Solution	Marks	Remarks
1. $\frac{d}{d\theta}\sec 6\theta$		
$=\lim_{h\to 0}\frac{\sec 6(\theta+h)-\sec 6\theta}{h}$	1M	
$=\lim_{h\to 0}\frac{\cos 6\theta-\cos 6(\theta+h)}{h\cos 6(\theta+h)\cos 6\theta}$	1M	
$=\lim_{h\to 0}\frac{2\sin(6\theta+3h)\sin 3h}{h\cos 6(\theta+h)\cos 6\theta}$	1M	
$=6\left(\lim_{h\to 0}\frac{\sin 3h}{3h}\right)\left(\lim_{h\to 0}\frac{\sin(6\theta+3h)}{\cos 6(\theta+h)\cos 6\theta}\right)$	1M	withhold 1M if the step is ski
$=6(1)\left(\frac{\sin 6\theta}{\cos^2 6\theta}\right)$	1A	
$=6\sec 6\theta\tan 6\theta$	----------(5)	
2. Note that $(1+ax)^8=1+C_1^8ax+C_2^8(ax)^2+\cdots+(ax)^8$ and	1M	
$(b+x)^9=b^9+C_1^9b^8x+C_2^9b^7x^2+\cdots+C_7^9b^2x^7+C_8^9bx^8+x^9$.	1M	
Also note that $\lambda_2:\mu_7=7:4$ and $\lambda_1+\mu_8+6=0$.		
Therefore, we have $\frac{C_2^8a^2}{C_7^9b^2}=\frac{7}{4}$ and $8a+9b+6=0$.	1M	for either one
So, we have $4a^2=9b^2$ and $8a+9b+6=0$.		
Hence, we have $4a^2-9\left(\frac{-8a-6}{9}\right)^2=0$.		
Simplifying, we have $7a^2+24a+9=0$.	1M	for $pa^2+qa+r=0$
Thus, we have $a=-3$ or $a=\frac{-3}{7}$.	1A	for both correct
	----------(5)	

Solution	Marks	Remarks
(a) $\overrightarrow{OP}$ $=\dfrac{2}{2+3}\mathbf{a}+\dfrac{3}{2+3}\mathbf{b}$		
$=\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}$	1A	
(b) (i) $\mathbf{a}\cdot\mathbf{b}$ $=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$	1M	
$=(45)(20)\left(\dfrac{1}{4}\right)$ $=225$	1A	
(ii) $\left\lvert\overrightarrow{OP}\right\rvert^2$ $=\left(\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}\right)\cdot\left(\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}\right)$ $=\dfrac{4}{25}\lvert\mathbf{a}\rvert^2+2\left(\dfrac{2}{5}\right)\left(\dfrac{3}{5}\right)\mathbf{a}\cdot\mathbf{b}+\dfrac{9}{25}\lvert\mathbf{b}\rvert^2$ $=324+108+144$ $=576$	1M	for using (b)(i)
$\left\lvert\overrightarrow{OP}\right\rvert$ $=\sqrt{576}$ $=24$	1A	
	----------(5)	
(a) $\int x^2e^{-x}\mathrm{d}x$ $=-\int x^2\mathrm{d}e^{-x}$		
$=-x^2e^{-x}+\int e^{-x}\mathrm{d}x^2$	1M	for integration by parts
$=-x^2e^{-x}+2\int xe^{-x}\mathrm{d}x$	1A	
$=-x^2e^{-x}-2\int x\mathrm{d}e^{-x}$ $=-x^2e^{-x}-2\left(xe^{-x}-\int e^{-x}\mathrm{d}x\right)$		
$=-x^2e^{-x}-2xe^{-x}-2e^{-x}+\text{constant}$	1A	
$=-e^{-x}(x^2+2x+2)+\text{constant}$		
(b) The required area $=\int_0^6 x^2e^{-x}\mathrm{d}x$	1M	
$=\left[-e^{-x}(x^2+2x+2)\right]_0^6$ (by (a))	1M	for using the result of (a)
$=2-\dfrac{50}{e^6}$	1A	
	----------(6)	

