香 港 考 試 及 評 核 局
HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

2 0 1 2 年 香 港 中 學 文 憑 考 試
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION 2012

數學 延伸部分
單元二（代數與微積分）

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)

評 卷 參 考

MARKING SCHEME

Solution	Marks	Remarks
1. $f(x) = e^{2x}$		
$f'(0) = \lim_{h\to 0} \frac{e^{2(0+h)} - e^{2(0)}}{h}$	1M	
$= \lim_{h\to 0} \frac{e^{2h}-1}{2h} \cdot 2$	1M	Accept $\lim_{h\to 0} \frac{e^{2h}-1}{h}$
$= 2$	1A	
	(3)	
2. $(1+ax)^n = 1 + nax + \frac{n(n-1)}{2}(ax)^2 + \cdots$	1M+1A	1A for $\frac{n(n-1)}{2}$
Hence $na = 6$ and $\frac{n(n-1)}{2}a^2 = 16$.	1M	
Solving, $\frac{n(n-1)}{2}\left(\frac{6}{n}\right)^2 = 16$		
$18(n-1) = 16n$		
$n = 9$	1A	
Therefore, $a = \frac{2}{3}$.	1A	
	(5)	
3. For $n = 1$,		
L.H.S. $= 1 \times 2 = 2$ and R.H.S. $= 1^2(1+1) = 2$		
$\therefore$ L.H.S. = R.H.S. and the statement is true for $n = 1$.	1	
Assume $1\times 2 + 2\times 5 + 3\times 8 + \cdots + k(3k-1) = k^2(k+1)$, where k is a positive integer.	1	
$1\times 2 + 2\times 5 + 3\times 8 + \cdots + k(3k-1) + (k+1)[3(k+1)-1]$		
$= k^2(k+1) + (k+1)(3k+2)$ by the assumption	1	
$= (k+1)(k^2+3k+2)$		
$= (k+1)^2(k+1+1)$	1	
Hence the statement is true for $n = k+1$.		
By the principle of mathematical induction, the statement is true for all positive integers n .	1	Follow through
	(5)	
4. (a) $\int \frac{x+1}{x} dx = \int \left(1 + \frac{1}{x}\right) dx$	1M	
$= x + \ln\lvert x\rvert + C$	1A	
(b) Let $u = x^2 - 1$.		
$du = 2x dx$		
$\int \frac{x^3}{x^2-1} dx = \int \frac{u+1}{u} \cdot \frac{du}{2}$	1A	
$= \frac{1}{2}u + \frac{1}{2}\ln\lvert u\rvert + C$ by (a)	1M	
$= \frac{1}{2}(x^2-1) + \frac{1}{2}\ln\lvert x^2-1\rvert + C$	1A	OR $\frac{1}{2}x^2 + \frac{1}{2}\ln\lvert x^2-1\rvert + C$
	(5)	

Solution	Marks	Remarks

5. $y=\dfrac{x^2+x+1}{x+1}$

$=x+\dfrac{1}{x+1}$ — 1M

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\dfrac{1}{(x+1)^2}$ — 1A — OR $\dfrac{x^2+2x}{(x+1)^2}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ when $x=-2$ or 0

x	$x<-2$	-2	$-2<x<-1$	-1	$-1<x<0$	0	$x>0$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	>0	0	<0	undefined	<0	0	>0

1M

Alternative Solution

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2}{(x+1)^3}$

When $x=0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2>0$, when $x=-2$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2<0$. — 1M

Hence $(0,1)$ is a minimum point. — 1A

The vertical asymptote is $x=-1$. — 1A

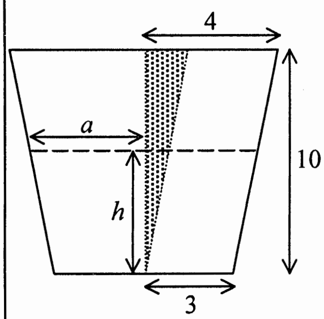
$\because\ \lim\limits_{x\to\pm\infty}\dfrac{1}{x+1}=0$ — Accept $\lim\limits_{x\to\infty}\dfrac{1}{x+1}=0$

$\therefore$ the oblique asymptote is $y=x$ — 1A

(6)

6. (a) Let the radius of the water surface be a cm.

