Marking Scheme

Module 2 (Algebra and Calculus)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

Solution	Marks	Remarks
1. (a) $(1-x)^4$		
$=1-4x+6x^2-4x^3+x^4$	1A	
(b) Note that $(1+kx)^9=1+C_1^9kx+C_2^9(kx)^2+\cdots+(kx)^9$.	1M	
$C_2^9k^2-4C_1^9k+6=-3$	1M	
$36k^2-36k+9=0$		
$(2k-1)^2=0$		
Thus, we have $k=\frac{1}{2}$.	1A	0.5
	----------(4)	
2. $f'(2)$		
$=\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$	1M	
$=\lim_{h\to 0}\frac{2+h-\sqrt{4+h}}{h\sqrt{4+h}}$		
$=\lim_{h\to 0}\frac{(2+h)^2-(4+h)}{h\sqrt{4+h}(2+h+\sqrt{4+h})}$	1M	
$=\lim_{h\to 0}\frac{h(h+3)}{h\sqrt{4+h}(2+h+\sqrt{4+h})}$		
$=\lim_{h\to 0}\frac{h+3}{\sqrt{4+h}(2+h+\sqrt{4+h})}$	1M	withhold 1M if the step is skipped
$=\frac{3}{8}$	1A	0.375
$f'(x)$		
$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$	1M	
$=\lim_{h\to 0}\frac{\frac{x+h}{\sqrt{2+x+h}}-\frac{x}{\sqrt{2+x}}}{h}$		
$=\lim_{h\to 0}\frac{(x+h)\sqrt{2+x}-x\sqrt{2+x+h}}{h\sqrt{2+x}\sqrt{2+x+h}}$		
$=\lim_{h\to 0}\frac{(2+x)(x+h)^2-x^2(2+x+h)}{h\sqrt{2+x}\sqrt{2+x+h}\,((x+h)\sqrt{2+x}+x\sqrt{2+x+h})}$	1M	
$=\lim_{h\to 0}\frac{h(x^2+4x+hx+2h)}{h\sqrt{2+x}\sqrt{2+x+h}\,((x+h)\sqrt{2+x}+x\sqrt{2+x+h})}$		
$=\lim_{h\to 0}\frac{x^2+4x+hx+2h}{\sqrt{2+x}\sqrt{2+x+h}\,((x+h)\sqrt{2+x}+x\sqrt{2+x+h})}$	1M	withhold 1M if the step is skipped
$=\frac{x^2+4x}{2x(2+x)\sqrt{2+x}}$		
Thus, we have $f'(2)=\frac{3}{8}$.	1A	0.375
	----------(4)	

Solution	Marks	Remarks
3. (a) (i) $\tan 3x$		
$= \tan(2x + x)$		
$= \dfrac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$	1M	either one
$= \dfrac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \left(\dfrac{2\tan x}{1-\tan^2 x}\right)\tan x}$		
$= \dfrac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x}$		
$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	1	
(ii) $\tan x \tan(60° - x)\tan(60° + x)$		
$= \tan x \left(\dfrac{\tan 60° - \tan x}{1 + \tan 60° \tan x}\right)\left(\dfrac{\tan 60° + \tan x}{1 - \tan 60° \tan x}\right)$		
$= \tan x \left(\dfrac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}\right)\left(\dfrac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right)$	1M	
$= \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$		
$= \tan 3x$	1	
(b) Putting $x = 5°$ in (a)(ii), we have		
$\tan 5° \tan(60° - 5°)\tan(60° + 5°) = \tan 3(5°)$	1M	
$\tan 5° \tan 55° \tan 65° = \tan 15°$		
$\left(\dfrac{1}{\tan 85°}\right)\tan 55° \tan 65° = \dfrac{1}{\tan 75°}$		
$\tan 55° \tan 65° \tan 75° = \tan 85°$	1	
Putting $x = 25°$ in (a)(ii), we have		
$\tan 25° \tan(60° - 25°)\tan(60° + 25°) = \tan 3(25°)$	1M	
$\tan 25° \tan 35° \tan 85° = \tan 75°$		
$\left(\dfrac{1}{\tan 65°}\right)\left(\dfrac{1}{\tan 55°}\right)\tan 85° = \tan 75°$		
$\tan 55° \tan 65° \tan 75° = \tan 85°$	1	
Putting $x = 95°$ in (a)(ii), we have		
$\tan 95° \tan(60° - 95°)\tan(60° + 95°) = \tan 3(95°)$	1M	
$\tan 95° \tan(-35°)\tan 155° = \tan 285°$		
$\tan 85°\left(\dfrac{1}{\tan 55°}\right)\left(\dfrac{1}{\tan 65°}\right) = \tan 75°$		
$\tan 55° \tan 65° \tan 75° = \tan 85°$	1	
	----------(6)	