Solution	Marks	Remarks
5. (a) (i) $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & -11 \\ 2 & 3 & h \end{vmatrix} \neq 0$	1M	
$8h-44-9+16+33-6h \neq 0$ $2h-4 \neq 0$ $h \neq 2$ $h<2$ or $h>2$	1A	
(ii) z $= \dfrac{\begin{vmatrix} 1 & 2 & 11 \\ 3 & 8 & 49 \\ 2 & 3 & k \end{vmatrix}}{2h-4}$	1M	
$= \dfrac{k-14}{h-2}$	1A	
(b) When $h=2$, the augmented matrix of (E) is $\left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 3 & 8 & -11 & 49 \\ 2 & 3 & 2 & k \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 2 & -1 & 11 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right) \sim \left(\begin{array}{ccc\|c} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & k-14 \end{array}\right)$.	1M	
Since (E) has infinitely many solutions, we have $h=2$ and $k=14$. Thus, the solution set of (E) is $\{(-7t-5, 4t+8, t) : t \in \mathbf{R}\}$.	1A	
	----------(6)	

Solution	Marks	Remarks
(a) Let r cm be the radius of the water surface in the container.		
Since $\frac{r}{h}=\frac{15}{20}$, we have $\frac{r}{h}=\frac{3}{4}$.	1M	
So, we have $r=\frac{3h}{4}$.		
A		
$=\pi\left(\frac{3h}{4}\right)\sqrt{h^2+\left(\frac{3h}{4}\right)^2}$	1M	
$=\pi\left(\frac{3h}{4}\right)\sqrt{\frac{25h^2}{16}}$		
$=\frac{15}{16}\pi h^2$	1	
(b) Let d cm be the depth of water when the volume of water in the container is $96\pi\ \text{cm}^3$.		
Note that $\frac{\pi d}{3}\left(\frac{3d}{4}\right)^2=96\pi$.	1M	
So, we have $d=8$.	1A	
By (a), we have $A=\frac{15}{16}\pi h^2$.		
At time t s , we have $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{15}{8}\pi h\frac{\mathrm{d}h}{\mathrm{d}t}$.	1M	
Also note that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{3}{\pi}$.		
Therefore, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=\frac{15}{8}\pi(8)\left(\frac{3}{\pi}\right)$.		
Hence, we have $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right\|_{h=8}=45$.	1A	
Thus, the required rate of change is $45\ \text{cm}^2/\text{s}$.	----------(7)	

Solution	Marks	Remarks
7. (a) $\sin 3x$		
$=\sin(x+2x)$	1M	
$=\sin x\cos 2x+\cos x\sin 2x$		
$=\sin x(\cos^2 x-\sin^2 x)+2\sin x\cos^2 x$		
$=\sin x(1-2\sin^2 x)+2\sin x(1-\sin^2 x)$	1	
$=3\sin x-4\sin^3 x$		
(b) (i) $\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$=\dfrac{\sin\left(3x-\frac{3\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}$		
$=\dfrac{\sin 3x\cos\frac{3\pi}{4}-\cos 3x\sin\frac{3\pi}{4}}{\sin x\cos\frac{\pi}{4}-\cos x\sin\frac{\pi}{4}}$	1M	
$=\dfrac{\frac{-1}{\sqrt{2}}(\sin 3x+\cos 3x)}{\frac{1}{\sqrt{2}}(\sin x-\cos x)}$		
$=\dfrac{\cos 3x+\sin 3x}{\cos x-\sin x}$	1	
(ii) $\dfrac{\cos 3x+\sin 3x}{\cos x-\sin x}=2$		
$\dfrac{\sin 3\left(x-\frac{\pi}{4}\right)}{\sin\left(x-\frac{\pi}{4}\right)}=2$ (by (b)(i))	1M	for using (b)(i)
Note that $\sin\left(x-\frac{\pi}{4}\right)\neq 0$.		
$3-4\sin^2\left(x-\frac{\pi}{4}\right)=2$ (by (a))	1M	for using (a)
$1-4\sin^2\left(x-\frac{\pi}{4}\right)=0$		
$\left(1-2\sin\left(x-\frac{\pi}{4}\right)\right)\left(1+2\sin\left(x-\frac{\pi}{4}\right)\right)=0$	1M	
Since $\frac{\pi}{4}<x<\frac{\pi}{2}$, we have $\sin\left(x-\frac{\pi}{4}\right)=\frac{1}{2}$.		
Therefore, we have $x-\frac{\pi}{4}=\frac{\pi}{6}$.		
Thus, we have $x=\frac{5\pi}{12}$.	1A	
	----------(8)	