By considering similar triangles, $\dfrac{a-3}{4-3}=\dfrac{h}{10}$. — 1M



i.e. $a=\dfrac{h+30}{10}$

$\therefore\ V=\dfrac{\pi}{3}h\left[3^2+3\left(\dfrac{h+30}{10}\right)+\left(\dfrac{h+30}{10}\right)^2\right]$ — 1M

$=\dfrac{\pi}{300}h[900+30(h+30)+(h^2+60h+900)]$

$=\dfrac{\pi}{300}(h^3+90h^2+2700h)$ — 1

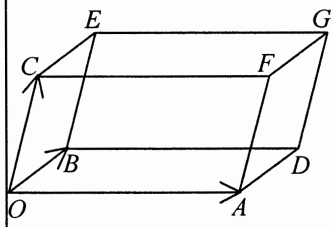
(b) $\dfrac{\mathrm{d}V}{\mathrm{d}t}=\dfrac{\pi}{300}(3h^2+180h+2700)\dfrac{\mathrm{d}h}{\mathrm{d}t}$ — 1M+1A

$\therefore\ 7\pi=\dfrac{\pi}{300}[3(5)^2+180(5)+2700]\dfrac{\mathrm{d}h}{\mathrm{d}t}$

$\dfrac{\mathrm{d}h}{\mathrm{d}t}=\dfrac{4}{7}$ — 1A

i.e. the rate of increase of depth of water is $\dfrac{4}{7}\ \mathrm{cm\,s^{-1}}$.

(6)

<table>
<tr><th colspan="3">Solution</th><th>Marks</th><th>Remarks</th></tr>
<tr><td>7.</td><td>(a)</td><td>The area of the parallelogram $OADB$</td><td></td><td rowspan="5"></td></tr>
<tr><td></td><td></td><td>$=\lvert(6\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+\mathbf{j})\rvert$</td><td>1M</td></tr>
<tr><td></td><td></td><td>$=\lvert\mathbf{i}-2\mathbf{j}+2\mathbf{k}\rvert$</td><td></td></tr>
<tr><td></td><td></td><td>$=\sqrt{1^2+2^2+2^2}$</td><td></td></tr>
<tr><td></td><td></td><td>$=3$</td><td>1A</td></tr>
<tr><td></td><td>(b)</td><td>The volume of the parallelepiped $OADBECFG$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$=(6\mathbf{i}+2\mathbf{j}-\mathbf{k})\times(2\mathbf{i}+\mathbf{j})\cdot(5\mathbf{i}-\mathbf{j}+2\mathbf{k})$</td><td>1M</td><td></td></tr>
<tr><td></td><td></td><td>$=(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\cdot(5\mathbf{i}-\mathbf{j}+2\mathbf{k})$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$=1\cdot 5+(-2)(-1)+2\cdot 2$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$=11$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>Hence, the distance between point C and plane $OADB$ is $\dfrac{11}{3}$.</td><td>1M+1A</td><td>1M for height $=\dfrac{\text{volume}}{\text{base}}$</td></tr>
<tr><td></td><td></td><td></td><td>(5)</td><td></td></tr>
<tr><td>8.</td><td>(a)</td><td>The augmented matrix is $\left(\begin{array}{ccc|c}1&1&1&0\\2&-1&5&6\end{array}\right)$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$\sim\left(\begin{array}{ccc|c}1&1&1&0\\0&-3&3&6\end{array}\right)$</td><td>1M</td><td></td></tr>
<tr><td></td><td></td><td>$\sim\left(\begin{array}{ccc|c}1&1&1&0\\0&1&-1&-2\end{array}\right)$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>Let $z=t$, where t is a real number. Then $y=t-2$ and $x=2-2t$.</td><td>1A</td><td>OR Solution Set = $\{(2-2t,t-2,t):t\in\mathbf{R}\}$</td></tr>
<tr><td></td><td>(b)</td><td>Substitute $(x,y,z)=(2-2t,t-2,t)$ into the last equation:</td><td>1M</td><td></td></tr>
<tr><td></td><td></td><td>$(2-2t)-(t-2)+\lambda(t)=4$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$(\lambda-3)t=0$</td><td></td><td></td></tr>
<tr><td></td><td></td><td>When $\lambda\neq 3$, $t=0$.</td><td></td><td></td></tr>
<tr><td></td><td></td><td>$\therefore$ $(x,y,z)=(2,-2,0)$</td><td>1A</td><td></td></tr>
<tr><td></td><td></td><td>When $\lambda=3$, t can be any real number.</td><td></td><td></td></tr>
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<tr><td></td><td></td><td>$\sim\left(\begin{array}{ccc|c}1&1&1&0\\0&1&-1&-2\\0&0&\lambda-3&0\end{array}\right)$</td><td></td><td></td></tr>
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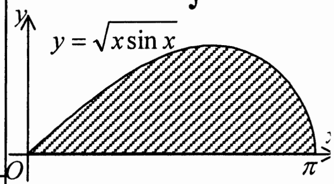
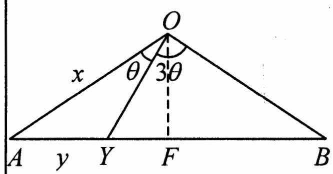
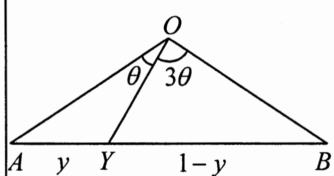
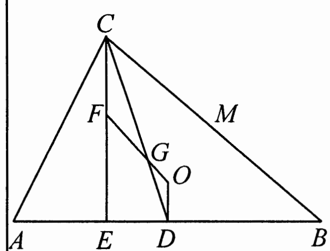
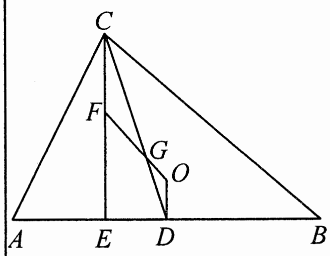
<tr><td></td><td></td><td>$\therefore$ $(x,y,z)=(2-2t,t-2,t)$, where t is a real number.</td><td>1A</td><td></td></tr>
<tr><td></td><td></td><td></td><td>(5)</td><td></td></tr>
</table>

