5. Definite Integration

(1979-CE-A MATH 2 #07) (20 marks)

7. (a) Evaluate $\displaystyle\int_1^5 \frac{x}{\sqrt{4x+5}}\,\mathrm{d}x$.

(b) Given that $x^2 + xy + y^2 = a^2$, where $a \neq 0$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and deduce that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{3x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y}{(x+2y)^2} .$$

Hence evaluate $(x+2y)^3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(1980-CE-A MATH 2 #12) (20 marks)

12. (a) Given that $\mathrm{f}(x) = \mathrm{f}(a-x)$ for all real values of x, by using the substitution $u = a - x$, show that

$$\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x = a\int_0^a \mathrm{f}(u)\,\mathrm{d}u - \int_0^a u\,\mathrm{f}(u)\,\mathrm{d}u .$$

Hence deduce that

$$\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x = \frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x .$$

(b) By using the substitution $u = x - \dfrac{\pi}{2}$, show that

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{\sin^4 u + \cos^4 u}\,\mathrm{d}u .$$

By using this result and

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x + \int_c^b \mathrm{f}(x)\,\mathrm{d}x ,$$

evaluate

$$\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x .$$

(c) Using (a) and (b), evaluate

$$\int_0^{\pi} \frac{x\sin^4 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x .$$

(1981-CE-A MATH 2 #03) (6 marks) (Modified)

3. Evaluate $\displaystyle\int_0^9 \frac{x}{\sqrt{9-x}}\,\mathrm{d}x$.

(1981-CE-A MATH 2 #08) (20 marks) (Modified)

8. (a) Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \, dx$.

(b) (i) Show that $\frac{1}{x^2+3} - \frac{1}{(x+1)^2} \equiv \frac{2(x-1)}{(x^2+3)(x+1)^2}$ for $x \neq -1$.

(ii) Using the substitution $x = \sqrt{3} \tan \theta$, show that $\int_0^3 \frac{dx}{x^2+3} = \frac{\pi\sqrt{3}}{9}$.

(iii) Using the results of (i) and (ii), evaluate $\int_0^3 \frac{2(x-1)}{(x^2+3)(x+1)^2} \, dx$.

(1982-CE-A MATH 2 #05) (6 marks) (Modified)

5. Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x + \sin x} \, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos x + \sin x} \, dx$. Evaluate I .

(1983-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\int_0^1 x^3 \sqrt{1+3x^2} \, dx$.

(1983-CE-A MATH 2 #11) (20 marks)

11. (a) Show that $\frac{\sin 3\theta}{\sin \theta} = 2\cos 2\theta + 1$.

By putting $\theta = \frac{\pi}{4} + \phi$ in the above identity, show that

$$\frac{\cos 3\phi - \sin 3\phi}{\cos \phi + \sin \phi} = 1 - 2\sin 2\phi \ .$$

(b) Using the substitution $\phi = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} \, d\phi = \int_{\frac{\pi}{2}}^0 \frac{\sin 3u}{\cos u + \sin u} \, du \ .$$

Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} \, d\phi = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos 3\phi - \sin 3\phi}{\cos \phi + \sin \phi} \, d\phi.$$

(c) Using the results in (a) and (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos \phi + \sin \phi} \, d\phi \ .$$

(1983-CE-A MATH 2 #12) (20 marks)

12. Let $f(x)$ be a function of x and let k and s be constants.

(a) By using the substitution $y = x + ks$, show that

$$\int_0^s f(x+ks)\,dx = \int_{ks}^{(k+1)s} f(x)\,dx \ .$$

Hence show that, for any positive integer n ,

$$\int_0^s \left[f(x) + f(x+s) + \cdots + f\left(x + (n-1)s\right)\right] dx = \int_0^{ns} f(x)\,dx \ .$$

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ by using the substitution $x = \sin\theta$.

Using the result together with (a), evaluate

$$\int_0^{\frac{1}{2n}} \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(x+\frac{1}{2n}\right)^2}} + \frac{1}{\sqrt{1-\left(x+\frac{2}{2n}\right)^2}} + \cdots + \frac{1}{\sqrt{1-\left(x+\frac{n-1}{2n}\right)^2}} \right) dx \ .$$

(1984-CE-A MATH 2 #05) (8 marks) (Modified)

5. By considering $\dfrac{d}{dx}(\tan^3\theta)$, find $\displaystyle\int \tan^2\theta \sec^2\theta \, d\theta$.

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4\theta \, d\theta$.