Solution	Marks	Remarks

(a) The slope of the tangent to Γ at P

$= f'(e^3)$

$= \dfrac{1}{e^3}\ln(e^3)^2$ — 1M

$= \dfrac{6}{e^3}$

The equation of the tangent to Γ at P is

$y-7=\dfrac{6}{e^3}(x-e^3)$

$6x-e^3y+e^3=0$ — 1A

(b) $\mathrm{f}(x)$

$=\displaystyle\int \frac{1}{x}\ln x^2\,\mathrm{d}x$

$=2\displaystyle\int \ln x\,\mathrm{d}\ln x$ — 1M

$=(\ln x)^2+C$

Since Γ passes through P, we have $7=(\ln e^3)^2+C$. — 1M

Solving, we have $C=-2$.

Thus, the equation of Γ is $y=(\ln x)^2-2$. — 1A

(c) Note that $\mathrm{f}''(x)=\dfrac{2-2\ln x}{x^2}$. — 1A

Therefore, we have $\mathrm{f}''(x)=0 \Leftrightarrow x=e$.

x	$(0,e)$	e	(e,∞)
$\mathrm{f}''(x)$	+	0	-

— 1M

Thus, the point of inflexion of Γ is $(e,-1)$. — 1A

----------(8)

Solution	Marks	Remarks
9. (a) The equation of the vertical asymptote is $x+4=0$.	1A	
Note that $f(x)=x-9+\frac{36}{x+4}$.	1M	
Thus, the equation of the oblique asymptote is $y=x-9$.	1A	
	----------(3)	
(b) $f'(x)$ $=\frac{d}{dx}\left(x-9+\frac{36}{x+4}\right)$	1M	
$=1+36(-1)(x+4)^{-2}$ $=1-\frac{36}{(x+4)^2}$	1A	
$f'(x)$ $=\frac{d}{dx}\left(\frac{x^2-5x}{x+4}\right)$ $=\frac{(x+4)(2x-5)-(x^2-5x)}{(x+4)^2}$	1M	
$=\frac{x^2+8x-20}{(x+4)^2}$	1A	
	----------(2)	

(c) Note that $f'(x)=\frac{(x+10)(x-2)}{(x+4)^2}$.

So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$. — 1A

x	$(-\infty,-10)$	-10	$(-10,-4)$	$(-4,2)$	2	$(2,\infty)$
$f'(x)$	+	0	−	−	0	+
$f(x)$	↗	−25	↘	↘	−1	↗

— 1M

Thus, the maximum point and the minimum point of G are $(-10,-25)$ and $(2,-1)$ respectively. — 1A, 1A

Note that $f'(x)=\frac{(x+10)(x-2)}{(x+4)^2}$ and $f''(x)=\frac{72}{(x+4)^3}$.

So, we have $f'(x)=0 \Leftrightarrow x=-10$ or $x=2$. — 1A

Also note that $f''(-10)=\frac{-1}{3}<0$ and $f''(2)=\frac{1}{3}>0$. — 1M

Further note that $f(-10)=-25$ and $f(2)=-1$.