Solution	Marks	Remarks
4. (a) $\int \sin^2\theta \, d\theta$		
$= \int \frac{1-\cos 2\theta}{2} \, d\theta$	1M	
$= \frac{\theta}{2} - \frac{1}{2}\int \cos 2\theta \, d\theta$		
$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + \text{constant}$	1A	
(b) The required volume		
$= \pi \int_0^1 \left(4x(1-x^2)^{\frac{1}{4}} \right)^2 dx$	1M	
$= \pi \int_0^{\frac{\pi}{2}} 16\sin^2 u \cos^2 u \, du$ (by letting $x = \sin u$)	1M	
$= \pi \int_0^{\frac{\pi}{2}} 4\sin^2 2u \, du$		
$= 2\pi \int_0^{\pi} \sin^2\theta \, d\theta$ (by letting $\theta = 2u$)		
$= \pi \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$ (by (a))	1M	for using the result of (a)
$= \pi^2$	1A	
	----------(6)	

Solution	Marks	Remarks
5. (a) Note that $\frac{1}{(1)(2)(3)}=\frac{(1)(4)}{4(2)(3)}$.		
Therefore, the statement is true for $n=1$.	1	
Assume that $\sum_{k=1}^{m}\frac{1}{k(k+1)(k+2)}=\frac{m(m+3)}{4(m+1)(m+2)}$, where m is a positive integer.	1M	
$\sum_{k=1}^{m+1}\frac{1}{k(k+1)(k+2)}$		
$=\sum_{k=1}^{m}\frac{1}{k(k+1)(k+2)}+\frac{1}{(m+1)(m+2)(m+3)}$		
$=\frac{m(m+3)}{4(m+1)(m+2)}+\frac{1}{(m+1)(m+2)(m+3)}$ (by induction assumption)	1M	for using induction assumption
$=\frac{m(m+3)^2+4}{4(m+1)(m+2)(m+3)}$		
$=\frac{m^3+6m^2+9m+4}{4(m+1)(m+2)(m+3)}$		
$=\frac{(m+1)^2(m+4)}{4(m+1)(m+2)(m+3)}$		
$=\frac{(m+1)(m+4)}{4(m+2)(m+3)}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, the statement is true for all positive integers n .	1	
(b) $\sum_{k=4}^{123}\frac{50}{k(k+1)(k+2)}$		
$=50\sum_{k=4}^{123}\frac{1}{k(k+1)(k+2)}$		
$=50\left(\sum_{k=1}^{123}\frac{1}{k(k+1)(k+2)}-\sum_{k=1}^{3}\frac{1}{k(k+1)(k+2)}\right)$	1M	
$=50\left(\frac{(123)(123+3)}{4(123+1)(123+2)}-\frac{1}{6}-\frac{1}{24}-\frac{1}{60}\right)$ (by (a))	1M	for using (a)
$=50\left(\frac{(123)(126)}{4(124)(125)}-\frac{9}{40}\right)$		
$=\frac{387}{310}$	1A	
	----------(7)	

