Marking Scheme

Module 2 (Algebra and Calculus)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles. All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.

Solution	Marks	Remarks
1. $\frac{\mathrm{d}}{\mathrm{d}x}(x^5+4)$		
$= \lim_{h\to 0}\frac{((x+h)^5+4)-(x^5+4)}{h}$	1M	
$= \lim_{h\to 0}\frac{x^5+5hx^4+10h^2x^3+10h^3x^2+5h^4x+h^5+4-x^5-4}{h}$	1M	for binomial expansion
$= \lim_{h\to 0}(5x^4+10hx^3+10h^2x^2+5h^3x+h^4)$	1M	withhold 1M if the step is skipped
$= 5x^4$	1A -----------(4)	
2. (a) $\frac{\mathrm{d}y}{\mathrm{d}x}$		
$= x\cos x+\sin x-\sin x$	1M	for product rule
$= x\cos x$	1A	
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$		
$= -x\sin x+\cos x$	1A	
(b) $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+k\frac{\mathrm{d}y}{\mathrm{d}x}+xy$		
$= x(-x\sin x+\cos x)+kx\cos x+x(x\sin x+\cos x)$ (by (a))	1M	for using the results of (a)
$= (2+k)x\cos x$		
Since $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+k\frac{\mathrm{d}y}{\mathrm{d}x}+xy=0$ for all real values of x, we have $k=-2$.	1A -----------(5)	

Solution	Marks	Remarks
3. (a) $\displaystyle\int \frac{1}{e^{2u}}\,\mathrm{d}u$		
$=\displaystyle\int e^{-2u}\,\mathrm{d}u$	1M	
$=\dfrac{-1}{2}e^{-2u}+\text{constant}$	1A	
(b) Let $u=\sqrt{x}$. Then, we have $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}$.	1M	
$\displaystyle\int_1^9 \frac{1}{\sqrt{x}\,e^{2\sqrt{x}}}\,\mathrm{d}x$		
$=\displaystyle\int_1^3 \frac{2}{e^{2u}}\,\mathrm{d}u$	1M+1A	
$=2\displaystyle\int_1^3 \frac{1}{e^{2u}}\,\mathrm{d}u$		
$=2\left[\dfrac{-1}{2}e^{-2u}\right]_1^3$ (by (a))	1M	for using the result of (a)
$=\dfrac{1}{e^2}-\dfrac{1}{e^6}$	1A	
	----------(7)	
4. (a) $\displaystyle\int x^2\ln x\,\mathrm{d}x$		
$=\dfrac{1}{3}\displaystyle\int \ln x\,\mathrm{d}x^3$		
$=\dfrac{1}{3}\left(x^3\ln x-\displaystyle\int x^3\,\mathrm{d}\ln x\right)$	1M	for integration by parts
$=\dfrac{1}{3}\left(x^3\ln x-\displaystyle\int x^2\,\mathrm{d}x\right)$	1A	
$=\dfrac{1}{3}x^3\ln x-\dfrac{1}{9}x^3+\text{constant}$	1A	
(b) y		
$=\displaystyle\int 9x^2\ln x\,\mathrm{d}x$	1M	
$=9\left(\dfrac{1}{3}x^3\ln x-\dfrac{1}{9}x^3\right)+C$ (by (a))	1M	for using the result of (a)
$=3x^3\ln x-x^3+C$, where C is a constant		
Since Γ passes through the point $(1,4)$, we have $4=3\ln 1-1+C$.	1M	
Solving, we have $C=5$.		
Thus, the equation of Γ is $y=3x^3\ln x-x^3+5$.	1A	
	----------(7)	

