3. Applications of Differentiation

3. (a) Tangents ~~and Normals~~ to curves

(1991-CE-A MATH 1 #06) (7 marks) (Modified)

6. Let C be the curve $y = \frac{1}{x} + x$, where $x \neq 0$. $P(1,2)$ and $Q\left(\frac{1}{2}, \frac{5}{2}\right)$ are two points on C .

   (a) Find equations of the tangent ~~and normal~~ to C at P .

   (b) Show that the tangent to C at Q passes through the point $A(0,4)$.

(1992-CE-A MATH 1 #05) (6 marks)

5. The curve $(x-2)(y^2+3) = -8$ cuts the y-axis at two points. Find

   (a) the coordinates of the two points;

   (b) the slope of the tangent to the curve at each of the two points.

(1993-CE-A MATH 1 #07) (7 marks)

7. Given the curve $C: x^2 - 2xy^2 + y^3 + 1 = 0$.

   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) Find the equation of the tangent to C at the point $(2,-1)$.

(1994-CE-A MATH 1 #06) (7 marks)

6. Given the curve $C: x^2 + y\cos x - y^2 = 0$.

   (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

   (b) $P\left(\frac{\pi}{2}, \frac{-\pi}{2}\right)$ is a point on the curve C . Find the equation of the tangent to the curve at P .

(1995-CE-A MATH 1 #06) (7 marks)

6. $P(4,1)$ is a point on the curve $y^2 + y\sqrt{x} = 3$, where $x > 0$.

   (a) Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at P .

   ~~(b) Find the equation of the normal to the curve at P .~~

(1996-CE-A MATH 1 #06) (7 marks)

6. Find the equations of the two tangents to the curve $C : y = \dfrac{6}{x+1}$ which are parallel to the line $x + 6y + 10 = 0$.

(1997-CE-A MATH 1 #02) (3 marks)

2. $P(8,1)$ is a point on the curve $y^2 + \sqrt[3]{x}\,y - 3 = 0$. Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at P .

(1998-CE-A MATH 1 #08) (7 marks)

8. $P(0,2)$ is a point on the curve $x^2 - xy + 3y^2 = 12$.

(a) Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at P .

~~(b) Find the equation of the normal to the curve at P .~~

(1999-CE-A MATH 1 #06) (6 marks)

6. The point $P(a, a)$ is on the curve $3x^2 - xy - y^2 - a^2 = 0$, where a is a non-zero constant.

(a) Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at P .

(b) Find the equation of the tangent to the curve at P .

(2000-CE-A MATH 1 #04) (5 marks)

4. $P(-1,2)$ is a point on the curve $(x+2)(y+3) = 5$. Find

(a) the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at P .

(b) the equation of the tangent to the curve at P .

(2001-CE-A MATH #07) (5 marks)

7. $P(2,0)$ is a point on the curve $x - (1 + \sin y)^5 = 1$. Find the equation of the tangent to the curve at P .

(2002-CE-A MATH #02) (4 marks)

2. Find the equation of the tangent to the curve $C : y = (x-1)^4 + 4$ which is parallel to the line $y = 4x + 8$.

(2004-CE-A MATH #09) (6 marks)

9.

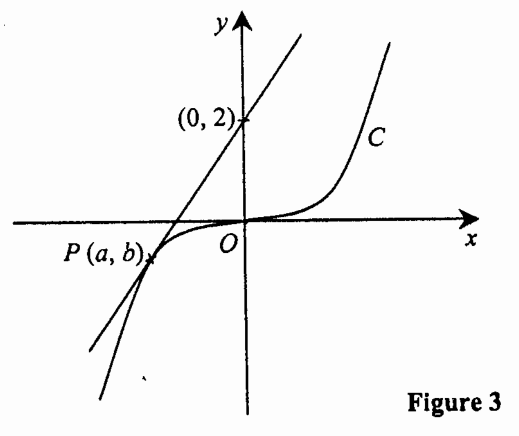


Figure 3

In Figure 3, $P(a, b)$ is a point on the curve $C : y = x^3$. The tangent to C at P passes through the point $(0,2)$.

(a) Show that $b = 3a^3 + 2$.

(b) Find the value of a and b .

