2013-DSE
MATH EP
M2

HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2013

Please stick the barcode label here.

Candidate Number									

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)

Question-Answer Book

8.30 am – 11.00 am (2½ hours)
This paper must be answered in English

INSTRUCTIONS

1. After the announcement of the start of the examination, you should first write your Candidate Number in the space provided on Page 1 and stick barcode labels in the spaces provided on Pages 1, 3, 5, 7, 9, 11, 13 and 15.

2. Answer **ALL** questions in this paper. Write your answers in the spaces provided in this Question-Answer Book. Do not write in the margins. Answers written in the margins will not be marked.

3. Graph paper and supplementary answer sheets will be supplied on request. Write your Candidate Number, mark the question number box and stick a barcode label on each sheet, and fasten them with string **INSIDE** this Book.

4. Unless otherwise specified, all working must be clearly shown.

5. Unless otherwise specified, numerical answers must be exact.

6. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\tilde{u}$ in their working.

7. The diagrams in this paper are not necessarily drawn to scale.

8. No extra time will be given to candidates for sticking on the barcode labels or filling in the question number boxes after the 'Time is up' announcement.

FORMULAS FOR REFERENCE

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$	$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$
$2\sin A \cos B = \sin(A+B) + \sin(A-B)$	$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$	
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$	

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Answers written in the margins will not be marked.

Section A (50 marks)

1. Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x)$ from first principles.

(4 marks)

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2. Suppose the coefficients of x and x^2 in the expansion of $(1+ax)^n$ are -20 and 180 respectively. Find the values of a and n.

(4 marks)

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3. Prove, by mathematical induction, that for all positive integers n,

$$1+\frac{1}{1\times 4}+\frac{1}{4\times 7}+\frac{1}{7\times 10}+\cdots+\frac{1}{(3n-2)(3n+1)}=\frac{4n+1}{3n+1} .$$

(5 marks)

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4. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = e^x - 1$. It is given that the curve passes through the point $(1, e)$.

(a) Find the equation of the curve.

(b) Find the equation of tangent to the curve at the point where the curve cuts the y-axis.

(5 marks)

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5. Consider a continuous function $f(x)=\dfrac{3-3x^2}{3+x^2}$. It is given that

x	$x<-1$	-1	$-1<x<0$	0	$0<x<1$	1	$x>1$
$f'(x)$	+	+	+	0	–	–	–
$f''(x)$	+	0	–	–	–	0	+

('+' and '–' denote 'positive value' and 'negative value' respectively.)

(a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.

(b) Find the asymptote(s) of the graph of $y=f(x)$.

(c) Sketch the graph of $y=f(x)$ on page 7.

(6 marks)

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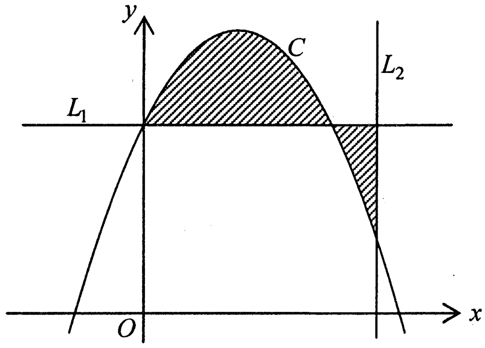

Figure 1

Figure 1 shows the shaded region with boundaries $C:\ y=\frac{-x^2}{2}+2x+4$, $L_1:\ y=4$ and $L_2:\ x=5$. It is given that C intersects L_1 at $(0,4)$ and $(4,4)$.

(a) Find the area of the shaded region.

(b) Find the volume of solid of revolution when the shaded region is revolved about L_1 .

(6 marks)

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7. (a) Prove the identity $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$.

(b) Using (a), prove the identity $\tan y = \dfrac{\sin 8y \cos 4y \cos 2y}{(1 + \cos 8y)(1 + \cos 4y)(1 + \cos 2y)}$.

(5 marks)

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8. Let M be the matrix $\begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ k & 0 & 0 \end{pmatrix}$, where $k \neq 0$.

(a) Find M^{-1} .