<table>
<tr><th>Solution</th><th>Marks</th><th>Remarks</th></tr>
<tr><td>5. (a) The augmented matrix is $\left(\begin{array}{ccc|c}1&1&1&2\\2&3&-3&4\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&1&2\\0&1&-5&0\end{array}\right)$.</td><td>1M</td><td></td></tr>
<tr><td>Thus, the solution set is $\{(2-6t,5t,t):t\in\mathbf{R}\}$.</td><td>1A</td><td></td></tr>
<tr><td>(b) Putting $(x,y,z)=(2-6t,5t,t)$ in the last equation, we have
$3(2-6t)+2(5t)+kt=6$.
So, we have $(k-8)t=0$.</td><td>1M</td><td></td></tr>
<tr><td>We now consider the cases $k=8$ and $k\neq 8$.</td><td>1M</td><td></td></tr>
<tr><td>Case 1: $k=8$
The system of linear equations in (b) is equivalent to the system of linear equations in (a).
Thus, the solution set is $\{(2-6t,5t,t):t\in\mathbf{R}\}$.</td><td>1M</td><td></td></tr>
<tr><td>Case 2: $k\neq 8$
So, we have $t=0$.
Thus, the solution is $(2,0,0)$.</td><td>1A</td><td></td></tr>
<tr><td>The augmented matrix is
$\left(\begin{array}{ccc|c}1&1&1&2\\2&3&-3&4\\3&2&k&6\end{array}\right)\sim\left(\begin{array}{ccc|c}1&1&1&2\\0&1&-5&0\\0&0&k-8&0\end{array}\right)$.</td><td>1M</td><td></td></tr>
<tr><td>We now consider the cases $k=8$ and $k\neq 8$.</td><td>1M</td><td></td></tr>
<tr><td>Case 1: $k=8$
In this case, the augmented matrix becomes $\left(\begin{array}{ccc|c}1&1&1&2\\0&1&-5&0\\0&0&0&0\end{array}\right)$.
Thus, the solution set is $\{(2-6t,5t,t):t\in\mathbf{R}\}$.</td><td>1M</td><td></td></tr>
<tr><td>Case 2: $k\neq 8$
The solution is $(2,0,0)$.</td><td>1A</td><td></td></tr>
<tr><td></td><td>----------(6)</td><td></td></tr>
</table>

Solution	Marks	Remarks
6. (a) Note that $\lvert M^T\rvert=\lvert M\rvert$ and $\lvert -M\rvert=-\lvert M\rvert$.	1M	either one
Since $\lvert M^T\rvert=\lvert -M\rvert$, we have $\lvert M\rvert=-\lvert M\rvert$.		
So, we have $2\lvert M\rvert=0$.		
Thus, we have $\lvert M\rvert=0$.	1	
(b) (i) $A+I$ $=\begin{pmatrix}0&a&b\\-a&0&-8\\-b&8&0\end{pmatrix}$		
So, we have $(A+I)^T=\begin{pmatrix}0&-a&-b\\a&0&8\\b&-8&0\end{pmatrix}=-(A+I)$.	1M	
By (a), we have $\lvert A+I\rvert=0$.	1	
$A+I$ $=\begin{pmatrix}0&a&b\\-a&0&-8\\-b&8&0\end{pmatrix}$ $\lvert A+I\rvert$ $=\begin{vmatrix}0&a&b\\-a&0&-8\\-b&8&0\end{vmatrix}$ $=0+8ab-8ab-0-0-0$	1M	
$=0$	1	
(ii) Note that $A^3+I=(A+I)(A^2-A+I)$.	1M	
$\lvert A^3+I\rvert$ $=\lvert A+I\rvert\lvert A^2-A+I\rvert$ $=(0)\lvert A^2-A+I\rvert$ (by (b)(i)) $=0$ Therefore, A^3+I is a singular matrix.		
Thus, the claim is agreed.	1A ------------(6)	f.t.