(2006-CE-A MATH #12) (5 marks)

12. (a) Let $x^2 - xy + y^2 = 7$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

~~(b) Find the equation of the normal to the curve $x^2 - xy + y^2 = 7$ at the point $(1,3)$.~~

(2008-CE-A MATH #06) (5 marks)

6. Find the equation of the tangent to the curve $y = \frac{3x}{x^2 + 2}$ at the point $(2,1)$.

(2009-CE-A MATH #06) (5 marks)

6. Let C be the curve $y^3 + x^3y = 10$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) Find the equation of the tangent to C at the point $(1,2)$.

(2010-CE-A MATH #10) (6 marks)

10. It is given that P is a point on the curve $C : y = x^3$. If the y-intercept of the tangent L to C at P is -16 , find the equation of L .

(2011-CE-A MATH #06) (5 marks)

~~6. Find the equation of the normal to the curve $y = \frac{x^2 + 1}{x + 1}$ at $x = 1$.~~

(SP-DSE-MATH-EP(M2) #06) (5 marks)

6. Let C be the curve $3e^{x-y} = x^2 + y^2 + 1$.

   Find the equation of the tangent to C at the point $(1,1)$.

(PP-DSE-MATH-EP(M2) #09) (6 marks)

9. Find the equation of the two tangents to the curve $x^2 - xy - 2y^2 - 1 = 0$ which are parallel to the straight line $y = 2x + 1$.

(2014-DSE-MATH-EP(M2) #03) (5 marks)

3. Find the equation of tangent to the curve $x \ln y + y = 2$ at the point where the curve cuts the y-axis.

ANSWERS

(1991-CE-A MATH 1 #06) (7 marks)

6. (a) Tangent is $y = 2$
   Normal is $x = 1$

(1992-CE-A MATH 1 #05) (6 marks)

5. (a) (0,1) and (0, − 1)
   (b) $\left.\frac{dy}{dx}\right|_{(0,1)} = 1$, $\left.\frac{dy}{dx}\right|_{(0,-1)} = -1$

(1993-CE-A MATH 1 #07) (7 marks)

7. (a) $\frac{dy}{dx} = \frac{2x - 2y^2}{4xy - 3y^2}$
   (b) $y = -\frac{2}{11}x - \frac{7}{11}$

(1994-CE-A MATH 1 #06) (7 marks)

6. (a) $\frac{dy}{dx} = \frac{y \sin x - 2x}{\cos x - 2y}$
   (b) $y = -\frac{3}{2}x + \frac{\pi}{4}$

(1995-CE-A MATH 1 #06) (7 marks)

6. (a) $\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-1}{16}$
   (b) $y = 16x - 63$

(1996-CE-A MATH 1 #06) (7 marks)

6. $x + 6y - 11 = 0$ or $x + 6y + 13 = 0$

(1998-CE-A MATH 1 #08) (7 marks)

8. (a) $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{1}{6}$
   (b) $6x + y - 2 = 0$

(1999-CE-A MATH 1 #06) (6 marks)

6. (a) $\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5}{3}$
   (b) $5x - 3y - 2a = 0$

(2000-CE-A MATH 1 #04) (5 marks)

4. (a) $\left.\frac{dy}{dx}\right|_{(-1,2)} = -5$
   (b) $5x + y + 3 = 0$

(2001-CE-A MATH #07) (5 marks)

7. $x - 5y - 2 = 0$

(2002-CE-A MATH #02) (4 marks)

2. $y = 4x - 3$

(2004-CE-A MATH #09) (6 marks)

9. (b) $a = b = -1$

(2006-CE-A MATH #12) (5 marks)

12. (a) $\frac{dy}{dx} = \frac{y - 2x}{2y - x}$
    (b) $y = -5x + 8$

(2008-CE-A MATH #06) (5 marks)

6. $x + 6y - 8 = 0$

(2009-CE-A MATH #06) (5 marks)

6. (a) $\frac{dy}{dx} = \frac{-3x^2y}{x^3 + 3y^2}$
   (b) $6x + 13y - 32 = 0$

(2010-CE-A MATH #10) (6 marks)

10. $y = 12x - 16$

(2011-CE-A MATH #06) (5 marks)

6. $2x + y - 3 = 0$

(SP-DSE-MATH-EP(M2) #06) (5 marks)

6. $x - 5y + 4 = 0$

(PP-DSE-MATH-EP(M2) #09) (6 marks)

9. $y = 2x + 2$ or $y = 2x - 2$

(2014-DSE-MATH-EP(M2) #03) (5 marks)

3. $y = -x \ln 2 + 2$

3. (b) Curve Sketching

(1991-CE-A MATH 1 #04) (7 marks)

4. Let $y = x + \sin 2x$, where $0 \le x \le \pi$.
   Find

   (a) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$,

   (b) the maximum and minimum values of y .