(1984-CE-A MATH 2 #07) (20 marks)

7. (a) Prove that $\dfrac{1}{x^3} + \dfrac{3}{(2-3x)^2} = \dfrac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}$.

Hence find the value of $\displaystyle\int_1^2 \frac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}\,dx$.

(b) (i) Find $\displaystyle\int \frac{\cos\phi}{\sin^4\phi}\,d\phi$.

(ii) Using the substitution $x = \tan\phi$ and the result of (i), evaluate $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \frac{3\sqrt{1+x^2}}{x^4}\,dx$.

(1985-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\displaystyle\int_3^4 \frac{x}{\sqrt{25-x^2}}\,dx$.

(1985-CE-A MATH 2 #08) (20 marks)

8. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 t\ \cos^4 t\ dt$.

(b) By using the substitution $t = \dfrac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \cos^3 t\ \sin^4 t\ dt = \int_0^{\frac{\pi}{2}} \sin^3 t\ \cos^4 t\ dt .$$

(c) Show that $\displaystyle\int_{-\frac{\pi}{2}}^0 \cos^3 t\ \sin^4 t\ dt = \int_0^{\frac{\pi}{2}} \cos^3 t\ \sin^4 t\ dt$ and $\displaystyle\int_{-\frac{\pi}{2}}^0 \sin^3 t\ \cos^4 t\ dt = -\int_0^{\frac{\pi}{2}} \sin^3 t\ \cos^4 t\ dt$.

(d) Using the above results, or otherwise, evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} \sin^3 2t\,(\sin t + \cos t)\,dt .$$

(1986-CE-A MATH 2 #08) (20 marks)

8. (a) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) Using the result in (a), or otherwise, evaluate the following integrals:

(i) $\displaystyle\int_0^{\pi} \cos^{2n+1} x\ dx$, where n is a positive integer,

(ii) $\displaystyle\int_0^{\pi} x \sin^2 x\,dx$,

(iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x\ dx}{\sin x + \cos x}$.

(1987-CE-A MATH 2 #04) (6 marks)

4. Using the substitution $x = \sin\theta$, evaluate $\int_0^{\frac{1}{2}} \frac{2x^2}{\sqrt{1-x^2}}\,\mathrm{d}x$.

(1987-CE-A MATH 2 #08) (20 marks)

8. (a) Using the substitution $u = \tan x$, find

$$\int \tan^{n-2}x \sec^2 x\,\mathrm{d}x ,$$

where n is an integer and $n \geq 2$.

(b) (i) By writing $\tan^n x$ as $\tan^{n-2}x \tan^2 x$, show that

$$\int_0^{\frac{\pi}{4}} \tan^n x\,\mathrm{d}x = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2}x\,\mathrm{d}x ,$$

where n is an integer and $n \geq 2$.

(ii) Evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

(c) Show that $\int_{-\frac{\pi}{4}}^{0} \tan^6 x\,\mathrm{d}x = \int_0^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

Hence evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^6 x\,\mathrm{d}x$.

(1988-CE-A MATH 2 #06) (6 marks)

6. Evaluate $\int_0^2 \frac{x^2\,\mathrm{d}x}{\sqrt{9-x^3}}$.

(1988-CE-A MATH 2 #08) (20 marks)

8. (a) Using the substitution $u = \sin x$, evaluate $\int_0^{\frac{\pi}{2}} \cos^7 x\,\mathrm{d}x$.

Leave the answer as a fraction.

(b) Let $y = \sin x \cos^{2n-1} x$, where n is a positive integer.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Hence show that

$$2n\int \cos^{2n} x\,\mathrm{d}x - (2n-1)\int \cos^{2n-2} x\,\mathrm{d}x = \sin x \cos^{2n-1} x + C ,$$

where C is a constant.

(c) (i) Using (b), show that

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x\,\mathrm{d}x = \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \cos^{2n-2} x\,\mathrm{d}x ,$$

where n is a positive integer.

(ii) Evaluate $\int_0^{\frac{\pi}{2}} \cos^6 x\,\mathrm{d}x$ in terms of π.

(d) Evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\,\mathrm{d}x$ in terms of π.

(1989-CE-A MATH 2 #03) (5 marks) (Modified)

3. Evaluate $\int_0^2 \frac{8x^3}{\sqrt{2x^2+1}}\,\mathrm{d}x$.

(1989-CE-A MATH 2 #09) (16 marks)

9. Let n be an integer greater than 1 .

(a) Using the substitution $x = \tan\theta$, evaluate $\int_0^1 \frac{\mathrm{d}x}{1+x^2}$.