Solution	Marks	Remarks
9. (a) $\int x\sin x\,dx = x(-\cos x) - \int(-\cos x)dx$	1M	
$= -x\cos x + \sin x + C$	1A	
(b) The volume $= \pi\int_0^{\pi} x\sin x\,dx$	1M	1M for $V = \pi\int y^2 dx$
$= \pi[-x\cos x + \sin x]_0^{\pi}$		
$= \pi^2$	1A	
	(4)	
10. (a) Let F be a point on AB such that $OF \perp AB$. Let OA be x .		
$\therefore$ $AF = \frac{1}{2}$ and $\angle AOF = 2\theta$ (properties of isos. Δ)		
In ΔOAF , $\sin 2\theta = \frac{\frac{1}{2}}{x}$	1M	
Alternative Solution In ΔOAB , $\frac{1}{\sin 4\theta} = \frac{x}{\sin(90° - 2\theta)}$	1M	OR $x^2 = x^2 + 1^2 - 2x\cos(90° - 2\theta)$
$x = \frac{1}{2\sin 2\theta}$ --- (1)		
In ΔOAY , $\frac{y}{\sin\theta} = \frac{x}{\sin(90° + \theta)}$ ---------------------------- (2)	1M	
Substitute (1) into (2) : $\frac{y}{\sin\theta} = \frac{1}{2\sin 2\theta}\cdot\frac{1}{\cos\theta}$	1M	
Alternative Solution In ΔOAY , $\frac{y}{\sin\theta} = \frac{OY}{\sin\angle OAY}$	1M	
In ΔOBY , $\frac{1-y}{\sin 3\theta} = \frac{OY}{\sin\angle OBY}$	1M	
$\because$ $\angle OAY = \angle OBY$ (base $\angle$s, isos. Δs)		
$\therefore$ $\frac{y}{\sin\theta} = \frac{1-y}{3\sin\theta - 4\sin^3\theta}$	1M	
$3y - 4y\sin^2\theta = 1 - y$		
$4y(1 - \sin^2\theta) = 1$		
$y = \frac{1}{4}\sec^2\theta$	1	
(b) $\because$ $0° < 4\theta < 180°$	1M	Accept using "≤" sign
$\therefore$ $0° < \theta < 45°$		
$\frac{1}{4}\sec^2 0° < y < \frac{1}{4}\sec^2 45°$ since $\sec^2\theta$ is an increasing function for $0° < \theta < 45°$		
i.e. $\frac{1}{4} < y < \frac{1}{2}$	1A	
	(6)	