Thus, the maximum point and the minimum point of G are $(-10,-25)$ and $(2,-1)$ respectively. — 1A, 1A

----------(4)

Solution	Marks	Remarks
(d) The required volume		
$=\pi\int_0^5\left(\frac{x^2-5x}{x+4}\right)^2 dx$	1M	
$=\pi\int_0^5\left(x-9+\frac{36}{x+4}\right)^2 dx$		
$=\pi\int_0^5\left(x^2-18x+81+\frac{72(x-9)}{x+4}+\frac{1296}{(x+4)^2}\right)dx$		
$=\pi\int_0^5\left(x^2-18x+153-\frac{936}{x+4}+\frac{1296}{(x+4)^2}\right)dx$	1M	
$=\pi\left[\frac{x^3}{3}-9x^2+153x-936\ln\lvert x+4\rvert-\frac{1296}{x+4}\right]_0^5$	1M	
$=\left(\frac{2285}{3}-1872\ln\left(\frac{3}{2}\right)\right)\pi$	1A	
The required volume		
$=\pi\int_0^5\left(\frac{x^2-5x}{x+4}\right)^2 dx$	1M	
$=\pi\int_4^9\frac{(x-4)^2(x-9)^2}{x^2}dx$		
$=\pi\int_4^9\left(\frac{x^4-26x^3+241x^2-936x+1296}{x^2}\right)dx$	1M	
$=\pi\int_4^9\left(x^2-26x+241-\frac{936}{x}+\frac{1296}{x^2}\right)dx$		
$=\pi\left[\frac{x^3}{3}-13x^2+241x-936\ln\lvert x\rvert-\frac{1296}{x}\right]_4^9$	1M	
$=\left(\frac{2285}{3}-1872\ln\left(\frac{3}{2}\right)\right)\pi$	1A	
	-----(4)	

Solution	Marks	Remarks
10. (a) Note that $\overrightarrow{AB}=2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\overrightarrow{AC}=6\mathbf{i}-6\mathbf{j}$.	1M	any one
$\overrightarrow{AE}$		
$=\dfrac{1}{1+r}\overrightarrow{AC}+\dfrac{r}{1+r}\overrightarrow{AB}$		
$=\dfrac{2r+6}{r+1}\mathbf{i}+\dfrac{r-6}{r+1}\mathbf{j}+\dfrac{r}{r+1}\mathbf{k}$		
Also note that $\overrightarrow{AE}=\dfrac{1}{11}\overrightarrow{AF}+\dfrac{10}{11}\overrightarrow{AD}$ and $\overrightarrow{AC}=2\overrightarrow{AD}$.		
$\overrightarrow{AF}$		
$=11\overrightarrow{AE}-5\overrightarrow{AC}$	1A	for both
$=\dfrac{-8r+36}{r+1}\mathbf{i}+\dfrac{41r-36}{r+1}\mathbf{j}+\dfrac{11r}{r+1}\mathbf{k}$	1M	
Since A , B and F are collinear, we have $\dfrac{2}{-8r+36}=\dfrac{1}{41r-36}=\dfrac{1}{11r}$.	1A	1.2
Solving, we have $r=\dfrac{6}{5}$.	----------(4)	
(b) (i) Note that $\overrightarrow{AD}=\dfrac{1}{2}\overrightarrow{AC}=3\mathbf{i}-3\mathbf{j}$.		
By (a), we have $\overrightarrow{AE}=\dfrac{1}{11}(42\mathbf{i}-24\mathbf{j}+6\mathbf{k})$.	1M	for using (a)
$\overrightarrow{AD}\cdot\overrightarrow{DE}$		
$=\overrightarrow{AD}\cdot(\overrightarrow{AE}-\overrightarrow{AD})$		
$=(3\mathbf{i}-3\mathbf{j})\cdot\left(\dfrac{1}{11}(9\mathbf{i}+9\mathbf{j}+6\mathbf{k})\right)$		
$=0$	1A	
(ii) $\overrightarrow{AB}\cdot\overrightarrow{BC}$	1M	
$=\overrightarrow{AB}\cdot(\overrightarrow{AC}-\overrightarrow{AB})$		
$=(2\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}-7\mathbf{j}-\mathbf{k})$		
$=0$		
Therefore, we have $\angle ABC=90^\circ=\angle ADE$.	1M	
So, we have $\angle CBF=90^\circ=\angle CDF$.		
Thus, B , D , C and F are concyclic.	1A	f.t.
	----------(5)	
(c) Note that $\overrightarrow{AF}=12\mathbf{i}+6\mathbf{j}+6\mathbf{k}$ and $\overrightarrow{AP}=\mathbf{i}+7\mathbf{j}-2\mathbf{k}$.		
Since $\angle CBF=90^\circ$, Q is the mid-point of CF .		
Therefore, we have $\overrightarrow{AQ}=\dfrac{1}{2}(\overrightarrow{AC}+\overrightarrow{AF})=9\mathbf{i}+3\mathbf{k}$.	1M	
The volume of the tetrahedron $ABPQ$		
$=\dfrac{1}{6}\left\|\overrightarrow{AQ}\cdot(\overrightarrow{AB}\times\overrightarrow{AP})\right\|$	1M	
$=\dfrac{1}{6}\left\|\begin{vmatrix}9&0&3\\2&1&1\\1&7&-2\end{vmatrix}\right\|$		
$=7$	1A	
	----------(3)	