Solution	Marks	Remarks
6. (a) Let A square units be the area of the circle with OP as a diameter.		
A $=\pi\left(\frac{\sqrt{u^2+v^2}}{2}\right)^2$ $=\frac{\pi}{4}(u^2+v^2)$	1M	
Therefore, we have $u^2+v^2=\frac{4A}{\pi}$.		
Hence, we have $S=\frac{4A}{\pi}$.		
At time t s , we have $\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{4}{\pi}\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)=\frac{4}{\pi}(5\pi)=20$.		
Note that 20 is a positive constant. Thus, S increases at a constant rate.	1A	f.t.
(b) Let R square units be the area of ΔOPQ .		
Note that $R=\frac{1}{2}u(2^u-2^{u-1})=u2^{u-2}$.	1M	
At time t s , we have $\frac{\mathrm{d}R}{\mathrm{d}t}=(1+u\ln 2)2^{u-2}\frac{\mathrm{d}u}{\mathrm{d}t}$.	1M	
Since $\frac{\mathrm{d}}{\mathrm{d}t}(u^2+v^2)=20$ (by (a)), we have $2u\frac{\mathrm{d}u}{\mathrm{d}t}+2v\frac{\mathrm{d}v}{\mathrm{d}t}=20$.	1M	
Therefore, we have $2u\frac{\mathrm{d}u}{\mathrm{d}t}+2(2^{u-1})^2\ln 2\frac{\mathrm{d}u}{\mathrm{d}t}=20$.		
Hence, we have $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{10}{u+2^{2u-2}\ln 2}$.		
$\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{u=2}$ $=\frac{10}{2+4\ln 2}$ $=\frac{5}{1+2\ln 2}$		
$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right\|_{u=2}$ $=(1+2\ln 2)\left.\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_{u=2}$	1M	
$=(1+2\ln 2)\frac{5}{1+2\ln 2}$ $=5$	1A	
Thus, the required rate of change is 5 square units per second.		
	----------(7)	

Solution	Marks	Remarks
7. (a) $\mathrm{f}(x)$		
$=\int(-2x+8)\,\mathrm{d}x$		
$=-x^2+8x+C$, where C is a constant	1M	
Since $\mathrm{f}(1)=2$, we have $-1^2+8(1)+C=2$.	1M	
Solving, we have $C=-5$.		
Thus, the equation of Γ is $y=-x^2+8x-5$.	1A	
(b) (i) Let (a,b) be the coordinates of P .		
Therefore, we have $\dfrac{b-14}{a-5}=-2a+8$ and $b=-a^2+8a-5$.	1M	for either one
$-2a^2+18a-26=-a^2+8a-5$		
$a^2-10a+21=0$		
$a=7$ or $a=3$		
Since $\mathrm{f}'(7)=-6<0$ and $\mathrm{f}'(3)=2>0$, we have $a=7$.		
Thus, the coordinates of P are $(7,2)$.	1A	
(ii) The slope of the tangent to Γ at P		
$=\mathrm{f}'(7)$		
$=-6$		
The slope of the normal to Γ at P		
$=\dfrac{1}{6}$	1M	
The equation of the normal to Γ at P is		
$y-2=\dfrac{1}{6}(x-7)$	1M	
$x-6y+5=0$	1A	
	----------(8)	