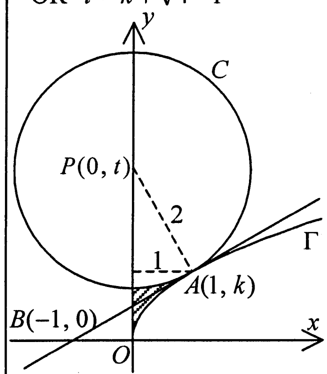
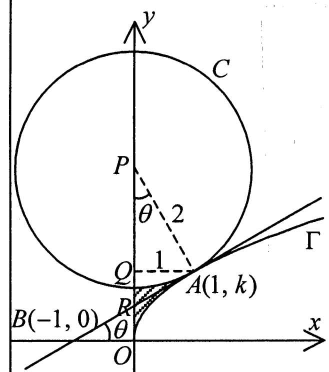
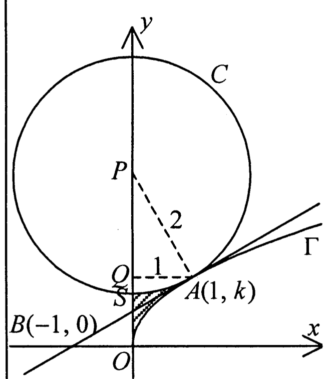
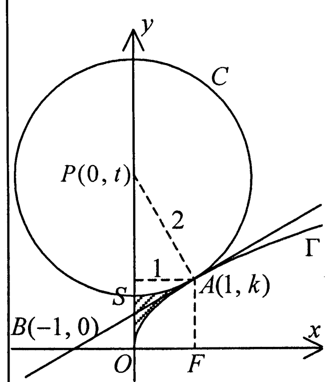
Solution	Marks	Remarks
11. (a) $\begin{vmatrix} 1-x & 4 \\ 2 & 3-x \end{vmatrix}=0$		
$(1-x)(3-x)-2\cdot 4=0$	1M	
$x^2-4x-5=0$		
$x=-1$ or 5	1A	
	(2)	
(b) (i) $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=-1\cdot\begin{pmatrix} a \\ b \end{pmatrix}$		
$\begin{cases} a+4b=-a \\ 2a+3b=-b \end{cases}$	1M	Either one
$a+2b=0$ ---------- (1)	1A	Either one
$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} c \\ 1 \end{pmatrix}=5\begin{pmatrix} c \\ 1 \end{pmatrix}$		
$\begin{cases} c+4=5c \\ 2c+3=5 \end{cases}$		
$c=1$ ---------- (2)		
$\begin{vmatrix} a & c \\ b & 1 \end{vmatrix}=1$		
By (2), $a-b=1$ ---------- (3)	1M	For $a-bc=1$
Solving (1) and (3), we have $a=\frac{2}{3}$ and $b=\frac{-1}{3}$.		
$\therefore\ P=\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$	1A	
(ii) $P^{-1}=\dfrac{1}{\frac{2}{3}+\frac{1}{3}}\begin{pmatrix} 1 & \frac{1}{3} \\ -1 & \frac{2}{3} \end{pmatrix}^t$		
$=\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$	1M	
$P^{-1}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}P=\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$		
$=\begin{pmatrix} -1 & 1 \\ \frac{5}{3} & \frac{10}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{3} & 1 \\ \frac{-1}{3} & 1 \end{pmatrix}$	1M	OR $\begin{pmatrix} 1 & -1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{-2}{3} & 5 \\ \frac{1}{3} & 5 \end{pmatrix}$
$=\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$	1A	