(1994-CE-A MATH 1 #09) (Modified) (16 marks)

9. Given the curve $C : y = \dfrac{x^2}{1+x} - \dfrac{4}{3}$, where $x \neq -1$.

   (a) Find the x- and y-intercepts of the curve C .

   (b) Find $\dfrac{dy}{dx}$ and show that $\dfrac{d^2y}{dx^2} = \dfrac{2}{(1+x)^3}$.

   (c) Find the turning point(s) of the curve C .
   For each turning point, test whether it is a maximum or a minimum point.

   (d) Sketch the curve C for
   (i) $-5 \le x < -1$;
   (ii) $-1 < x \le 3$.

(1995-CE-A MATH 1 #03) (5 marks)

3. Using the information in the following table, sketch the graph of $y = f(x)$, where $f(x)$ is a polynomial.

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f(x)$		1		2		1	
$f'(x)$	< 0	0	> 0	0	< 0	0	> 0

(1996-CE-A MATH 1 #09) (Modified) (16 marks)

9. C_1 is the curve $y = \dfrac{4x-3}{x^2+1}$.

(a) Find

(i) the x- and y-intercepts of the curve C_1 ;

(ii) the range of values of x for which $\dfrac{4x-3}{x^2+1}$ is decreasing;

(iii) the turning point(s) of C_1 , stating whether each point is a maximum or a minimum point.

(Testing for maximum/minimum is not required.)

(b) Sketch the curve C_1 for $-10 \le x \le 10$.

(c) C_2 is the curve $y = \dfrac{|4x-3|}{x^2+1}$.

Using the result of (b), sketch the curve C_2 for $-10 \le x \le 10$ on the same graph.

Hence write down the greatest and least values of $\dfrac{|4x-3|}{x^2+1}$ for $-10 \le x \le 10$.

(1997-CE-A MATH 1 #10) (Modified) (16 marks)

10. The function $f(x) = \dfrac{x^2+kx+9}{x^2+1}$, where k is a constant, attains a stationary value at $x = 3$.

(a) Find $f'(x)$ in terms of k and x .

Hence show that $k = -6$.

(b) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the maximum and minimum points of the curve $y = f(x)$.

(c) Sketch the graph of $y = f(x)$ for $-6 \le x \le 6$.

Hence sketch the graph of $y = -f(x) - 1$ for $-6 \le x \le 6$ on the same graph.

(2000-AL-P MATH 2 #09) (Modified) (12 marks)

9. Let $f(x) = \dfrac{x}{(1+x^2)^2}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Determine the values of x for each of the following cases:

(i) $f'(x) > 0$,

(ii) $f''(x) > 0$.

(c) Find all relative extreme points, points of inflexion and asymptotes of $y = f(x)$.

(d) Sketch the graph of $f(x)$.

(2000-CE-A MATH 1 #10) (Modified) (16 marks)

10. Let $f(x) = \dfrac{7-4x}{x^2+2}$.

(a) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the range of values of x for which $f(x)$ is decreasing.

(iii) Show that the maximum and minimum values of $f(x)$ are 4 and $\dfrac{-1}{2}$ respectively.

(b) Sketch the curve $y = f(x)$ for $-2 \le x \le 5$.

(c) Let $p = \dfrac{7-4\sin\theta}{\sin^2\theta+2}$, where θ is real.

From the graph in (b), a student concludes that the greatest and least values of p are 4 and $\dfrac{-1}{2}$ respectively.

Explain whether the student is correct. If not, what should be the greatest and least values of p ?

(2001-CE-A MATH #18) (12 marks)

18. Let $f(x)$ be a polynomial, where $-2 \le x \le 10$. Figure 5 (a) shows a sketch of the curve $y = f'(x)$, where $f'(x)$ denotes the first derivative of $f(x)$.