(b) By differentiating $\frac{x}{(1+x^2)^{n-1}}$ with respect to x , show that

$$\int \frac{x^2}{(1+x^2)^n}\,\mathrm{d}x = \frac{1}{2(n-1)}\left[\int \frac{\mathrm{d}x}{(1+x^2)^{n-1}} - \frac{x}{(1+x^2)^{n-1}}\right] .$$

(c) Using the identity $\frac{1}{(1+x^2)^n} \equiv \frac{1}{(1+x^2)^{n-1}} - \frac{x^2}{(1+x^2)^n}$, show that

$$\int \frac{\mathrm{d}x}{(1+x^2)^n} = \frac{2n-3}{2n-2}\int \frac{\mathrm{d}x}{(1+x^2)^{n-1}} + \frac{1}{2(n-1)}\cdot\frac{x}{(1+x^2)^{n-1}} .$$

(d) Using the above results or otherwise, evaluate

(i) $\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^2}$,

(ii) $\int_0^1 \frac{\mathrm{d}x}{(1+x^2)^3}$.

(1990-CE-A MATH 2 #09) (16 marks)

9. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$,

evaluate $\int_0^{\pi} x\cos^2 x \, dx$.

(b) Show that

(i) $\int_{\pi}^{2\pi} x\cos^2 x \, dx = \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x\cos^2 x \, dx$.

(ii) $\int_0^{2\pi} x\cos^2 x \, dx = \pi^2$.

(c) Using the result of (b) (ii),

evaluate $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$.

(1990-AL-P MATH 2 #03) (5 marks)

3. Suppose $f(x)$ and $g(x)$ are real-valued continuous functions on $[0,a]$ satisfying the conditions that $f(x) = f(a - x)$ and $g(x) + g(a - x) = K$ where K is a constant.

Show that $\int_0^{a} f(x)\, g(x) \, dx = \frac{K}{2} \int_0^{a} f(x) \, dx$. Hence, or otherwise, evaluate $\int_0^{\pi} x \sin x \cos^4 x \, dx$.

(1991-CE-A MATH 2 #02) (5 marks)

2. Evaluate $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$.

(1991-CE-A MATH 2 #12) (16 marks)

12. Let m , n be positive integers.

(a) Given that $y = (1+x)^{m+1}(1-x)^n$. Find $\frac{dy}{dx}$.

Hence show that

$$(m+1)\int (1+x)^m(1-x)^n \, dx = (1+x)^{m+1}(1-x)^n + n\int (1+x)^{m+1}(1-x)^{n-1} \, dx \ .$$

(b) Using the result of (a), show that

$$\int_{-1}^{1} (1+x)^m (1-x)^n \, dx = \frac{n}{m+1}\int_{-1}^{1} (1+x)^{m+1} (1-x)^{n-1} \, dx \ .$$

(c) Without using a binomial expansion, evaluate

$$\int_{-1}^{1} (1+x)^8 \, dx \ .$$

(d) Using the substitution $x = \tan\theta$, show that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta (1+\tan\theta)^4}{\cos^6\theta} \, d\theta = \int_{-1}^{1} (1+x)^6 (1-x)^2 \, dx \ .$$

Hence, using the results of (b) and (c), evaluate

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^2 2\theta (1+\tan\theta)^4}{\cos^6\theta} \, d\theta \ .$$

(1992-CE-A MATH 2 #08) (16 marks)

8. (a) Let $y = \dfrac{\sin x}{2 + \cos x}$.

Show that $\dfrac{dy}{dx} = \dfrac{2}{2 + \cos x} - \dfrac{3}{(2 + \cos x)^2}$.

(b) Using the substitution $t = \sqrt{3} \tan \theta$, evaluate

$$\int_0^1 \frac{dt}{t^2 + 3} .$$

(c) Using the substitution $t = \tan \dfrac{x}{2}$ and the result of (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \cos x} .$$

(d) Using the results of (a) and (c), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(2 + \cos x)^2} .$$

(1993-CE-A MATH 2 #09) (16 marks)

9. Let m , n be integers such that $m > 1$ and $n \geq 0$.

(a) Find $\dfrac{d}{dx}(\sin^{m-1}x \cos^{n+1}x)$.