Solution	Marks	Remarks
(a) $\int_0^1 \frac{1}{x^2+2x+3}\,dx$		
$=\int_0^1 \frac{1}{(x+1)^2+2}\,dx$	1M	
$=\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right]_0^1$	1M	
$=\frac{\sqrt{2}}{2}\left(\tan^{-1}\sqrt{2}-\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$		
$=\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$	1A	
	----------(3)	
(b) (i) $\frac{2\tan\theta}{1+\tan^2\theta}$		
$=\frac{\frac{2\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$		
$=2\sin\theta\cos\theta$		
$=\sin 2\theta$	1	
$\frac{1-\tan^2\theta}{1+\tan^2\theta}$		
$=\frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$		
$=\cos^2\theta-\sin^2\theta$		
$=\cos 2\theta$	1	
(ii) Let $t=\tan\theta$. Then, we have $\frac{d\theta}{dt}=\frac{1}{1+t^2}$.	1M	
Note that $\frac{1}{\sin 2\theta+\cos 2\theta+2}=\frac{1}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+2}=\frac{1+t^2}{t^2+2t+3}$.		
$\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\int_0^1 \frac{1+t^2}{t^2+2t+3}\left(\frac{1}{1+t^2}\right)dt$	1M	
$=\int_0^1 \frac{1}{t^2+2t+3}\,dt$		
$=\int_0^1 \frac{1}{x^2+2x+3}\,dx$		
$=\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (a))	1M	(a)
	----------(5)	

Solution	Marks	Remarks
(c) Let $y=\frac{\pi}{4}-\theta$. Then, we have $\frac{d\theta}{dy}=-1$.	1M	
$\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=-\int_{\frac{\pi}{4}}^{0}\frac{\sin\left(\frac{\pi}{2}-2y\right)+1}{\sin\left(\frac{\pi}{2}-2y\right)+\cos\left(\frac{\pi}{2}-2y\right)+2}\,dy$		
$=\int_0^{\frac{\pi}{4}}\frac{\cos 2y+1}{\cos 2y+\sin 2y+2}\,dy$		
$=\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$	1	
	----------(2)	
(d) $\int_0^{\frac{\pi}{4}}\frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\int_0^{\frac{\pi}{4}}\frac{4(\sin 2\theta+1)+4(\sin 2\theta+1)+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+4\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$ (by (c))	1M	for using (c)
$=4\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+\cos 2\theta+2}{\sin 2\theta+\cos 2\theta+2}\,d\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$	1M	
$=4\int_0^{\frac{\pi}{4}}d\theta+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	π + (b)(ii)
Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$ and $J=\int_0^{\frac{\pi}{4}}\frac{\cos 2\theta+1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$.		
Note that $I+J=\int_0^{\frac{\pi}{4}}d\theta=\frac{\pi}{4}$.	1M	
By (c), we have $I=J=\frac{\pi}{8}$.	1M	for using (c)
$\int_0^{\frac{\pi}{4}}\frac{8\sin 2\theta+9}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=8I+\int_0^{\frac{\pi}{4}}\frac{1}{\sin 2\theta+\cos 2\theta+2}\,d\theta$		
$=\pi+\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$ (by (b)(ii))	1M	π + (b)(ii)
	----------(3)	