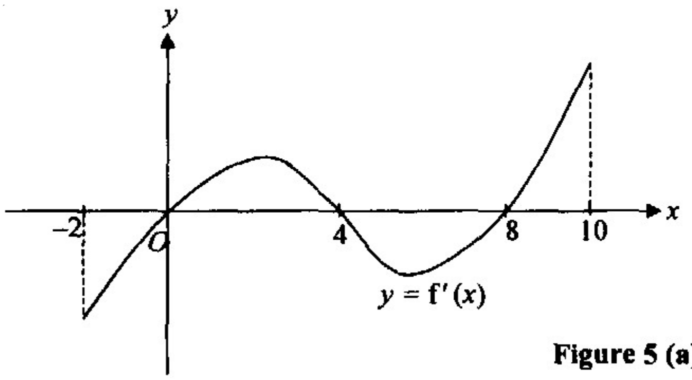


Figure 5 (a)

(a) (i) Write down the range of values of x for which $f(x)$ is increasing.

(ii) Find the x-coordinates of the maximum and minimum points of the curve $y = f(x)$.

(iii) In Figure 5 (b), draw a possible sketch of the curve $y = f(x)$.

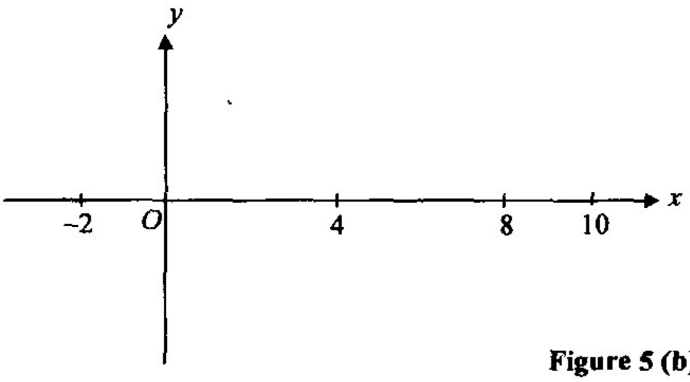


Figure 5 (b)

(b) In Figure 5 (c), sketch the curve $y = f''(x)$.

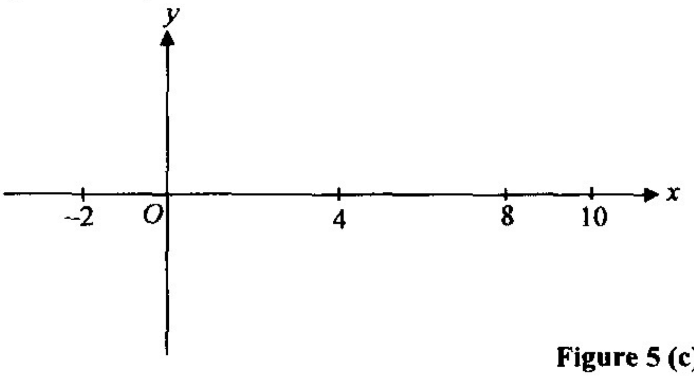


Figure 5 (c)

(c) Let $g(x) = f(x) + x$, where $-2 \le x \le 10$.

(i) In Figure 5 (a), sketch the curve $y = g'(x)$.

(ii) A student makes the following note:

Since the functions $f(x)$ and $g(x)$ are different, the graphs of $y = f''(x)$ and $y = g''(x)$ should be different.

Explain whether the student is correct or not.

(2002-AL-P MATH 2 #08) (Modified) (10 marks)

8. Let $f(x) = x^2 - \dfrac{8}{x-1}$. ($x \neq 1$)

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Determine the range of values of x for each of the following cases:

   (i) $f'(x) > 0$,

   (ii) $f'(x) < 0$,

   (iii) $f''(x) > 0$,

   (iv) $f''(x) < 0$.

   (c) Find the relative extreme point(s) and point(s) of inflexion of $f(x)$.

   (d) Find the asymptote(s) of the graph of $f(x)$.

   (e) Sketch the graph of $f(x)$.

(2003-CE-A MATH #13) (7 marks)

13. Let $f(x) = 2\sin x - x$ for $0 \leq x \leq \pi$. Find the greatest and least values of $f(x)$.

(2007-AL-P MATH 2 #07)

7. Let $f(x) = \dfrac{(x+15)(x+1)^2}{(x-6)^2}$. ($x \neq 6$)

   (a) Find $f'(x)$ and $f''(x)$.

   (b) Solve each of the following inequalities:

   (i) $f'(x) > 0$,

   (ii) $f''(x) > 0$.

   (c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$.

   (d) Find the asymptote(s) of the graph of $y = f(x)$.

   (e) Sketch the graph of $y = f(x)$.