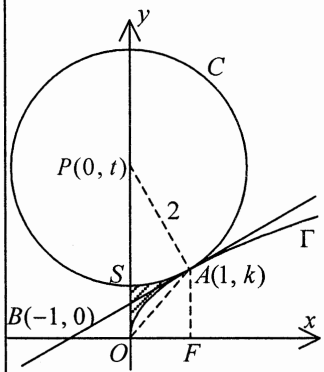
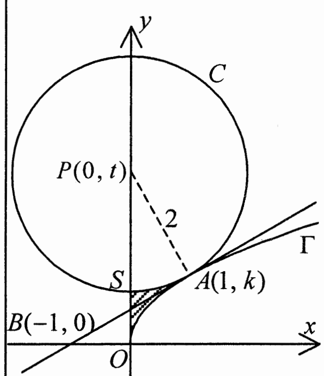
Solution	Marks	Remarks
(iii) $P^{-1}\begin{pmatrix}1&4\\2&3\end{pmatrix}P=\begin{pmatrix}-1&0\\0&5\end{pmatrix}$		
$\begin{pmatrix}1&4\\2&3\end{pmatrix}=P\begin{pmatrix}-1&0\\0&5\end{pmatrix}P^{-1}$	1M	
$\begin{pmatrix}1&4\\2&3\end{pmatrix}^{12}=\underbrace{P\begin{pmatrix}-1&0\\0&5\end{pmatrix}P^{-1}P\begin{pmatrix}-1&0\\0&5\end{pmatrix}P^{-1}\cdots P\begin{pmatrix}-1&0\\0&5\end{pmatrix}P^{-1}}_{12\text{ times}}$		
$=P\begin{pmatrix}-1&0\\0&5\end{pmatrix}^{12}P^{-1}$	1M	
$=\begin{pmatrix}\frac{2}{3}&1\\\frac{-1}{3}&1\end{pmatrix}\begin{pmatrix}1&0\\0&5^{12}\end{pmatrix}\begin{pmatrix}1&-1\\\frac{1}{3}&\frac{2}{3}\end{pmatrix}$	1M	For $\begin{pmatrix}1&0\\0&5^{12}\end{pmatrix}$
$=\begin{pmatrix}\frac{2}{3}&5^{12}\\\frac{-1}{3}&5^{12}\end{pmatrix}\begin{pmatrix}1&-1\\\frac{1}{3}&\frac{2}{3}\end{pmatrix}$		OR $\begin{pmatrix}\frac{2}{3}&1\\\frac{-1}{3}&1\end{pmatrix}\begin{pmatrix}1&-1\\\frac{5^{12}}{3}&\frac{2\cdot5^{12}}{3}\end{pmatrix}$
$=\begin{pmatrix}\frac{5^{12}+2}{3}&\frac{2\cdot5^{12}-2}{3}\\\frac{5^{12}-1}{3}&\frac{2\cdot5^{12}+1}{3}\end{pmatrix}$	1A	OR $\begin{pmatrix}81380209&162760416\\81380208&162760417\end{pmatrix}$
	(11)	
12. (a) Let $\overrightarrow{AG}=\frac{\overrightarrow{AC}+\lambda\overrightarrow{AD}}{1+\lambda}$. Since $\overrightarrow{AG}$ lies on a median, $\overrightarrow{AG}=k\frac{\overrightarrow{AC}+\overrightarrow{AB}}{2}=k\frac{\overrightarrow{AC}+2\overrightarrow{AD}}{2}$ for some k . Comparing the two expressions of $\overrightarrow{AG}$, we get $\lambda=2$.		
$\overrightarrow{AG}=\frac{\overrightarrow{AC}+2\overrightarrow{AD}}{3}$	1A	
$=\frac{\overrightarrow{AC}+\overrightarrow{AB}}{3}$		
$=\frac{(\mathbf{c}-\mathbf{a})+(\mathbf{b}-\mathbf{a})}{3}$	1M	For tip-to-tail method
<u>Alternative Solution 1</u> Let M be the mid-point of BC .		
$\overrightarrow{AG}=\frac{2}{3}\overrightarrow{AM}$	1A	
$=\frac{2}{3}\cdot\frac{\overrightarrow{AC}+\overrightarrow{AB}}{2}$		
$=\frac{(\mathbf{c}-\mathbf{a})+(\mathbf{b}-\mathbf{a})}{3}$	1M	For tip-to-tail method
<u>Alternative Solution 2</u> $\overrightarrow{AG}=\overrightarrow{OG}-\overrightarrow{OA}$	1M	
$=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}-\mathbf{a}$	1A	For $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$
$=\frac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3}$	1A	
	(3)	