Solution	Marks	Remarks
7. (a) $\sin^2 x\cos^2 x$		
$=\frac{(2\sin x\cos x)^2}{4}$		
$=\frac{\sin^2 2x}{4}$	1M	
$=\frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)$		
$=\frac{1-\cos 4x}{8}$	1	
(b) (i) $\mathrm{f}(x)$		
$=\sin^4 x+\cos^4 x$		
$=(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x$	1M	
$=1^2-2\left(\frac{1-\cos 4x}{8}\right)$ (by (a))	1M	for using (a)
$=\frac{1}{4}\cos 4x+\frac{3}{4}$	1A	
(ii) $8\,\mathrm{f}(x)=7$		
$8\left(\frac{1}{4}\cos 4x+\frac{3}{4}\right)=7$ (by (b)(i))	1M	for using the result of (b)(i)
$2\cos 4x+6=7$		
$\cos 4x=\frac{1}{2}$		
$4x=\frac{\pi}{3}$ or $4x=\frac{5\pi}{3}$		
$x=\frac{\pi}{12}$ or $x=\frac{5\pi}{12}$	1A	for both
	----------(7)	

Solution	Marks	Remarks
8. (a) Note that $\sin\frac{x}{2}\cos\frac{(1+1)x}{2}=\sin\frac{x}{2}\cos x$.	1	
So, the statement is true for $n=1$.		
Assume that $\sin\frac{x}{2}\sum_{k=1}^{m}\cos kx=\sin\frac{mx}{2}\cos\frac{(m+1)x}{2}$ for some positive integer m .	1M	
$\sin\frac{x}{2}\sum_{k=1}^{m+1}\cos kx$		
$=\sin\frac{x}{2}\sum_{k=1}^{m}\cos kx+\sin\frac{x}{2}\cos(m+1)x$		
$=\sin\frac{mx}{2}\cos\frac{(m+1)x}{2}+\sin\frac{x}{2}\cos(m+1)x$ (by induction assumption)	1M	
$=\frac{1}{2}\left(\sin\frac{(2m+1)x}{2}-\sin\frac{x}{2}\right)+\frac{1}{2}\left(\sin\frac{(2m+3)x}{2}-\sin\frac{(2m+1)x}{2}\right)$	1M	
$=\frac{1}{2}\left(\sin\frac{(2m+3)x}{2}-\sin\frac{x}{2}\right)$		
$=\cos\frac{(2m+4)x}{4}\sin\frac{(2m+2)x}{4}$	1M	
$=\sin\frac{(m+1)x}{2}\cos\frac{(m+2)x}{2}$		
So, the statement is true for $n=m+1$ if it is true for $n=m$.		
By mathematical induction, we have $\sin\frac{x}{2}\sum_{k=1}^{n}\cos kx=\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}$ for all positive integers n .	1	
(b) Putting $x=\frac{\pi}{7}$ and $n=567$ in (a), we have		
$\sum_{k=1}^{567}\cos\frac{k\pi}{7}$		
$=\frac{\sin\frac{(567)(\pi)}{(2)(7)}\cos\frac{(568)(\pi)}{(2)(7)}}{\sin\frac{\pi}{(2)(7)}}$	1M	
$=\frac{\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2}+\frac{\pi}{14}\right)}{\sin\frac{\pi}{14}}$		
$=\frac{-\sin\frac{\pi}{14}}{\sin\frac{\pi}{14}}$		
$=-1$	1A	
	----------(8)	