(b) Using the result of (a), show that

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cos^n x \, dx .$$

(c) Using the result of (b) and the substitution $x = \dfrac{\pi}{2} - y$, show that

$$\int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx = \frac{m-1}{m+n} \int_0^{\frac{\pi}{2}} \sin^n x \cos^{m-2} x \, dx .$$

(d) Using the results of (b) and (c), evaluate

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx .$$

(1994-CE-A MATH 2 #10) (16 marks)

10. (a) Using the substitution $x = \tan\theta$, evaluate

$$\int_0^1 \frac{dx}{1+x^2} .$$

(b) Given $-\pi < x < \pi$ and $t = \tan\frac{x}{2}$. By expressing $\sin x$ and $\cos x$ in terms of t , show that

$$3 + 2\sin x + \cos x = \frac{2(2+2t+t^2)}{1+t^2} .$$

Hence show that

$$\int \frac{dx}{3+2\sin x+\cos x} = \int \frac{dt}{1+(1+t)^2} .$$

(c) Using (b), evaluate

$$\int_{\frac{-\pi}{2}}^{0} \frac{dx}{3+2\sin x+\cos x} .$$

(d) Using the result of (c), evaluate

$$\int_{\frac{-\pi}{2}}^{0} \frac{(2\sin x+\cos x)\,dx}{3+2\sin x+\cos x} .$$

(1995-CE-A MATH 2 #08) (16 marks)

8. Let n be an integer greater than 1 .

(a) Show that

$$\frac{d}{dx}\left[x^{n-1}(1-x^2)^{\frac{3}{2}}\right] = (n-1)x^{n-2}\sqrt{1-x^2} - (n+2)x^n\sqrt{1-x^2} .$$

(b) Using (a), show that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \frac{n-1}{n+2}\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx .$$

(c) Using the substitution $x = \sin\theta$, evaluate

$$\int_0^1 \sqrt{1-x^2}\,dx .$$

(d) Using (b) and (c), evaluate the following integrals:

(i) $\int_0^1 x^4\sqrt{1-x^2}\,dx$,

(ii) $\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\,d\theta$.

(1996-AL-P MATH 2 #03) (6 marks)

3. (a) Suppose $f(x)$ is continuous on $[0,a]$. Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Furthermore, if $f(x) + f(a-x) = K$ for all $x \in [0,a]$, where K is a constant, prove that

(i) $K = 2\,f\left(\frac{a}{2}\right)$;

(ii) $\int_0^a f(x)\,dx = a\,f\left(\frac{a}{2}\right)$.

(b) Hence, or otherwise, evaluate $\int_0^{2\pi} \frac{1}{e^{\sin x}+1}\,dx$.

(1996-CE-A MATH 2 #09) (16 marks)

9. (a) Evaluate $\int_0^{\pi} \sin^5 x\,dx$.

(Hint: Let $t = \cos x$.)

(b) Using the substitution $u = \pi - x$ and the result of (a), evaluate

$$\int_0^{\pi} x\sin^5 x\,dx \ .$$

(c) By differentiating $y = x^2\sin^5 x$ with respect to x and using the result of (b), evaluate

$$I_1 = \int_0^{\pi} x^2\sin^4 x\cos x\,dx \ .$$

(d) Let $I_2 = \int_0^{\pi} x^2\sin^4 x\cos|x|\,dx$.

State, with a reason, whether I_2 is smaller than, equal to or larger than I_1 in (c).

(1997-AL-P MATH 2 #04) (6 marks)

4. Show that $\left(\sin 2x + \sin 4x + \ldots \sin 2nx\right) \sin x = \sin nx \sin(n+1)x$.

Hence or otherwise, evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 6x \sin 7x}{\sin x}\, \mathrm{d}x$.

(1997-CE-A MATH 2 #11) (16 marks)

11. (a) Using the substitution $u = \cot\theta$, find

$$\int \cot^n\theta \operatorname{cosec}^2\theta \, \mathrm{d}\theta \ ,$$

where n is a non-negative integer.

(b) By writing $\cot^{n+2}\theta$ as $\cot^n\theta\cot^2\theta$, show that

$$\int \cot^{n+2}\theta \, \mathrm{d}\theta = -\frac{\cot^{n+1}\theta}{n+1} - \int \cot^n\theta \, \mathrm{d}\theta \ ,$$

where n is a non-negative integer.

(c) Using (b), or otherwise, show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot^2\theta \, \mathrm{d}\theta = 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{12} \ .$$

(d) Using the substitution $x = \sec\theta$, evaluate

$$\int_{\sqrt{2}}^{2} \frac{\mathrm{d}x}{x\sqrt{(x^2-1)^5}} \ .$$

(1998-CE-A MATH 2 #06) (6 marks)

6. Using the substitution $u = \sin\theta$, evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5\theta \sin^2\theta \, d\theta \; .$$

(1998-CE-A MATH 2 #09) (16 marks)

9. (a) Let a be a positive number.