(b) If $M\begin{pmatrix} x \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find the value of k .

(5 marks)

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9. Consider the following system of linear equations in x, y and z

$$(E)\quad \begin{cases} x - ay + z = 2 \\ 2x + (1-2a)y + (2-b)z = a+4 \\ 3x + (1-3a)y + (3-ab)z = 4 \end{cases}, \text{ where } a \text{ and } b \text{ are real numbers.}$$

It is given that (E) has infinitely many solutions.

(a) Find the values of a and b.

(b) Solve (E).

(5 marks)

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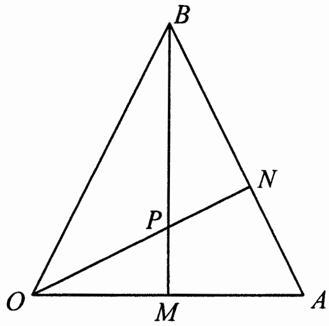

Figure 2

Let $\overrightarrow{OA} = 2\mathbf{i}$ and $\overrightarrow{OB} = \mathbf{i} + 2\mathbf{j}$. M is the mid-point of OA and N lies on AB such that $BN : NA = k : 1$. BM intersects ON at P (see Figure 2).

(a) Express $\overrightarrow{ON}$ in terms of k .

(b) If A , N , P and M are concyclic, find the value of k .

(5 marks)

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Section B (50 marks)

11. (a) Let $0 < \theta < \frac{\pi}{2}$. By finding $\frac{d}{d\theta}\ln(\sec\theta + \tan\theta)$, or otherwise, show that $\int \sec\theta \, d\theta = \ln(\sec\theta + \tan\theta) + C$, where C is any constant.

(2 marks)

(b) (i) Using (a) and a suitable substitution, show that $\int \frac{du}{\sqrt{u^2 - 1}} = \ln(u + \sqrt{u^2 - 1}) + C$ for $u > 1$.

(ii) Using (b)(i), show that $\int_0^1 \frac{2x}{\sqrt{x^4 + 4x^2 + 3}} dx = \ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$.

(5 marks)

(c) Let $t = \tan\phi$. Show that $\frac{d\phi}{dt} = \frac{1}{1+t^2}$.

Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1 + 2\cos^2\phi}} d\phi$.

(5 marks)

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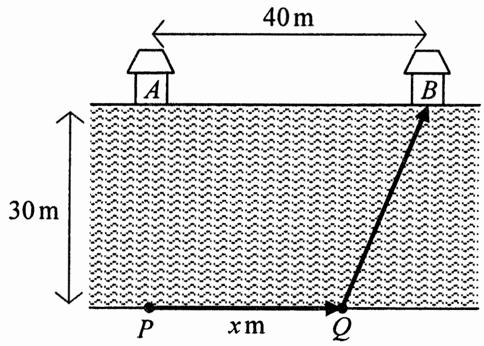

Figure 3

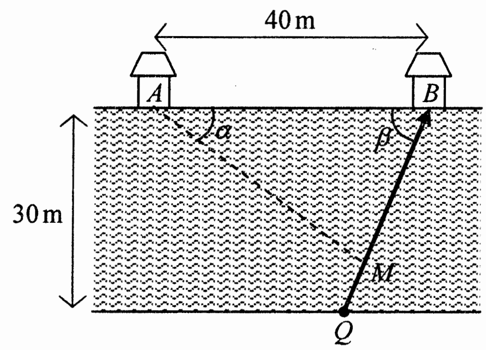

Figure 4

In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m . The width of the river is always 30 m . In the beginning, Mike stands at the starting point P in the opposite bank which is 30 m from A . Mike's wife, situated at A , is watching him running along the bank for x m at a constant speed of $7 \mathrm{~m\,s^{-1}}$ to point Q and then swimming at a constant speed of $1.4 \mathrm{~m\,s^{-1}}$ along a straight path to reach B .

(a) Let T seconds be the time that Mike travels from P to B .

(i) Express T in terms of x .

(ii) When T is minimum, show that x satisfies the equation $2x^2-160x+3125=0$.