(2007-CE-A MATH #10) (5 marks)

10.

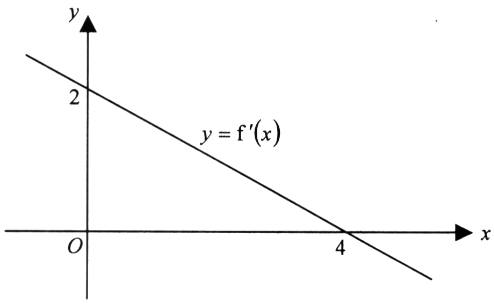


Figure 4

Let $f(x)$ be a function of x . Figure 4 shows the graph of $y = f'(x)$ which is a straight line with x- and y-intercepts 4 and 2 respectively.

(a) Find the slope of the tangent to the curve $y = f(x)$ at $x = 1$.

(b) Find the x-coordinate(s) of all the turning point(s) of the curve $y = f(x)$. For each turning point, determine whether it is a minimum point or a maximum point.

(2008-CE-A MATH #13) (7 marks)

13. Let $f(x) = x(x-6)^2$.

(a) Find the maximum and minimum points of the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$.

(2010-AL-P MATH 2 #07) (15 marks)

7. Let $f : \mathbb{R}\backslash\{-3\} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x-1}{(x+3)^3}$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Solve

(i) $f(x) > 0$,

(ii) $f'(x) > 0$,

(iii) $f''(x) > 0$.

(c) Find the relative extreme point(s) and point(s) of inflexion of the graph of $y = f(x)$.

(d) Find the asymptote(s) of the graph of $y = f(x)$.

(e) Sketch the graph of $y = f(x)$.

(f) Let $n(k)$ be the number of points of intersection of the graph of $y = f(x)$ and the horizontal line $y = k$. Using the graph of $y = f(x)$, find $n(k)$ for any $k \in \mathbb{R}$.

(2010-CE-A MATH #08) (4 marks)

2. It is given that $f(x)$ is a polynomial with the following properties:

(1) $f(0) = f(2) = f(4) = 0$;

(2) For $0 \le x \le 4$, the minimum and maximum values of $f(x)$ are -2 and 2 respectively;

(3)

	$0 \le x < 1$	$x = 1$	$1 < x < 3$	$x = 3$	$3 < x \le 4$
$f'(x)$	> 0	0	< 0	0	> 0

Using the above information, sketch the graph of $y = f(x)$ for $0 \le x \le 4$.

(2011-CE-A MATH #08) (6 marks)

8. Let $f(x) = (x + 2)(x^2 + 1)$.

(a) Find the maximum and minimum points of the graph of $y = f(x)$.

(b) Sketch the graph of $y = f(x)$.

(2012-DSE-MATH-EP(M2) #05) (6 marks)

5. Find the minimum point(s) and asymptote(s) of the graph of $y = \dfrac{x^2 + x + 1}{x + 1}$.

(2013-DSE-MATH-EP(M2) #05) (6 marks)

5. Consider a continuous function $f(x) = \dfrac{3 - 3x^2}{3 + x^2}$. It is given that

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$+$	0	$-$	$-$	$-$	0	$+$

(‘+’ and ‘$-$’ denote ‘positive value’ and ‘negative value’ respectively.)

(a) Find all the maximum and/or minimum point(s) and point(s) of inflexion.

(b) Find the asymptote(s) of the graph of $y = f(x)$.

(c) Sketch the graph of $y = f(x)$.

(2014-DSE-MATH-EP(M2) #02) (5 marks)

2. Consider the curve $C : y = x^3 - 3x$.

(a) Find $\dfrac{dy}{dx}$ from first principles.

(b) Find the range of x where C is decreasing.

(2016-DSE-MATH-EP(M2) #04) (7 marks)

4. Define $\mathrm{f}(x)=\dfrac{2x^2+x+1}{x-1}$ for all $x \neq 1$. Denote the graph of $y=\mathrm{f}(x)$ by G . Find

(a) the asymptote(s) of G ,

(b) the slope of the normal to G at the point $(2, 11)$.

(2018-DSE-MATH-EP(M2) #08) (8 marks)

8. Define $\mathrm{f}(x)=\dfrac{A}{x^2-4x+7}$ for all real numbers x , where A is a constant. It is given that the extreme value of $\mathrm{f}(x)$ is 4 .