<table>
<tr><th>Solution</th><th>Marks</th><th>Remarks</th></tr>
<tr><td>9. (a) $\mathrm{f}'(x)$</td><td></td><td></td></tr>
<tr><td>$=\dfrac{(x-2)(2x)-(x^2+12)}{(x-2)^2}$</td><td>1M</td><td>for quotient rule</td></tr>
<tr><td>$=\dfrac{x^2-4x-12}{(x-2)^2}$</td><td>1A</td><td></td></tr>
<tr><td></td><td>----------(2)</td><td></td></tr>
<tr><td>(b) Note that $\mathrm{f}'(x)=\dfrac{(x+2)(x-6)}{(x-2)^2}$.</td><td></td><td></td></tr>
<tr><td>So, we have $\mathrm{f}'(x)=0 \iff x=-2$ or $x=6$.</td><td>1A</td><td></td></tr>
<tr><td><table>
<tr><td>x</td><td>$(-\infty,-2)$</td><td>-2</td><td>$(-2,2)$</td><td>$(2,6)$</td><td>6</td><td>$(6,\infty)$</td></tr>
<tr><td>$\mathrm{f}'(x)$</td><td>$+$</td><td>0</td><td>$-$</td><td>$-$</td><td>0</td><td>$+$</td></tr>
<tr><td>$\mathrm{f}(x)$</td><td>↗</td><td>-4</td><td>↘</td><td>↘</td><td>12</td><td>↗</td></tr>
</table></td><td>1M</td><td></td></tr>
<tr><td>Thus, the maximum value and the minimum value of $\mathrm{f}(x)$ are -4</td><td>1</td><td></td></tr>
<tr><td>and 12 respectively.</td><td>1</td><td></td></tr>
<tr><td>Note that $\mathrm{f}'(x)=\dfrac{(x+2)(x-6)}{(x-2)^2}$ and $\mathrm{f}''(x)=\dfrac{32}{(x-2)^3}$.</td><td></td><td></td></tr>
<tr><td>So, we have $\mathrm{f}'(x)=0 \iff x=-2$ or $x=6$.</td><td>1A</td><td></td></tr>
<tr><td>Also note that $\mathrm{f}''(-2)=\dfrac{-1}{2}<0$ and $\mathrm{f}''(6)=\dfrac{1}{2}>0$.</td><td>1M</td><td></td></tr>
<tr><td>Further note that $\mathrm{f}(-2)=-4$ and $\mathrm{f}(6)=12$.</td><td></td><td></td></tr>
<tr><td>Thus, the maximum value and the minimum value of $\mathrm{f}(x)$ are -4</td><td>1</td><td></td></tr>
<tr><td>and 12 respectively.</td><td>1</td><td></td></tr>
<tr><td></td><td>----------(4)</td><td></td></tr>
<tr><td>(c) The equation of the vertical asymptote is $x-2=0$.</td><td>1A</td><td></td></tr>
<tr><td>Note that $\mathrm{f}(x)=x+2+\dfrac{16}{x-2}$.</td><td>1M</td><td></td></tr>
<tr><td>Thus, the equation of the oblique asymptote is $y=x+2$.</td><td>1A</td><td></td></tr>
<tr><td></td><td>----------(3)</td><td></td></tr>
<tr><td>(d) $\dfrac{x^2+12}{x-2}=14$
$x^2-14x+40=0$</td><td></td><td></td></tr>
<tr><td>$x=4$ or $x=10$</td><td>1A</td><td>can be absorbed</td></tr>
<tr><td>The required area</td><td></td><td></td></tr>
<tr><td>$=\displaystyle\int_4^{10}\left(14-\frac{x^2+12}{x-2}\right)\mathrm{d}x$</td><td>1M</td><td></td></tr>
<tr><td>$=\displaystyle\int_4^{10}\left(12-x-\frac{16}{x-2}\right)\mathrm{d}x$</td><td></td><td></td></tr>
<tr><td>$=\left[12x-\dfrac{x^2}{2}-16\ln(x-2)\right]_4^{10}$</td><td>1M</td><td></td></tr>
<tr><td>$=30-32\ln 2$</td><td>1A</td><td></td></tr>
<tr><td></td><td>----------(4)</td><td></td></tr>
</table>