Hence show that $QB=\dfrac{25\sqrt{6}}{2}$ m .

(6 marks)

(b) In Figure 4, Mike is swimming from Q to B with QB equal to the value mentioned in (a)(ii).
Let $\angle MAB=\alpha$ and $\angle ABM=\beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that $MB=\dfrac{200\tan\alpha}{\tan\alpha+2\sqrt{6}}$.

(ii) Find the rate of change of α when $\alpha=0.2$ radian . Correct your answer to 4 decimal places.

(7 marks)

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13. For any matrix $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define $\operatorname{tr}(M)=a+d$.

Let A and B be 2×2 matrices such that $BAB^{-1}=\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(a) (i) For any matrix $N=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, prove that $\operatorname{tr}(MN)=\operatorname{tr}(NM)$.

(ii) Show that $\operatorname{tr}(A)=4$.

(iii) Find the value of $|A|$.

(6 marks)

(b) Let $C=\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. It is given that $C\begin{pmatrix} x \\ y \end{pmatrix}=\lambda_1\begin{pmatrix} x \\ y \end{pmatrix}$ and $C\begin{pmatrix} x \\ y \end{pmatrix}=\lambda_2\begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$ and distinct scalars λ_1 and λ_2 .

(i) Prove that $\begin{vmatrix} p-\lambda_1 & q \\ r & s-\lambda_1 \end{vmatrix}=0$ and $\begin{vmatrix} p-\lambda_2 & q \\ r & s-\lambda_2 \end{vmatrix}=0$.

(ii) Prove that λ_1 and λ_2 are the roots of the equation $\lambda^2-\operatorname{tr}(C)\cdot\lambda+|C|=0$.

(5 marks)

(c) Find the two values of λ such that $A\begin{pmatrix} x \\ y \end{pmatrix}=\lambda\begin{pmatrix} x \\ y \end{pmatrix}$ for some non-zero matrices $\begin{pmatrix} x \\ y \end{pmatrix}$.

(2 marks)

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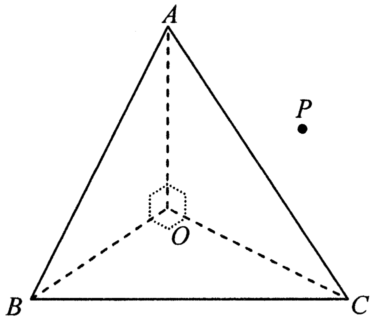

Figure 5

Figure 5 shows a fixed tetrahedron $OABC$ with $\angle AOB = \angle BOC = \angle COA = 90°$. P is a variable point such that $\overrightarrow{AP}\cdot\overrightarrow{BP}+\overrightarrow{BP}\cdot\overrightarrow{CP}+\overrightarrow{CP}\cdot\overrightarrow{AP}=0$. Let D be the fixed point such that $\overrightarrow{OD}=\dfrac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$. Let $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$, $\overrightarrow{OC}=\mathbf{c}$, $\overrightarrow{OP}=\mathbf{p}$ and $\overrightarrow{OD}=\mathbf{d}$.

(a) (i) Show that $\overrightarrow{AP}\cdot\overrightarrow{BP}=\mathbf{p}\cdot\mathbf{p}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{p}$.

(ii) Using (a)(i), show that $\mathbf{p}\cdot\mathbf{p}=2\mathbf{p}\cdot\mathbf{d}$.

(iii) Show that $|\mathbf{p}-\mathbf{d}|=|\mathbf{d}|$.
Hence show that P lies on the sphere centred at D with fixed radius.

(8 marks)

(b) (i) Alice claims that O lies on the sphere mentioned in (a)(iii). Do you agree? Explain your answer.

(ii) Suppose P_1 , P_2 and P_3 are three distinct points on the sphere in (a)(iii) such that $\overrightarrow{DP_1}\times\overrightarrow{DP_2}=\overrightarrow{DP_2}\times\overrightarrow{DP_3}$. Alice claims that the radius of the circle passing through P_1 , P_2 and P_3 is OD . Do you agree? Explain your answer.

(4 marks)

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