(a) Find $\mathrm{f}'(x)$.

(b) Someone claims that there are at least two asymptotes of the graph of $y=\mathrm{f}(x)$. Do you agree? Explain your answer.

(c) Find the point(s) of inflexion of the graph of $y=\mathrm{f}(x)$.

(2021-DSE-MATH-EP(M2) #05) (7 marks)

5. Define $r(x)=\dfrac{x^3-x^2-2x+3}{(x-1)^2}$ for all real numbers $x \neq 1$.

(a) Find the asymptote(s) of the graph of $y=\mathrm{r}(x)$.

(b) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{r}(x)$.

(c) Someone claims that there is only one point of inflexion of the graph of $y=\mathrm{r}(x)$. Do you agree? Explain your answer.

ANSWERS

(1991-CE-A MATH 1 #04) (7 marks)

4. (a) $\dfrac{dy}{dx} = 1 + 2\cos 2x$

$\dfrac{d^2y}{dx^2} = -4\sin 2x$

(b) $y_{max} = \dfrac{\pi}{3} + \dfrac{\sqrt{3}}{2}$, $y_{min} = \dfrac{2\pi}{3} - \dfrac{\sqrt{3}}{2}$

(1994-CE-A MATH 1 #09) (16 marks)

9. (a) x-intercept $= 2$ or $\dfrac{-2}{3}$, y-intercept $= \dfrac{-4}{3}$

(b) $\dfrac{dy}{dx} = \dfrac{2x + x^2}{(1+x)^2}$

(c) Minimum point $= \left(0, \dfrac{-4}{3}\right)$

Maximum point $= \left(-2, \dfrac{-16}{3}\right)$

(1996-CE-A MATH 1 #09) (16 marks)

9. (a) (i) x-intercept $= \dfrac{3}{4}$, y-intercept $= -3$

(ii) $x \geq 2$ or $x \leq \dfrac{-1}{2}$

(iii) Minimum point $= \left(\dfrac{-1}{2}, -4\right)$

Maximum point $= (2,1)$

(1997-CE-A MATH 1 #10) (16 marks)

10. (a) $f'(x) = \dfrac{-kx^2 - 16x + k}{(x^2+1)^2}$

(b) (i) x-intercept $= 3$, y-intercept $= 9$

(ii) Minimum point $= (3,0)$

Maximum point $= \left(\dfrac{-1}{3}, 10\right)$

(2000-AL-P MATH 2 #09) (12 marks)

9. (a) $f'(x) = \dfrac{1 - 3x^2}{(1+x^2)^3}$

$f''(x) = \dfrac{-12x(1-x^2)}{(1+x^2)^4}$

(b) (i) $-\dfrac{\sqrt{3}}{3} < x < \dfrac{\sqrt{3}}{3}$

(ii) $-1 < x < 0$ or $x > 1$

(c) Minimum point $= \left(\dfrac{-\sqrt{3}}{3}, \dfrac{-3\sqrt{3}}{16}\right)$

Maximum point $= \left(\dfrac{-1}{3}, 10\right)$

(2000-CE-A MATH 1 #10) (16 marks)

10. (a) (i) x-intercept $= \dfrac{7}{4}$, y-intercept $= \dfrac{7}{2}$

(ii) $\dfrac{-1}{2} \leq x \leq 4$

(c) Greatest value of $p = 4$

Least value of $p = 1$

(2001-CE-A MATH #18) (12 marks)

18. (a) (i) $0 \leq x \leq 4$, $x \geq 8$

(ii) Minimum point at $x = 0$ and 8

Maximum point at $x = 4$

(2002-AL-P MATH 2 #08)

8. (a) $f'(x) = 2x + \dfrac{8}{(x-1)^2}$

$f''(x) = 2 + \dfrac{16}{(1-x)^3}$

(b) (i) $x > -1$ and $x \neq 1$

(ii) $x < -1$

(iii) $x < 1$ or $x > 3$

(iv) $1 < x < 3$

(c) Minimum point $= (-1,5)$

Point of inflexion $= (3,5)$

(d) Vertical asymptote is $x = 1$

(2003-CE-A MATH #13) (7 marks)

13. Greatest value $= \sqrt{3} - \frac{\pi}{3}$

Least values of $= -\pi$

(2007-AL-P MATH 2 #07)

