexion of the graph of $y=(7x-2x^2)e^{-x}$. Do you agree? Explain your answer.

(6 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

5. Let n be an integer greater than 1 . Define $(a+x)^n = \sum_{k=0}^{n} \mu_k x^k$, where a is a constant. It is given that $\mu_2 = -10$.

(a) Explain why a is a negative number and n is an odd number.

(b) Let $(bx-1)^n = \sum_{k=0}^{n} \lambda_k x^k$, where b is a constant. If $\lambda_0 = \mu_0$ and $\lambda_1 = 2\mu_1$, find a , b and n .

(6 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

6. (a) Using integration by substitution, prove that $\int \frac{1}{x^2+2x+5}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + \text{constant}$.

(b) At any point (x, y) on the curve G, the slope of the tangent to G is $\frac{2x+1}{x^2+2x+5}$. Given that G passes through the point $(-3, \ln 2)$, does G pass through the point $\left(-1, \frac{-\pi}{8}\right)$? Explain your answer.

(7 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

7. Consider the curve $\Gamma : y = \ln(x + 2)$, where $x > 0$. Let P be a moving point on Γ with h as its x-coordinate. Denote the tangent to Γ at P by L and the area of the region bounded by Γ, L and the y-axis by A square units.

(a) Prove that $A = \dfrac{h^2 + 4h}{2h+4} - 2\ln(h+2) + 2\ln 2$.

(b) If $h = 3^{-t}$, where t is the time measured in seconds, find the rate of change of A when $t = 1$.

(8 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

8. Consider the system of linear equations in real variables x, y and z

$$(E):\begin{cases} ax + 2y - z = 4k \\ -x + ay + 2z = 4 \\ 2x - y + az = k^2 \end{cases}, \text{ where } a, k \in \mathbf{R}.$$

(a) Assume that (E) has a unique solution. Express y in terms of a and k.

(b) Assume that (E) has infinitely many solutions. Solve (E).

(7 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

SECTION B (50 marks)

9. Let $f(x)=\dfrac{x^2+3x}{x-1}$, where $x \neq 1$. Denote the graph of $y=f(x)$ by H .

(a) Find the asymptote(s) of H . (3 marks)

(b) Find the maximum point(s) and minimum point(s) of H . (4 marks)

(c) Sketch H . (3 marks)

(d) Let R be the region bounded by H and the straight line $y=10$. Find the volume of the solid of revolution generated by revolving R about the straight line $y=10$. (3 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

10. Let $g(x)=\cos^2 x\cos 2x$.

(a) Prove that $\int g(x)\,dx=\frac{\sin 2x\cos^2 x}{2}+\frac{1}{2}\int \sin^2 2x\,dx$. (2 marks)

(b) Evaluate $\int_0^{\pi} g(x)\,dx$. (2 marks)

(c) Using integration by substitution, evaluate $\int_0^{\pi} xg(x)\,dx$. (4 marks)

(d) Evaluate $\int_{-\pi}^{2\pi} xg(x)\,dx$. (4 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

11. (a) Let n be a positive integer. Denote the 2×2 identity matrix by I.

(i) Let A be a 2×2 matrix. Simplify $(I-A)(I+A+A^2+\cdots+A^n)$.

(ii) Let $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$, where θ is not a multiple of 2π.

It is given that $A^n=\begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix}$.

(1) Prove that $(I-A)^{-1}=\dfrac{1}{2\sin\frac{\theta}{2}}\begin{pmatrix}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\\ \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\end{pmatrix}$.

(2) Using the result of (a)(i) and (a)(ii)(1),

prove that $I+A+A^2+\cdots+A^n=\dfrac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\begin{pmatrix}\cos\frac{n\theta}{2} & -\sin\frac{n\theta}{2}\\ \sin\frac{n\theta}{2} & \cos\frac{n\theta}{2}\end{pmatrix}$.

(7 marks)

(b) Using (a)(ii), evaluate

(i) $\cos\frac{5\pi}{18}+\cos\frac{5\pi}{9}+\cos\frac{5\pi}{6}+\cdots+\cos 25\pi$;

(ii) $\cos^2\frac{\pi}{7}+\cos^2\frac{2\pi}{7}+\cos^2\frac{3\pi}{7}+\cdots+\cos^2 7\pi$.

(6 marks)

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

Answers written in the margins will not be marked.

12. Consider ΔABC. Denote the origin by O.

(a) Let D be a point lying on BC such that AD is the angle bisector of $\angle BAC$. Define $BC = a$, $AC = b$ and $AB = c$.

(i) Using the fact that $BD : DC = c : b$, prove that $\overrightarrow{AD} = -\overrightarrow{OA} + \frac{b}{b+c}\overrightarrow{OB} + \frac{c}{b+c}\overrightarrow{OC}$.

(ii) Let E be a point lying on AC such that BE is the angle bisector of $\angle ABC$.
Define $\overrightarrow{OJ} = \frac{a}{a+b+c}\overrightarrow{OA} + \frac{b}{a+b+c}\overrightarrow{OB} + \frac{c}{a+b+c}\overrightarrow{OC}$.
Prove that J lies on AD. Hence, deduce that AD and BE intersect at J.

(7 marks)

(b) Suppose that $\overrightarrow{OA} = 35\mathbf{i} + 9\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = 40\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = -3\mathbf{j} + \mathbf{k}$. Let I be the incentre of ΔABC.

(i) Find $\overrightarrow{OI}$.

(ii) By considering $\overrightarrow{AI} \times \overrightarrow{AB}$, find the radius of the inscribed circle of ΔABC.

(5 marks)