Solution	Marks	Remarks
10. (a) (i) $\overrightarrow{OQ}$		
$=\frac{2}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}$	1M	
$\overrightarrow{OC}$		
$=(1-t)\overrightarrow{OP}+t\overrightarrow{OR}$		any one
$=(1-t)\left(\frac{1}{2}\overrightarrow{OA}\right)+t\left(\frac{3}{4}\overrightarrow{OB}\right)$		
$=\frac{1-t}{2}\overrightarrow{OA}+\frac{3t}{4}\overrightarrow{OB}$	1A	can be absorbed
Note that C lies on OQ .		
So, we have $\frac{1-t}{2}:\frac{2}{3}=\frac{3t}{4}:\frac{1}{3}$.	1M	
Thus, we have $t=\frac{1}{4}$.	1A	
(ii) By (a)(i), we have $\overrightarrow{OC}=\frac{3}{8}\overrightarrow{OA}+\frac{3}{16}\overrightarrow{OB}$.	1M	for using (a)(i)
Therefore, we have $\overrightarrow{OC}=\frac{9}{16}\overrightarrow{OQ}$.	1M	
Hence, we have $\lvert\overrightarrow{OC}\rvert=\frac{9}{16}\lvert\overrightarrow{OQ}\rvert$.		
So, we have $OC:OQ=9:16$.		
Thus, we have $CQ:OQ=7:16$.	1A	
	----------(7)	
(b) (i) The area of ΔOAB		
$=\frac{1}{2}\lvert\overrightarrow{OA}\times\overrightarrow{OB}\rvert$	1M	
$=\frac{1}{2}\left\lVert\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 20 & -6 & -12\\ 16 & -16 & 0\end{vmatrix}\right\rVert$		
$=\frac{1}{2}\lvert -192\mathbf{i}-192\mathbf{j}-224\mathbf{k}\rvert$		
$=176$	1A	
(ii) The volume of the tetrahedron $ABCD$		
$=\frac{1}{6}\lvert\overrightarrow{CD}\cdot(\overrightarrow{CA}\times\overrightarrow{CB})\rvert$	1M	
$=\left(\frac{1}{6}\right)\left(\frac{7}{16}\right)\lvert\overrightarrow{OD}\cdot(\overrightarrow{OA}\times\overrightarrow{OB})\rvert$ (by (a)(ii))	1M	for using the result of (a)(ii)
$=\frac{7}{96}\lvert(\mathbf{i}+3\mathbf{j}-6\mathbf{k})\cdot(-192\mathbf{i}-192\mathbf{j}-224\mathbf{k})\rvert$ (by (b)(i))		
$=\left(\frac{7}{96}\right)(576)$		
$=42$	1A	

Solution	Marks	Remarks
Since $\overrightarrow{OC}=\frac{3}{8}\overrightarrow{OA}+\frac{3}{16}\overrightarrow{OB}$, we have $\overrightarrow{OC}=\frac{21}{2}\mathbf{i}-\frac{21}{4}\mathbf{j}-\frac{9}{2}\mathbf{k}$.	1M	
$\overrightarrow{CA}=\frac{19}{2}\mathbf{i}-\frac{3}{4}\mathbf{j}-\frac{15}{2}\mathbf{k}$ $\overrightarrow{CB}=\frac{11}{2}\mathbf{i}-\frac{43}{4}\mathbf{j}+\frac{9}{2}\mathbf{k}$ $\overrightarrow{CD}=\frac{-19}{2}\mathbf{i}+\frac{33}{4}\mathbf{j}-\frac{3}{2}\mathbf{k}$		
The volume of the tetrahedron $ABCD$ $=\frac{1}{6}\left\lvert\overrightarrow{CD}\cdot(\overrightarrow{CA}\times\overrightarrow{CB})\right\rvert$	1M	
$=\frac{1}{6}\left\lvert\begin{vmatrix} \frac{-19}{2} & \frac{33}{4} & \frac{-3}{2} \\ \frac{19}{2} & \frac{-3}{4} & \frac{-15}{2} \\ \frac{11}{2} & \frac{-43}{4} & \frac{9}{2} \end{vmatrix}\right\rvert$ $=\frac{252}{6}$		
$=42$	1A	
	----------(5)	