Solution	Marks	Remarks
(b) (i) Since O is the circumcentre of the ΔABC, $OD \perp AB$.		
$\therefore\ OD // CE$	1	
$\angle DOG = \angle CFG$ (alt. $\angle$s, $OD // CF$)		
$\angle ODG = \angle FCG$ (alt. $\angle$s, $OD // CF$)		
$\angle OGD = \angle FGC$ (vert. opp. $\angle$s)		
$\therefore\ \Delta DOG \sim \Delta CFG$ (A.A.A.)	1	
$FG : GO = CG : GD$ (corr. sides, $\sim \Delta$s)		
$= 2 : 1$	1A	
(ii) $\overrightarrow{AG} = \dfrac{\overrightarrow{AF} + 2\overrightarrow{AO}}{3}$	1M	For using (b)(i)
$\overrightarrow{AF} = 3\overrightarrow{AG} - 2\overrightarrow{AO}$		
$= 3 \cdot \dfrac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3} - 2(-\mathbf{a})$	1M	For using (a)
Alternative Solution 1		
$\overrightarrow{AF} = \overrightarrow{AG} + \overrightarrow{GF}$		
$= \overrightarrow{AG} + 2\overrightarrow{OG}$	1M	For using (b)(i)
$= \dfrac{\mathbf{b}+\mathbf{c}-2\mathbf{a}}{3} + 2 \cdot \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$	1M	For using (a)
Alternative Solution 2		
$\overrightarrow{AF} = \overrightarrow{AC} + \overrightarrow{CF}$		
$= \overrightarrow{AC} + 2\overrightarrow{OD}$	1M	
$= \overrightarrow{AC} + \overrightarrow{OA} + \overrightarrow{OB}$	1M	
$= (\mathbf{c}-\mathbf{a}) + \mathbf{a} + \mathbf{b}$		
Alternative Solution 3		
$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA}$		
$= 3\overrightarrow{OG} - \overrightarrow{OA}$	1M	For using (b)(i)
$= 3 \cdot \dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3} - \mathbf{a}$	1M	
$= \mathbf{b} + \mathbf{c}$	1	
$\overrightarrow{AF} \cdot \overrightarrow{BC} = (\mathbf{b}+\mathbf{c}) \cdot (\mathbf{c}-\mathbf{b})$	1M	
$= \lvert\mathbf{c}\rvert^2 - \lvert\mathbf{b}\rvert^2$		
$= 0$ ($\because$ O is the circumcentre)	1A	
$\therefore\ AF \perp BC$		
$\therefore\ AF$ is another altitude of ΔABC.		
Alternative Solution		
$\overrightarrow{BF} \cdot \overrightarrow{AC} = (\overrightarrow{BA} + \overrightarrow{AF}) \cdot \overrightarrow{AC}$	1M	
$= (\mathbf{a}-\mathbf{b}+\mathbf{b}+\mathbf{c}) \cdot (\mathbf{c}-\mathbf{a})$		
$= \lvert\mathbf{c}\rvert^2 - \lvert\mathbf{a}\rvert^2$		
$= 0$ ($\because$ O is the circumcentre)	1A	
$\therefore\ BF \perp AC$		
$\therefore\ BF$ is another altitude of ΔABC.		
$\therefore\ F$ is the orthocentre of ΔABC.	1	
	(9)	

Solution	Marks	Remarks
13. (a) (i) $\tan u = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$		
$= \dfrac{-1+1-2\sin^2\frac{\pi}{5}}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$	1M	
$= -\tan\frac{\pi}{5}$		
$= \tan\frac{-\pi}{5}$		
$\therefore\ u = \frac{-\pi}{5}$ for $\frac{-\pi}{2} < u < \frac{\pi}{2}$	1	
(ii) $\tan v = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$		
$= \dfrac{1+2\cos^2\frac{\pi}{5}-1}{2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$	1M	
$= \cot\frac{\pi}{5}$		
$= \tan\left(\frac{\pi}{2}-\frac{\pi}{5}\right)$		
$\therefore\ v = \frac{3\pi}{10}$ for $\frac{-\pi}{2} < v < \frac{\pi}{2}$	1A	
	(4)	
(b) (i) $x^2+2x\cos\frac{2\pi}{5}+1$		
$x^2+2x\cos\frac{2\pi}{5}+\cos^2\frac{2\pi}{5}+\sin^2\frac{2\pi}{5}$		
$=(x+\cos\frac{2\pi}{5})^2+\sin^2\frac{2\pi}{5}$	1A	
(ii) $\displaystyle\int_{-1}^{1}\frac{\sin\frac{2\pi}{5}}{x^2+2x\cos\frac{2\pi}{5}+1}\,dx = \int_{-1}^{1}\frac{\sin\frac{2\pi}{5}}{(x+\cos\frac{2\pi}{5})^2+\sin^2\frac{2\pi}{5}}\,dx$		
Let $x+\cos\frac{2\pi}{5} = \sin\frac{2\pi}{5}\tan\theta$	1M	
$\therefore\ dx = \sin\frac{2\pi}{5}\sec^2\theta\, d\theta$	1A	
When $x=-1$, $\tan\theta = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$ which gives $\theta = \frac{-\pi}{5}$ (by (a)(i))	1M	For using (a)
When $x=1$, $\tan\theta = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$ which gives $\theta = \frac{3\pi}{10}$ (by (a)(ii))		