(i) Show that $\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx$.

(ii) If $f(x) = f(-x)$ for $-a \le x \le a$, show that

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \; .$$

(b) Using the substitution $t = \frac{\sqrt{3}}{3}\tan\theta$, show that

$$\int_0^1 \frac{dt}{1+3t^2} = \frac{\sqrt{3}\pi}{9} \; .$$

(c) Given $I_1 = \int_0^1 \frac{1-t^2}{1+3t^2}\,dt$ and $I_2 = \int_0^1 \frac{t^2}{1+3t^2}\,dt$.

(i) Without evaluating I_1 and I_2 ,

(1) show that $I_1 + 4I_2 = 1$, and

(2) using the result of (b), evaluate $I_1 + I_2$.

(ii) Using the result of (c) (i), or otherwise, evaluate I_2 .

(d) Evaluate $\int_{-1}^{1} \frac{1+t^2}{1+3t^2}\,dt$.

(1999-AL-P MATH 2 #02) (6 marks)

2. (a) Let f be a continuous function. Show that $\int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$.

(b) Evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

(1999-CE-A MATH 2 #01) (3 marks)

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$.

(1999-CE-A MATH 2 #12) (16 marks)

12. (a) Prove, by mathematical induction, that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta} \ ,$$

where $\sin\theta \neq 0$, for all positive integers n .

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x} \ ,$$

where $\cos x \neq 0$.

(c) Using (a) and (b), evaluate

$$\int_{0.1}^{0.5} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} \right)^2 \mathrm{d}x \ ,$$

giving your answer correct to two significant figures.

(d) Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x)\,\mathrm{d}x \ .$$

(2002-CE-A MATH #04) (4 marks)

4. Find $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$.

(Hint: Let $x = \sin\theta$.)

(2004-AL-P MATH 2 #04) (Part)

4. Using the substitution $u = \frac{1}{x}$, prove that $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\ln x}{1+x^2}\,\mathrm{d}x = 0$.

(2006-AL-P MATH 2 #03) (7 marks)

3. For any positive integers m and n , define $I_{m,n} = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^m\theta}{\cos^n\theta}\mathrm{d}\theta$.

(a) Prove that $I_{m+2,n+2} = \dfrac{1}{n+1}\left(\dfrac{1}{\sqrt{2}}\right)^{m-n} - \dfrac{m+1}{n+1} I_{m,n}$.

(b) Using the substitution $u = \cos\theta$, evaluate $I_{3,1}$.

(c) Using the results of (a) and (b), evaluate $I_{7,5}$.

(SP-DSE-MATH-EP(M2) #13) (14 marks)

13. (a) Let $a > 0$ and $\mathrm{f}(x)$ be a continuous function.

Prove that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\,\mathrm{d}x$.

Hence, prove that $\int_0^a \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{2}\int_0^a \left[\mathrm{f}(x)+\mathrm{f}(a-x)\right]\mathrm{d}x$.

(b) Show that $\int_0^1 \frac{\mathrm{d}x}{x^2-x+1} = \frac{2\sqrt{3}\pi}{9}$.

(c) Using (a) and (b), or otherwise, evaluate $\int_0^1 \frac{\mathrm{d}x}{\left(x^2-x+1\right)\left(e^{2x-1}+1\right)}$.

(PP-DSE-MATH-EP(M2) #13) (10 marks)

13. (a) Let $\mathrm{f}(x)$ be an odd function for $-p \le x \le p$, where p is a positive constant.

Prove that $\int_0^{2p} \mathrm{f}(x-p)\,\mathrm{d}x = 0$.

Hence evaluate $\int_0^{2p} \left[\mathrm{f}(x-p)+q\right]\,\mathrm{d}x$, where q is a constant.

(b) Prove that $\frac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \frac{1+\sqrt{3}\tan x}{2}$.

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,\mathrm{d}x$.

(2012-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (i) Suppose $\tan u = \dfrac{-1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$, where $\frac{-\pi}{2} < u < \frac{\pi}{2}$. Show that $u = \frac{-\pi}{5}$.

(ii) Suppose $\tan v = \dfrac{1+\cos\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}$. Find v , where $\frac{-\pi}{2} < v < \frac{\pi}{2}$.

(b) (i) Express $x^2 + 2x\cos\frac{2\pi}{5} + 1$ in the form $(x+a)^2 + b^2$, where a and b are constants.