Solution	Marks	Remarks
11. (a) (i) AB $=\frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}$	1M	either one
$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ BA $=\frac{1}{(\lambda-\mu+2)^2}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}$ $=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	1A	for both
$A+B$ $=\frac{1}{\lambda-\mu+2}\left[\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}+\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}\right]$ $=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	1A	
(ii) A^2 $=A(I-B)$ (by (a)(i))	1M	either one
$=A-AB$ $=A-0$ (by (a)(i)) $=A$ B^2 $=B(I-A)$ (by (a)(i)) $=B-BA$ $=B-0$ (by (a)(i)) $=B$	1	for both
(iii) $(\lambda+1)A+(\mu-1)B$ $=\frac{\lambda+1}{\lambda-\mu+2}(I-\mu I+M)+\frac{\mu-1}{\lambda-\mu+2}(I+\lambda I-M)$ $=\frac{-\mu\lambda+\lambda-\mu+1}{\lambda-\mu+2}I+\frac{\lambda+1}{\lambda-\mu+2}M+\frac{\mu\lambda-\lambda+\mu-1}{\lambda-\mu+2}I-\frac{\mu-1}{\lambda-\mu+2}M$ $=M$	1	
So, we have $M=(\lambda+1)A+(\mu-1)B$. M^2 $=((\lambda+1)A+(\mu-1)B)((\lambda+1)A+(\mu-1)B)$ $=(\lambda+1)^2A^2+(\lambda+1)(\mu-1)AB+(\lambda+1)(\mu-1)BA+(\mu-1)^2B^2$ $=(\lambda+1)^2A+(\mu-1)^2B$ (by (a)(i) and (a)(ii))	1M	for using (a)(i) and (a)(ii)
M^3 $=M^2M$ $=((\lambda+1)^2A+(\mu-1)^2B)((\lambda+1)A+(\mu-1)B)$ $=(\lambda+1)^3A^2+(\lambda+1)^2(\mu-1)AB+(\lambda+1)(\mu-1)^2BA+(\mu-1)^3B^2$ $=(\lambda+1)^3A+(\mu-1)^3B$ (by (a)(i) and (a)(ii)) Thus, we have $M^n=(\lambda+1)^nA+(\mu-1)^nB$.	1	

Solution	Marks	Remarks
$(\lambda+1)A+(\mu-1)B$		
$=\frac{\lambda+1}{\lambda-\mu+2}\begin{pmatrix}\lambda-\mu+1 & 1\\ \lambda-\mu+1 & 1\end{pmatrix}+\frac{\mu-1}{\lambda-\mu+2}\begin{pmatrix}1 & -1\\ -\lambda+\mu-1 & \lambda-\mu+1\end{pmatrix}$		
$=\frac{1}{\lambda-\mu+2}\begin{pmatrix}\lambda^2-\lambda\mu+2\lambda & \lambda-\mu+2\\ (\lambda-\mu+1)(\lambda-\mu+2) & \lambda\mu-\mu^2+2\mu\end{pmatrix}$		
$=M$	1	
So, the statement is true for $n=1$.		
Assume that $M^k=(\lambda+1)^k A+(\mu-1)^k B$, where k is a positive integer.		
M^{k+1}		
$=MM^k$		
$=((\lambda+1)A+(\mu-1)B)((\lambda+1)^k A+(\mu-1)^k B)$		
$=(\lambda+1)^{k+1}A^2+(\lambda+1)(\mu-1)^k AB+(\lambda+1)^k(\mu-1)BA+(\mu-1)^{k+1}B^2$		
$=(\lambda+1)^{k+1}A+(\mu-1)^{k+1}B$ (by (a)(i) and (a)(ii))	1M	
So, the statement is true for $n=k+1$ if it is true for $n=k$.		
By mathematical induction, we have $M^n=(\lambda+1)^n A+(\mu-1)^n B$.	1	
	----------(8)	
(b) Note that $\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}=2^{315}\begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}^{315}$.	1M	
Also note that $3-2=1\neq 2$ and $2-3+1=0$.	1M	withhold 1M if checking is omitte
Putting $\lambda=2$, $\mu=3$ and $n=315$ in (a)(iii), we have	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}$		
$=\frac{(2^{315})(3^{315})}{1}\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}+\frac{(2^{315})(2^{315})}{1}\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$		
$=\begin{pmatrix}2^{630} & 6^{315}-2^{630}\\ 0 & 6^{315}\end{pmatrix}$.	1A	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}$	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^2=\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4^2 & 6^2-4^2\\ 0 & 6^2\end{pmatrix}$	1M	
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^3=\begin{pmatrix}4^2 & 6^2-4^2\\ 0 & 6^2\end{pmatrix}\begin{pmatrix}4 & 6-4\\ 0 & 6\end{pmatrix}=\begin{pmatrix}4^3 & 6^3-4^3\\ 0 & 6^3\end{pmatrix}$	1M	withhold 1M if the step is skipped
$\begin{pmatrix}4 & 2\\ 0 & 6\end{pmatrix}^{315}=\begin{pmatrix}4^{315} & 6^{315}-4^{315}\\ 0 & 6^{315}\end{pmatrix}=\begin{pmatrix}2^{630} & 6^{315}-2^{630}\\ 0 & 6^{315}\end{pmatrix}$	1A	
	----------(4)	