7. (a) $f'(x) = \frac{(x+1)(x+8)(x-27)}{(x-6)^3}$

$f''(x) = \frac{686(x+3)}{(x-6)^4}$

(b) (i) $x < -8, -1 < x < 6$ or $x > 27$

(ii) $-3 < x < 6$ or $x > 6$

(c) Minimum points = $(-1,0)$ and $\left(27, \frac{224}{3}\right)$

Maximum point = $\left(-8, \frac{7}{4}\right)$

Point of inflexion = $\left(-3, \frac{16}{27}\right)$

(d) Vertical asymptote is $x = 6$

Oblique asymptote is $y = x + 29$

(2007-CE-A MATH #10) (5 marks)

10. (a) Slope $= f'(1) = \frac{3}{2}$ at $x = 1$

(b) x-coordinate = 4, maximum point

(2008-CE-A MATH #13) (7 marks)

13. (a) Maximum point = (2, 32)

Minimum point = (6, 0)

(2010-AL-P MATH 2 #07) (15 marks)

7. (a) $f'(x) = \frac{-2(x-3)}{(x+3)^4}$

$f''(x) = \frac{6(x-5)}{(x+3)^5}$

(b) (i) $x < -3$ or $x > 1$

(ii) $x < -3$ or $-3 < x < 3$

(iii) $x < -3$ or $x > 5$

(c) Maximum point = $\left(3, \frac{1}{108}\right)$

Point of inflexion = $\left(5, \frac{1}{128}\right)$

(d) Vertical asymptote is $x = -3$

Horizontal asymptote os $y = 0$

(f) $n(k) = \begin{cases} 1 \text{ when } k \le 0 \text{ or } k > \frac{1}{108} \\ 2 \text{ when } k = \frac{1}{108} \\ 3 \text{ when } 0 < k < \frac{1}{108} \end{cases}$

(2011-CE-A MATH #08) (6 marks)

8. (a) Maximum point = $(-1,2)$

Minimum point = $\left(\frac{-1}{3}, \frac{50}{27}\right)$

(2012-DSE-MATH-EP(M2) #05) (6 marks)

5. Minimum point = (0,1)

Vertical asymptote is $x = -1$

Oblique asymptote $y = x$

(2013-DSE-MATH-EP(M2) #05) (6 marks)

5. (a) Maximum point = (0,1)

Points of inflexion = (1,0) and $(-1,0)$

(b) Horizontal asymptote is $y = -3$

(2014-DSE-MATH-EP(M2) #02) (5 marks)

2. (a) $\frac{dy}{dx} = 3x^2 - 3$

(b) $-1 \le x \le 1$

(2016-DSE-MATH-EP(M2) #04) (7 marks)

4. (a) Vertical asymptote is $x = 1$

Oblique asymptote is $y = 2x + 3$

(b) Slope $= \frac{1}{2}$

(2018-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) $\frac{48 - 24x}{(x^2 - 4x + 7)^2}$

(b) Horizontal asymptote is $y = 0$

The claim is disagreed.

(c) (1,3) and (3,3)

(2021-DSE-MATH-EP(M2) #05) (7 marks)

5. (a) Vertical asymptote is $x = 1$

Oblique asymptote is $y = x + 1$

(b) $1 + \frac{x - 3}{(x - 1)^3}$

(c) The claim is agreed.

3. (c) Optimization and Rates of Change problems

(1991-CE-A MATH 1 #11) (Modified) (16 marks)

11.

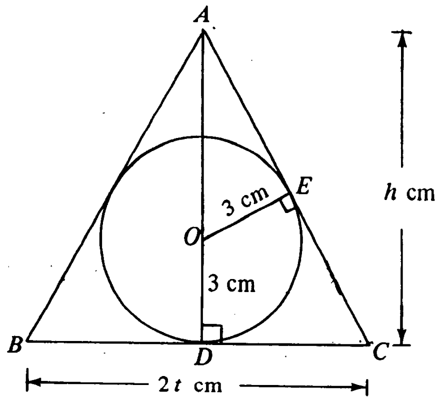


Figure 2(a)

ABC is a variable isosceles triangle with $AB = AC$ such that the radius of its inscribed circle is 3 cm. The height AD and the base BC of ΔABC are h cm and $2t$ cm respectively, where $h > 6$. (See Figure 2(a).) Let p cm be the perimeter of ΔABC.

(a) Show that $t^2 = \dfrac{9h}{h-6}$.