(ii) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{2\pi}{5}}{x^2 + 2x\cos\frac{2\pi}{5} + 1}\,dx$.

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{\sin\frac{7\pi}{5}}{x^2 + 2x\cos\frac{7\pi}{5} + 1}\,dx$.

(2013-DSE-MATH-EP(M2) #11) (12 marks)

11. (a) Let $0 < \theta < \frac{\pi}{2}$. By finding $\frac{d}{d\theta}\ln(\sec\theta + \tan\theta)$, or otherwise, show that

$$\int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta) + C$$, where C is any constant.

(b) (i) Using (a) and a suitable substitution, show that $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \ln(u + \sqrt{u^2-1}) + C$ for $u > 1$.

(ii) Using (b)(i), show that $\displaystyle\int_0^1 \frac{2x}{\sqrt{x^4+4x^2+3}}\,dx = \ln\left(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6}\right)$.

(c) Let $t = \tan\phi$. Show that $\dfrac{d\phi}{dt} = \dfrac{1}{1+t^2}$.

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan\phi}{\sqrt{1+2\cos^2\phi}}\,d\phi$.

(2015-DSE-MATH-EP(M2) #03) (7 marks)

3. (a) Find $\int \frac{1}{e^{2u}}\,du$.

(b) Using integration by substitution, evaluate $\int_1^9 \frac{1}{\sqrt{x}e^{2\sqrt{x}}}\,dx$.

(2016-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) Let $f(x)$ be a continuous function defined on the interval $(0, a)$, where a is a positive constant. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

(b) Prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$.

(c) Using (b), prove that $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{\pi \ln 2}{8}$.

(d) Using integration by parts, evaluate $\int_0^{\frac{\pi}{4}} \frac{x\sec^2 x}{1+\tan x}\,dx$.

(2017-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) Using $\tan^{-1}\sqrt{2} - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$, evaluate $\int_0^1 \frac{1}{x^2+2x+3}\,dx$.

(b) (i) Let $0 \le \theta \le \frac{\pi}{4}$. Prove that $\frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$ and $\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$.

(ii) Using the substitution $t = \tan\theta$, evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(c) Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(d) Evaluate $\int_0^{\frac{\pi}{4}} \frac{8\sin 2\theta + 9}{\sin 2\theta + \cos 2\theta + 2}\,d\theta$.

(2019-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) Using integration by parts, find $\int e^x \sin \pi x\,dx$.

(b) Using integration by substitution, evaluate $\int_0^3 e^{3-x} \sin \pi x\,dx$.

(2019-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) Let $0 \le x \le \frac{\pi}{4}$. Prove that $\frac{1}{2+\cos 2x} = \frac{\sec^2 x}{2+\sec^2 x}$.

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{2+\cos 2x} \, dx$.

(c) Let $f(x)$ be a continuous function defined on $\mathbf{R}$ such that $f(-x) = -f(x)$ for all $x \in \mathbf{R}$.

Prove that $\int_{-a}^{a} f(x) \ln(1+e^x) \, dx = \int_0^a x \, f(x) \, dx$ for any $a \in \mathbf{R}$.

(d) Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin 2x}{(2+\cos 2x)^2} \ln(1+e^x) \, dx$.

(2020-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) Using integration by substitution, prove that

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin\left(\frac{\pi}{4}-x\right)\right) dx = \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln\left(\sin x\right) dx \ .$$

(b) Using (a), evaluate $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \ln(\cot x - 1) \, dx$.

(c) (i) Using $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$, or otherwise, prove that $\cot \frac{\pi}{12} = 2+\sqrt{3}$.

(ii) Using integration by parts, prove that $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \frac{x \csc^2 x}{\cot x - 1} dx = \frac{\pi}{8} \ln\left(2+\sqrt{3}\right)$.

(2021-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) Let $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

(i) Find $\frac{d}{d\theta} \ln(\sec\theta + \tan\theta)$.

(ii) Using the result of (a) (i), find $\int \sec\theta \, d\theta$. Hence, find $\int \sec^3\theta \, d\theta$.

(b) Let $g(x)$ and $h(x)$ be continuous functions defined on $\mathbf{R}$ such that $g(x) + g(-x) = 1$ and $h(x) = h(-x)$ for all $x \in \mathbf{R}$.

Using integration by substitution, prove that $\int_{-a}^{a} g(x) \, h(x) \, dx = \int_0^a h(x) \, dx$ for any $a \in \mathbf{R}$.