Solution	Marks	Remarks
$\therefore \int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2+2x\cos\frac{2\pi}{5}+1}\,dx = \int_{\frac{-\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin^2\frac{2\pi}{5}\sec^2\theta}{\sin^2\frac{2\pi}{5}(\tan^2\theta+1)}\,d\theta$	1A	For integrand
$=[\theta]_{\frac{-\pi}{5}}^{\frac{3\pi}{10}}$		
$=\frac{\pi}{2}$	1A	
	(6)	
(c) $\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,dx = \int_{-1}^{1} \frac{-\sin\frac{2\pi}{5}}{x^2-2x\cos\frac{2\pi}{5}+1}\,dx$	1A	
Let $y=-x$.	1M	
$dy=-dx$		
When $x=-1$, $y=1$; when $x=1$, $y=-1$.		
$\therefore \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,dx = \int_{1}^{-1} \frac{-\sin\frac{2\pi}{5}}{y^2+2y\cos\frac{2\pi}{5}+1}\cdot -dy$		
<u>Alternative Solution</u>		
$\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,dx = \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{(x+\cos\frac{7\pi}{5})^2+\sin^2\frac{7\pi}{5}}\,dx$		
Let $x+\cos\frac{7\pi}{5}=\sin\frac{7\pi}{5}\tan\theta$	1M	
$\therefore\ dx=\sin\frac{7\pi}{5}\sec^2\theta\,d\theta$		
When $x=-1$, $\theta=\frac{3\pi}{10}$; when $x=1$, $\theta=\frac{-\pi}{5}$.	1A	
$\therefore \int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2+2x\cos\frac{7\pi}{5}+1}\,dx = \int_{\frac{3\pi}{10}}^{\frac{-\pi}{5}} \frac{\sin^2\frac{7\pi}{5}\sec^2\theta}{\sin^2\frac{7\pi}{5}(\tan^2\theta+1)}\,d\theta$		
$=\frac{-\pi}{2}$ by (b)(ii)	1A	
	(3)	
14. (a) $y=kx^p$		
$\frac{dy}{dx}=kpx^{p-1}$	1A	
The slope of the tangent to Γ at A is kpa^{p-1} .		
$\therefore \frac{ka^p-0}{a-(-a)}=kpa^{p-1}$	1M	
$p=\frac{1}{2}$	1	
	(3)	

Solution	Marks	Remarks
(b) (i) Let $P(0, t)$ be the centre of C. $\because\ AP = 2$ $\therefore\ (k-t)^2 + (1-0)^2 = 2^2$	1M	OR $t = k + \sqrt{4-1}$
$k - t = -\sqrt{3}$ $\boxed{\text{or } \sqrt{3} \text{ (rejected)}}$ ---------------------------- (1) Slope of $AP = \dfrac{k-t}{1-0}$		
Slope of $AB = \dfrac{k}{2}$ (by (a))	1M	
$\therefore\ (k-t)\dfrac{k}{2} = -1$ -- (2)	1A	
Substitute (1) into (2): $(-\sqrt{3})\dfrac{k}{2} = -1$		
Alternative Solution According to the figure, $\angle ROB = \angle RAP = \dfrac{\pi}{2}$. $\angle ORB = \angle ARP$ (vert. opp. $\angle$ s)		
$\therefore\ \angle RBO = \angle RPA$ and let the angles be θ .	1M	
Since $PA = 2$ and $QA = 1$, $\theta = \dfrac{\pi}{6}$	1A	
Slope of $AB = \dfrac{k}{2}$ (by (a))	1M	
$\therefore\ \tan\dfrac{\pi}{6} = \dfrac{k}{2}$		
$k = \dfrac{2\sqrt{3}}{3}$	1	
(ii) The shaded area = area of ΔPQA + area on the left of Γ from O to A – area of sector PAS	1M	
$= \dfrac{1}{2}\cdot 1 \cdot \sqrt{4-1} + \displaystyle\int_0^{\frac{2\sqrt{3}}{3}} \left(\dfrac{y}{\frac{2\sqrt{3}}{3}}\right)^2 \mathrm{d}y - \dfrac{1}{2}(2)^2\dfrac{\pi}{6}$	1M+1A	
$= \dfrac{\sqrt{3}}{2} + \dfrac{3}{4}\left[\dfrac{y^3}{3}\right]_0^{\frac{2\sqrt{3}}{3}} - \dfrac{\pi}{3}$	1M	
Alternative Solution 1 $t = k + \sqrt{3}$ $= \dfrac{5\sqrt{3}}{3}$ The shaded area = area of trapezium $OFAP$ – area of sector PAS – area under Γ from O to A	1M	
$= \dfrac{1}{2}\left(\dfrac{2\sqrt{3}}{3} + \dfrac{5\sqrt{3}}{3}\right)(1) - \dfrac{1}{2}(2)^2\dfrac{\pi}{6} - \displaystyle\int_0^1 \dfrac{2\sqrt{3}}{3}x^{\frac{1}{2}}\mathrm{d}x$	1M+1A	
$= \dfrac{7\sqrt{3}}{6} - \dfrac{\pi}{3} - \dfrac{4\sqrt{3}}{9}\left[x^{\frac{3}{2}}\right]_0^1$	1M	