Solution	Marks	Remarks
8. (a) $\begin{pmatrix}-5 & -2\\ 15 & 6\end{pmatrix}\begin{pmatrix}1 & a\\ b & c\end{pmatrix}=\begin{pmatrix}1 & a\\ b & c\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$		
$\begin{pmatrix}-5-2b & -5a-2c\\ 15+6b & 15a+6c\end{pmatrix}=\begin{pmatrix}1 & 0\\ b & 0\end{pmatrix}$		
So, we have $-5-2b=1$, $-5a-2c=0$, $15+6b=b$ and $15a+6c=0$.	1M	for either one
Therefore, we have $b=-3$ and $c=\frac{-5}{2}a$.		
Since $\lvert M \rvert=1$, we have $c-ab=1$.	1M	
Hence, we have $\frac{-5a}{2}+3a=1$.		
Thus, we have $a=2$, $b=-3$ and $c=-5$.	1A	for all correct
(b) (i) By (a), we have $M=\begin{pmatrix}1 & 2\\ -3 & -5\end{pmatrix}$.		
M^{-1} $=\frac{1}{1}\begin{pmatrix}-5 & -2\\ 3 & 1\end{pmatrix}$ $=\begin{pmatrix}-5 & -2\\ 3 & 1\end{pmatrix}$	1M	
$M^{-1}RM$ $=\begin{pmatrix}-5 & -2\\ 3 & 1\end{pmatrix}\begin{pmatrix}6 & 2\\ -15 & -5\end{pmatrix}\begin{pmatrix}1 & 2\\ -3 & -5\end{pmatrix}$ $=\begin{pmatrix}0 & 0\\ 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ -3 & -5\end{pmatrix}$ $=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$	1A	
(ii) Note that $P=M\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}M^{-1}$.		
By (b)(i), we have $R=M\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}M^{-1}$.		
$(\alpha P+\beta R)^{99}$ $=\left[\alpha M\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}M^{-1}+\beta M\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}M^{-1}\right]^{99}$	1M	
$=\left[M\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}M^{-1}\right]^{99}$		
$=M\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}^{99}M^{-1}$	1M	
$=M\begin{pmatrix}\alpha^{99} & 0\\ 0 & \beta^{99}\end{pmatrix}M^{-1}$		
$=\alpha^{99}M\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}M^{-1}+\beta^{99}M\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}M^{-1}$		
$=\alpha^{99}P+\beta^{99}R$	1	
	----------(8)	

Solution	Marks	Remarks

9. (a) The equation of the vertical asymptote is $x-4=0$. — 1A

Note that $\dfrac{(x+4)^3}{(x-4)^2}=x+20+\dfrac{192x-256}{(x-4)^2}$. — 1M

Thus, the equation of the oblique asymptote is $y=x+20$. — 1A

----------(3)

(b) $f'(x)$

$$=\frac{(x-4)^2(3(x+4)^2)-(x+4)^3(2(x-4))}{(x-4)^4}$$ — 1M

$$=\frac{(x+4)^2(x-20)}{(x-4)^3}$$

$f''(x)$

$$=\frac{384(x+4)}{(x-4)^4}$$ — 1A

----------(2)

(c) By (b), we have $f'(x)=\dfrac{(x+4)^2(x-20)}{(x-4)^3}$.

So, we have $f'(x)=0 \iff x=-4$ or $x=20$.

x	$(-\infty,-4)$	-4	$(-4,4)$	$(4,20)$	20	$(20,\infty)$
$f'(x)$	+	0	+	–	0	+
$f(x)$	↗	0	↗	↘	54	↗

— 1M

Therefore, there is only one turning point of H.
Thus, the claim is disagreed. — 1A — f.t.

----------(2)

(d) By (b), we have $f''(x)=\dfrac{384(x+4)}{(x-4)^4}$.

So, we have $f''(x)=0 \iff x=-4$.

x	$(-\infty,-4)$	-4	$(-4,4)$
$f''(x)$	–	0	+

— 1M

Thus, the point of inflexion of H is $(-4,0)$. — 1A

----------(2)

(e) The required area

$$=\int_{-4}^{0}\frac{(x+4)^3}{(x-4)^2}\,dx$$ — 1M

$$=\int_{-8}^{-4}\frac{(u+8)^3}{u^2}\,du \qquad (\text{ by letting } u=x-4\ )$$

$$=\int_{-8}^{-4}\left(u+24+\frac{192}{u}+\frac{512}{u^2}\right)du$$ — 1M

$$=\left[\frac{u^2}{2}+24u+192\ln|u|-\frac{512}{u}\right]_{-8}^{-4}$$

$$=136-192\ln 2$$ — 1A

----------(3)

Solution	Marks	Remarks

10. (a) Let $u=\frac{\pi}{4}-x$. Then, we have $\frac{du}{dx}=-1$. **1M**

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)dx$$

$$=-\int_{\frac{\pi}{6}}^{\frac{\pi}{12}} \ln(\sin u)\,du$$ **1M**

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin u)\,du$$

$$=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$ **1**

----------(3)

(b) $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right)dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$ (by (a))