(c) Evaluate $\int_{-1}^{1} \frac{3^x x^2}{(3^x + 3^{-x})\sqrt{x^2+1}} \, dx$.

ANSWERS

(1979-CE-A MATH 2 #07) (20 marks)

7. (a) $\frac{17}{6}$

(b) $\frac{dy}{dx} = \frac{-2x - y}{x + 2y}$

$(x + 2y)^3 \frac{d^2y}{dx^2} = -6x^2 - 6xy - 6y^2$

(1980-CE-A MATH 2 #12) (20 marks)

12. (b) $\int_0^{\pi} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx = \frac{\pi}{2}$

(c) $\frac{\pi^2}{4}$

(1981-CE-A MATH 2 #03) (6 marks) (Modified)

3. 36

(1981-CE-A MATH 2 #08) (20 marks) (Modified)

8. (a) $\frac{2}{15}$

(b) (iii) $\frac{\pi\sqrt{3}}{9} - \frac{3}{4}$

(1982-CE-A MATH 2 #05) (6 marks) (Modified)

5. $\frac{1}{2}\left(\frac{\pi}{2} - \frac{1}{2}\right)$

(1983-CE-A MATH 2 #03) (5 marks) (Modified)

3. $\frac{58}{135}$

(1983-CE-A MATH 2 #11) (20 marks)

11. (c) $\frac{\pi}{4} - 1$

(1983-CE-A MATH 2 #12) (20 marks)

12. (b) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{6}$

The required value $= \frac{\pi}{6}$

(1984-CE-A MATH 2 #05) (8 marks) (Modified)

5. $\int \tan^2\theta \sec^2\theta \, d\theta = \frac{1}{3}\tan^3\theta + \text{constant}$

$\int_0^{\frac{\pi}{3}} \tan^4\theta \, d\theta = \frac{\pi}{3}$

(1984-CE-A MATH 2 #07) (20 marks)

7. (a) $\frac{9}{8}$

(b) (i) $\frac{-1}{3\sin^3\phi} + \text{constant}$

(ii) $8 - 2\sqrt{2}$

(1985-CE-A MATH 2 #03) (5 marks) (Modified)

3. 1

(1985-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{2}{35}$

(d) $\frac{4}{35}$

(1986-CE-A MATH 2 #08) (20 marks)

8. (b) (i) 0

(ii) $\frac{\pi^2}{4}$

(iii) $\frac{\pi}{4}$

(1987-CE-A MATH 2 #04) (6 marks)

4. $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$

(1987-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{\tan^{n-1}x}{n-1} + C$

(b) (ii) $\left(\frac{13}{15} - \frac{\pi}{4}\right)$

(c) $2\left(\frac{13}{15} - \frac{\pi}{4}\right)$

(1988-CE-A MATH 2 #06) (6 marks)

6. $\frac{4}{3}$

(1988-CE-A MATH 2 #08) (20 marks)

8. (a) $\frac{16}{35}$

(b) $\frac{dy}{dx} = \cos^{2n} x - (2n-1)\cos^{2n-2} x \sin^2 x$

(c) (ii) $\frac{5\pi}{32}$

(d) $\frac{5\pi}{32}$

(1989-CE-A MATH 2 #03) (5 marks) (Modified)

3. $\frac{40}{3}$

(1989-CE-A MATH 2 #09) (16 marks)

9. (a) $\frac{\pi}{4}$

(d) (i) $\frac{1}{8}(\pi + 2)$

(ii) $\frac{1}{32}(3\pi + 8)$

(1990-CE-A MATH 2 #09) (16 marks)

9. (a) (i) $\frac{\pi}{2}$

(ii) $\frac{\pi^2}{4}$

(c) $\frac{\pi^2}{2}$

(1990-AL-P MATH 2 #03) (5 marks)

3. $\frac{\pi}{5}$

(1991-CE-A MATH 2 #02) (5 marks)

2. $\frac{\pi}{2} + 1$

(1991-CE-A MATH 2 #12) (16 marks)

12. (a)

$\frac{dy}{dx} = (m+1)(1+x)^n(1-x)^n - n(1+x)^{n+1}(1-x)^{n-1}$

(c) $\frac{512}{9}$

(d) $\frac{128}{63}$

(1992-CE-A MATH 2 #08) (16 marks)

8. (b) $\frac{\sqrt{3}\pi}{18}$

(c) $\frac{\sqrt{3}\pi}{9}$

(d) $\frac{2\sqrt{3}\pi}{27} - \frac{1}{6}$

(1993-CE-A MATH 2 #09) (16 marks)