Solution	Marks	Remarks
(a) A $=\begin{pmatrix}3&1\\0&3\end{pmatrix}$ $=3^1\begin{pmatrix}1&0\\0&1\end{pmatrix}+3^0(1)\begin{pmatrix}0&1\\0&0\end{pmatrix}$ So, the statement is true for $n=1$.	1	
Assume that $A^k=3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}$, where k is a positive integer.	1M	
A^{k+1} $=A^kA$ $=\left(3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}3&1\\0&3\end{pmatrix}$ (by induction assumption)	1M	for using induction assump
$=\left(3^kI+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\left(3\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)$ $=3^{k+1}I+3^kk\begin{pmatrix}0&1\\0&0\end{pmatrix}+3^k\begin{pmatrix}0&1\\0&0\end{pmatrix}+3^{k-1}k\begin{pmatrix}0&1\\0&0\end{pmatrix}^2$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}$ $=3^{k+1}I+3^k(k+1)\begin{pmatrix}0&1\\0&0\end{pmatrix}$		
Therefore, the statement is true for $n=k+1$ if it is true for $n=k$. By mathematical induction, the statement is true for all positive integers n .	1 ----------(4)	
(b) (i) Note that $P^{-1}=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$.	1A	
$P^{-1}BP$ $=\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}\begin{pmatrix}5&1\\-4&1\end{pmatrix}\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$ $=\begin{pmatrix}3&1\\0&3\end{pmatrix}$ $=A$	1A	
(ii) By (b)(i), we have $P^{-1}BP=A$. So, we have $(P^{-1}BP)^n=A^n$. Therefore, we have $P^{-1}B^nP=A^n$. Hence, we have $B^n=PA^nP^{-1}$.	1M	
B^n $=\begin{pmatrix}-1&0\\2&-1\end{pmatrix}\left(3^nI+3^{n-1}n\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$	1M	
$=3^nI+3^{n-1}n\begin{pmatrix}-1&0\\2&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}-1&0\\-2&-1\end{pmatrix}$		
$=3^nI+3^{n-1}n\begin{pmatrix}2&1\\-4&-2\end{pmatrix}$	1	

Solution

B
$=\begin{pmatrix}5 & 1\\-4 & 1\end{pmatrix}$
$=3^1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+3^0(1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$

So, the statement is true for $n=1$.

Assume that $B^k=3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$, where k is a positive integer.

B^{k+1}
$=B^kB$
$=\left(3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)\begin{pmatrix}5 & 1\\-4 & 1\end{pmatrix}$ (by induction assumption) 1M for using induction assumption
$=\left(3^kI+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)\left(3I+\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right)$ 1M
$=3^{k+1}I+3^kk\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+3^k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+3^{k-1}k\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}^2$
$=3^{k+1}I+3^k(k+1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}+\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$
$=3^{k+1}I+3^k(k+1)\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}$

Therefore, the statement is true for $n=k+1$ if it is true for $n=k$. 1
By mathematical induction, the statement is true for all positive integers n.

(iii) $|A^m-B^m|=4m^2$

$\left|3^{m-1}m\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}-3^{m-1}m\begin{pmatrix}2 & 1\\-4 & -2\end{pmatrix}\right|=4m^2$

$(3^{m-1})^2m^2\begin{vmatrix}-2 & 0\\4 & 2\end{vmatrix}=4m^2$ 1M

$-4m^2(3^{2(m-1)})=4m^2$

$3^{2(m-1)}=-1$ 1M

Note that $-1<0<3^{2(m-1)}$.

Thus, there does not exist a positive integer m such that $|A^m-B^m|=4m^2$. 1A f.t.

----------(8)