Solution	Marks	Remarks
12. (a) (i) Solving $\begin{cases}3x+y-9=0\\x^2-4y+8=0\end{cases}$, we have $x^2+12x-28=0$.	1M	
So, we have $x=2$ or $x=-14$ (rejected) .		
Thus, the coordinates of B are $(2,3)$.	1A	
(ii) The required capacity		
$=\int_0^3 \pi\left(\frac{9-y}{3}\right)^2 \mathrm{d}y+\int_3^h \pi(4y-8)\,\mathrm{d}y$	1M+1M+1A	
$=\pi\int_0^3\left(9-2y+\frac{y^2}{9}\right)\mathrm{d}y+\pi\int_3^h(4y-8)\,\mathrm{d}y$		
$=\pi\left[9y-y^2+\frac{y^3}{27}\right]_0^3+\pi\left[2y^2-8y\right]_3^h$	1M	either one
$=\pi(2h^2-8h+25)$	1	
	-----------(7)	
(b) (i) Putting $x=6$ in $x^2-4y+8=0$, we have $y=11$.	1M	
The required capacity		
$=\pi(2(11)^2-8(11)+25)$ (by (a)(ii))		
$=179\pi\text{ cm}^3$	1A	
(ii) Let h cm be the depth of water in the cup at time t s .		
Also let p cm be the depth of water when the volume of water in the cup is $35\pi\text{ cm}^3$.		
Note that the volume of the frustum is $19\pi\text{ cm}^3$.		
Since $35\pi>19\pi$, we have $p>3$.	1M	withhold 1M if checking is omitte
By (a)(ii), we have $\pi(2p^2-8p+25)=35\pi$.		
Simplifying, we have $p^2-4p-5=0$.	1M	for $k_1p^2+k_2p+k_3=0$
Solving, we have $p=5$ or $p=-1$ (rejected as $3<p\le 11$) .		
Hence, we have $p=5$.		
Let $V\text{ cm}^3$ be the volume of water in the cup at time t s .		
For $3<h\le 11$, we have $V=\pi(2h^2-8h+25)$ (by (a)(ii)) .		
So, we have $\frac{\mathrm{d}V}{\mathrm{d}t}=\pi(4h-8)\frac{\mathrm{d}h}{\mathrm{d}t}$ for $3<h\le 11$.	1M	
Since $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right\vert_{h=5}=24\pi$, we have $24\pi=\pi(4(5)-8)\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\vert_{h=5}$.		
Therefore, we have $\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right\vert_{h=5}=2$.	1A	
Thus, the required rate of change is 2 cm/s .		
	-----------(6)	