9. (a) $(m-1)\sin^{m-2} x \cos^{n+2} x - (n+1)\sin^m x \cos^n x$

(d) $\frac{3\pi}{512}$

(1994-CE-A MATH 2 #10) (16 marks)

10. (a) $\frac{\pi}{4}$

(c) $\frac{\pi}{4}$

(d) $-\frac{\pi}{4}$

(1995-CE-A MATH 2 #08) (16 marks)

8. (c) $\frac{\pi}{4}$

(d) (i) $\frac{\pi}{32}$

(ii) $\frac{5\pi}{256}$

(1996-AL-P MATH 2 #03) (6 marks)

3. (b) π

(1996-CE-A MATH 2 #09) (16 marks)

9. (a) $\frac{16}{15}$

(b) $\frac{8\pi}{15}$

(c) $\frac{-16\pi}{75}$

(d) I_2 is equal to I_1 because $|x| = x$

(1997-AL-P MATH 2 #04) (6 marks)

4. $\frac{1}{10}$

(1997-CE-A MATH 2 #11) (16 marks)

11. (a) $-\frac{\cot^{n+1}\theta}{n+1} + C$

(d) $\frac{-2}{3} + \frac{8\sqrt{3}}{27} + \frac{\pi}{12}$

(1998-CE-A MATH 2 #06) (6 marks)

6. $\frac{8}{105}$

(1998-CE-A MATH 2 #09) (16 marks)

9. (c) (i) (2) $\frac{\sqrt{3}\pi}{9}$

(ii) $\frac{1}{3} - \frac{\sqrt{3}\pi}{27}$

(d) $\frac{2}{3} + \frac{4\sqrt{3}\pi}{27}$

(1999-AL-P MATH 2 #02) (6 marks)

2. (b) $\frac{\pi^2}{4}$

(1999-CE-A MATH 2 #01) (3 marks)

1. $\frac{\pi}{4}$

(1999-CE-A MATH 2 #12) (16 marks)

12. (c) 0.046

(d) $\frac{1}{2}$

(2002-CE-A MATH #04) (4 marks)

4. $\frac{\pi}{6}$

(2006-AL-P MATH 2 #03) (7 marks)

3. (b) $\frac{1}{2}\ln 2 - \frac{1}{4}$

(c) $\frac{3}{2}\ln 2 - 1$

(SP-DSE-MATH-EP(M2) #13) (14 marks)

13. (c) $\frac{\sqrt{3}\pi}{9}$

(PP-DSE-MATH-EP(M2) #13) (10 marks)

13. (a) $2pq$

(c) $\frac{\pi \ln 2}{3}$

(2012-DSE-MATH-EP(M2) #13) (13 marks)

13. (a) (ii) $v = \frac{3\pi}{10}$

(b) (i) $\left(x + \cos\frac{2\pi}{5}\right)^2 + \sin^2\frac{2\pi}{5}$

(ii) $\frac{\pi}{2}$

(c) $-\frac{\pi}{2}$

(2013-DSE-MATH-EP(M2) #11) (12 marks)

11. (c) $\frac{1}{2}\ln(6 + 4\sqrt{2} - 3\sqrt{3} - 2\sqrt{6})$

(2015-DSE-MATH-EP(M2) #03) (7 marks)

3. (a) $\frac{-1}{2}e^{-2u}$+constant

(b) $\frac{1}{e^2} - \frac{1}{e^6}$

(2016-DSE-MATH-EP(M2) #10) (12 marks)

10. (d) $\frac{\pi \ln 2}{8}$

(2017-DSE-MATH-EP(M2) #11) (13 marks)

11. (a) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(b) (ii) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(d) $\pi + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)$

(2019-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) $\frac{e^x \sin \pi x - \pi e^x \cos \pi x}{1+\pi^2}$ + constant

(b) $\frac{\pi\left(1+e^3\right)}{1+\pi^2}$

(2019-DSE-MATH-EP(M2) #10) (13 marks)

10. (b) $\frac{\sqrt{3}\pi}{18}$

(d) $\frac{\pi}{16} - \frac{\sqrt{3}\pi}{36}$

(2020-DSE-MATH-EP(M2) #10) (13 marks)

10. (b) $\frac{\pi \ln 2}{24}$

(2020-DSE-MATH-EP(M2) #09) (12 marks)

9. (a) (i) $\sec\theta$

(ii)

$$\int \sec\theta \, d\theta = \ln(\sec\theta + \tan\theta) + \text{ constant}$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)) + \text{ constant}$$

(c) $\frac{1}{2}\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)$