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\frac{\pi}{4}\cos x-\cos\frac{\pi}{4}\sin x\right)dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$ **1M**

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\frac{1}{\sqrt{2}}(\cos x-\sin x)\right)dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \left(\ln\left(\frac{1}{\sqrt{2}}\right)+\ln(\cos x-\sin x)\right)dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\sin x)\,dx$$ **1M**

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\frac{\cos x-\sin x}{\sin x}\right)dx=\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\sqrt{2}\,dx$$

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x-1)\,dx=\frac{\pi\ln 2}{24}$$ **1A**

----------(3)

(c) (i) $\cot\frac{\pi}{12}$

$$=\cot\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$=\frac{\cot\frac{\pi}{4}\cot\frac{\pi}{6}+1}{\cot\frac{\pi}{6}-\cot\frac{\pi}{4}}$$ **1M**

$$=\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$=\frac{3+2\sqrt{3}+1}{2}$$

$$=2+\sqrt{3}$$ **1**

Solution	Marks	Remarks
$\cot\left(\frac{\pi}{12}-\left(\frac{-\pi}{12}\right)\right)=\cot\frac{\pi}{6}$		
$\dfrac{\cot\frac{\pi}{12}\cot\left(\frac{-\pi}{12}\right)+1}{\cot\left(\frac{-\pi}{12}\right)-\cot\frac{\pi}{12}}=\cot\frac{\pi}{6}$	1M	
$\dfrac{-\cot^2\frac{\pi}{12}+1}{-2\cot\frac{\pi}{12}}=\sqrt{3}$		
$\cot^2\frac{\pi}{12}-2\sqrt{3}\cot\frac{\pi}{12}-1=0$		
$\cot\frac{\pi}{12}=\dfrac{2\sqrt{3}\pm\sqrt{(-2\sqrt{3})^2-4(1)(-1)}}{2}$		
$\cot\frac{\pi}{12}=2+\sqrt{3}$ or $\cot\frac{\pi}{12}=-2+\sqrt{3}$ (rejected)		
Thus, we have $\cot\frac{\pi}{12}=2+\sqrt{3}$.	1	
(ii) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\frac{x\csc^2 x}{\cot x-1}\,dx$		
$=-\Big[x\ln(\cot x-1)\Big]_{\frac{\pi}{12}}^{\frac{\pi}{6}}+\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{6}}\ln(\cot x-1)\,dx$	1M	
$=-\left(\frac{\pi}{6}\ln\left(\cot\frac{\pi}{6}-1\right)-\frac{\pi}{12}\ln\left(\cot\frac{\pi}{12}-1\right)\right)+\frac{\pi\ln 2}{24}$ (by (b))	1M	for using the result of (b)
$=\frac{\pi}{12}(\ln(\sqrt{3}+1)-2\ln(\sqrt{3}-1))+\frac{\pi\ln 2}{24}$ (by (c)(i))	1M	for using (c)(i)
$=\frac{\pi}{12}\ln\left(\frac{\sqrt{3}+1}{(\sqrt{3}-1)^2}\right)+\frac{\pi\ln 2}{24}$	1M	
$=\frac{\pi}{12}\ln\left(\frac{(\sqrt{3}+1)^3}{4}\right)+\frac{\pi\ln 2}{24}$		
$=\frac{\pi}{24}\ln\left(2\left(\frac{(\sqrt{3}+1)^3}{4}\right)^2\right)$		
$=\frac{\pi}{24}\ln\left(\frac{(\sqrt{3}+1)^6}{8}\right)$		
$=\frac{\pi}{8}\ln\left(\frac{(\sqrt{3}+1)^2}{2}\right)$		
$=\frac{\pi}{8}\ln\left(\frac{4+2\sqrt{3}}{2}\right)$		
$=\frac{\pi}{8}\ln(2+\sqrt{3})$	1	
	----------(7)	