Solution	Marks	Remarks
<u>Alternative Solution 2</u>		
$t=k+\sqrt{3}$ $=\dfrac{5\sqrt{3}}{3}$		
The shaded area = area of ΔOAP + area of ΔOAF − area of sector PAS − area under Γ from O to A	1M	
$=\dfrac{1}{2}\left(\dfrac{5\sqrt{3}}{3}\right)(1)+\dfrac{1}{2}(1)\left(\dfrac{2\sqrt{3}}{3}\right)-\dfrac{1}{2}(2)^2\dfrac{\pi}{6}-\displaystyle\int_0^1\dfrac{2\sqrt{3}}{3}x^{\frac{1}{2}}\,\mathrm{d}x$	1M+1A	
$=\dfrac{5\sqrt{3}}{6}+\dfrac{\sqrt{3}}{3}-\dfrac{\pi}{3}-\dfrac{4\sqrt{3}}{9}\left[x^{\frac{3}{2}}\right]_0^1$	1M	
<u>Alternative Solution 3</u>		
$t=k+\sqrt{3}$ $=\dfrac{5\sqrt{3}}{3}$		
The equation of C is $x^2+\left(y-\dfrac{5\sqrt{3}}{3}\right)^2=4$.	1A	
Hence, the equation of $\overset{\frown}{AS}$ is $y=\dfrac{5\sqrt{3}}{3}-\sqrt{4-x^2}$.		
The shaded area $=\displaystyle\int_0^1\left(\dfrac{5\sqrt{3}}{3}-\sqrt{4-x^2}-\dfrac{2\sqrt{3}}{3}x^{\frac{1}{2}}\right)\mathrm{d}x$	1M	For $A=\displaystyle\int y\,\mathrm{d}x$ 
For $\displaystyle\int_0^1\sqrt{4-x^2}\,\mathrm{d}x$, let $x=2\sin\phi$. $\therefore$ $\mathrm{d}x=2\cos\phi\,\mathrm{d}\phi$ When $x=1$, $\phi=\dfrac{\pi}{6}$; when $x=0$, $\phi=0$.		
$\displaystyle\int_0^1\sqrt{4-x^2}\,\mathrm{d}x=\int_0^{\frac{\pi}{6}}\sqrt{4-4\sin^2\phi}\;2\cos\phi\,\mathrm{d}\phi$	1M	
$=\displaystyle\int_0^{\frac{\pi}{6}}2(1+\cos 2\phi)\,\mathrm{d}\phi$ $=[2\phi+\sin 2\phi]_0^{\frac{\pi}{6}}$ $=\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}$		
Hence, the shaded area $=\left[\dfrac{5\sqrt{3}}{3}x-\dfrac{4\sqrt{3}}{9}x^{\frac{3}{2}}\right]_0^1-\left(\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}\right)$	1M	
$=\dfrac{13\sqrt{3}}{18}-\dfrac{\pi}{3}$	1A	
	(9)	