Solution	Marks	Remarks
11. (a) (i) (1) Note that $\begin{vmatrix} 1 & -1 & -2 \\ 1 & -2 & h \\ 4 & h & -7 \end{vmatrix}$		
$= (-2)(-7)+(-1)(h)(4)+(-2)(h)-(-2)(-2)(4)-(h)(h)-(-1)(-7)$	1A	
$= -(h+3)^2$		
Since (E) has a unique solution, we have $\begin{vmatrix} 1 & -1 & -2 \\ 1 & -2 & h \\ 4 & h & -7 \end{vmatrix} \neq 0$.	1M	
So, we have $-(h+3)^2 \neq 0$.		
Thus, we have $h \neq -3$.	1	
The augmented matrix of (E) is		
$\left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 1 & -2 & h & k \\ 4 & h & -7 & 7 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 0 & -1 & h+2 & k-1 \\ 0 & h+4 & 1 & 3 \end{array}\right)$	1M	
$\sim \left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 0 & -1 & h+2 & k-1 \\ 0 & 0 & (h+3)^2 & hk-h+4k-1 \end{array}\right)$	1A	
Since (E) has a unique solution, we have $(h+3)^2 \neq 0$.		
Thus, we have $h \neq -3$.	1	
(2) Since (E) has a unique solution, we have		
$x = \dfrac{\begin{vmatrix} 1 & -1 & -2 \\ k & -2 & h \\ 7 & h & -7 \end{vmatrix}}{-(h+3)^2}$	1M	for Cramer's Rule
$= \dfrac{h^2+2hk+7h+7k+14}{(h+3)^2}$		
$y = \dfrac{\begin{vmatrix} 1 & 1 & -2 \\ 1 & k & h \\ 4 & 7 & -7 \end{vmatrix}}{-(h+3)^2}$		
$= \dfrac{3h-k+7}{(h+3)^2}$		
$z = \dfrac{\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & k \\ 4 & h & 7 \end{vmatrix}}{-(h+3)^2}$		
$= \dfrac{hk-h+4k-1}{(h+3)^2}$	1A+1A	1A for any one + 1A for all

Solution	Marks	Remarks
Since (E) has a unique solution, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 0 & -1 & h+2 & k-1 \\ 0 & 0 & (h+3)^2 & hk-h+4k-1 \end{array}\right)$ $\sim\left(\begin{array}{ccc\|c} 1 & 0 & 0 & \dfrac{h^2+2hk+7h+7k+14}{(h+3)^2} \\ 0 & 1 & 0 & \dfrac{3h-k+7}{(h+3)^2} \\ 0 & 0 & 1 & \dfrac{hk-h+4k-1}{(h+3)^2} \end{array}\right)$	1M	
Thus, we have $x=\dfrac{h^2+2hk+7h+7k+14}{(h+3)^2}$, $y=\dfrac{3h-k+7}{(h+3)^2}$ and $z=\dfrac{hk-h+4k-1}{(h+3)^2}$.	1A+1A	1A for any one + 1A for all
(ii) (1) When $h=-3$, the augmented matrix of (E) is $\left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 1 & -2 & -3 & k \\ 4 & -3 & -7 & 7 \end{array}\right)\sim\left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 0 & -1 & -1 & k-1 \\ 0 & 0 & 0 & k+2 \end{array}\right)$	1M	either one
Since (E) is consistent, we have $k=-2$.	1	
(2) When $h=-3$ and $k=-2$, the augmented matrix of (E) $\sim\left(\begin{array}{ccc\|c} 1 & -1 & -2 & 1 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Thus, the solution set of (E) is $\{(t+4,-t+3,t):\ t\in\mathbf{R}\}$.	1A ----------(9)	
(b) When $h=-3$, the solution set of (F) is $\{(t+4,-t+3,t):\ t\in\mathbf{R}\}$. $3x^2+4y^2-7z^2$ $=3(t+4)^2+4(-t+3)^2-7t^2$ $=84$ $\neq 1$	1M	
When $h=-3$, (F) does not have a real solution (x,y,z) satisfying $3x^2+4y^2-7z^2=1$.	1A	
When $h\neq -3$, the solution of (F) is $x=\dfrac{h}{h+3}$, $y=\dfrac{3}{h+3}$ and $z=\dfrac{-3}{h+3}$. $3x^2+4y^2-7z^2$ $=3\left(\dfrac{h}{h+3}\right)^2+4\left(\dfrac{3}{h+3}\right)^2-7\left(\dfrac{-3}{h+3}\right)^2$ $=\dfrac{3(h-3)}{h+3}$	1M	
Hence, we have $3x^2+4y^2-7z^2=1 \Leftrightarrow \dfrac{3(h-3)}{h+3}=1$. Therefore, we have $3x^2+4y^2-7z^2=1 \Leftrightarrow h=6$. So, there is only one value of h such that (F) has a real solution (x,y,z) satisfying $3x^2+4y^2-7z^2=1$. Thus, the claim is disagreed.	1A ----------(4)	f.t.

Solution	Marks	Remarks
12. (a) Note that $\overrightarrow{OR}=\frac{3}{4}\overrightarrow{OP}+\frac{1}{4}\overrightarrow{OQ}$.	1M	
So, we have $\overrightarrow{OR}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.		
$\overrightarrow{OP}\times\overrightarrow{OR}$ $=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&4\\2&-1&2\end{vmatrix}$		
$=6\mathbf{i}+6\mathbf{j}-3\mathbf{k}$	1A ----------(2)	
(b) Note that the quadrilateral $OPSR$ is a parallelogram. The area of the quadrilateral $OPSR$		
$=\left\|\overrightarrow{OP}\times\overrightarrow{OR}\right\|$	1M	
$=\left\|6\mathbf{i}+6\mathbf{j}-3\mathbf{k}\right\|$		
$=\sqrt{6^2+6^2+(-3)^2}$		
$=9$	1A ----------(2)	
(c) (i) Note that $\overrightarrow{NR}=\overrightarrow{OR}-\overrightarrow{ON}=(2-6\lambda)\mathbf{i}-(1+6\lambda)\mathbf{j}+(2+3\lambda)\mathbf{k}$	1M	
and $\overrightarrow{PQ}=4\mathbf{i}-8\mathbf{j}-8\mathbf{k}$.		
So, we have $\overrightarrow{NR}\cdot\overrightarrow{PQ}=4(2-6\lambda)+8(1+6\lambda)-8(2+3\lambda)=0$.		
Thus, $\overrightarrow{NR}$ is perpendicular to $\overrightarrow{PQ}$.	1A	f.t.
(ii) (1) Note that $\overrightarrow{NQ}=\overrightarrow{OQ}-\overrightarrow{ON}=(5-6\lambda)\mathbf{i}-(6\lambda+7)\mathbf{j}+(3\lambda-4)\mathbf{k}$.		
Also note that $\overrightarrow{NQ}$ is parallel to $11\mathbf{i}+\mu\mathbf{j}-10\mathbf{k}$.		
Since $\mu\neq 0$, we have $\frac{6\lambda-5}{11}=\frac{6\lambda+7}{\mu}=\frac{3\lambda-4}{10}$.	1M	
Solving, we have $\lambda=\frac{2}{9}$ and $\mu=-25$.	1A+1A	
(2) Note that PQ is the line of intersection of ΔOPQ and ΔNPQ .		
Since $\overrightarrow{OR}\cdot\overrightarrow{PQ}=0$, OR is perpendicular to PQ . By (c)(i), NR is perpendicular to PQ . So, we have $\theta=\angle ORN$.	1M	for identifying θ
Therefore, we have $\tan\theta=\tan\angle ORN=\frac{\left\|\overrightarrow{ON}\right\|}{\left\|\overrightarrow{OR}\right\|}$.		
By (c)(ii)(1), we have $\tan\theta=\frac{\frac{2}{9}\left\|\overrightarrow{OP}\times\overrightarrow{OR}\right\|}{\left\|\overrightarrow{OR}\right\|}$.	1M	for using the result of (c)(ii)(1)
By (b), we have $\tan\theta=\frac{\frac{2}{9}(9)}{\sqrt{2^2+(-1)^2+2^2}}=\frac{2}{3}$.	1A ----------(8)	