(b) Show that $p = \dfrac{2h^{\frac{3}{2}}}{(h-6)^{\frac{1}{2}}}$.

(c) Find

(i) the range of values of h for which $\dfrac{\mathrm{d}p}{\mathrm{d}h}$ is positive.

(ii) the minimum value of p.

(d) (i) Sketch the graph of p against h for $h > 6$.

(ii) Hence write down the range of values of p for which two different isosceles triangles whose inscribed circles are of radii 3 cm can have the same perimeter p cm.

(1991-CE-A MATH 1 #12) (16 marks)

12.

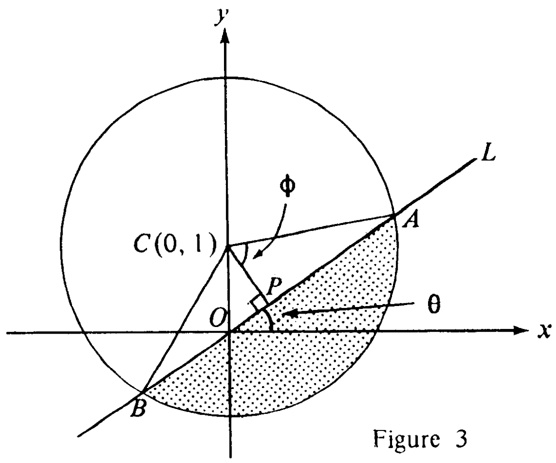


Figure 3

Figure 3 shows a circle of radius 2 centred at the point $C(0,1)$. A variable straight line L with positive slope passes through the origin O and makes an angle θ with the positive x-axis. L intersects the circle at points A and B . Let S be the area of the shaded segment. P is the point on L such that CP is perpendicular to AB . Let $\angle PCA = \varphi$.

(a) (i) Find the length of CP in terms of θ .

Hence show that $\cos\theta = 2\cos\varphi$.

(ii) Show that $S = 4\varphi - 2\sin 2\varphi$.

(b) (i) Find $\dfrac{d\varphi}{d\theta}$ in terms of θ and φ .

(ii) Hence find $\dfrac{dS}{d\theta}$ in terms of θ .

(c) L rotates about O in the clockwise direction such that θ decreases steadily at a rate of $\dfrac{1}{30}$ radian per second.

Find the rate of change of S with respect to time when $\theta = \dfrac{\pi}{3}$.

(1992-CE-A MATH 1 #07) (7 marks)

7.

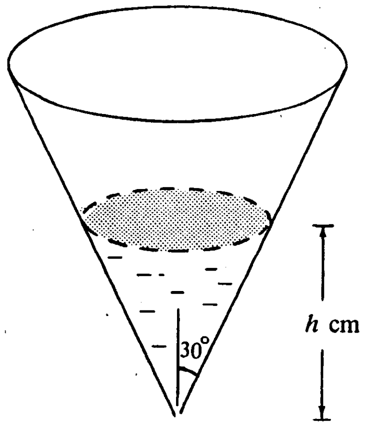


Figure 2

Figure 2 shows a vessel in the shape of a right circular cone with semi-vertical angle $30°$. Water is flowing out of the cone through its apex at a constant rate of π cm^3 s^{-1}.

(a) Let V cm^3 be the volume of water in the vessel when the depth of water is h cm. Express V in terms of h.

(b) How fast is the water level falling when the depth of water is 4 cm ?

(1992-CE-A MATH 1 #11) (16 marks)

11.

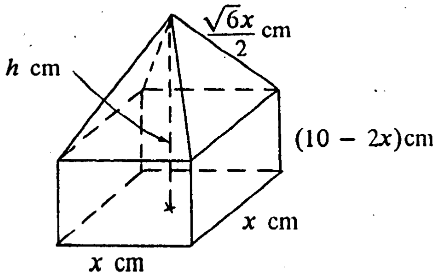


Figure 4 (a)

Figure 4 (a) shows a solid consisting of a right pyramid and a cuboid with a common face which is a square of side x cm .The slant edge of the pyramid is $\frac{\sqrt{6}x}{2}$ cm and the height of the cuboid is $(10-2x)$ cm , where $0 < x < 5$.

(a) Let h cm be the height of the solid. Show that $h = 10 - x$.

(b) Let V cm³ be the volume of the solid.

(i) Show that $V = 10x^2 - \frac{5}{3}x^3$.

(ii) Find the range of values of x for which V is increasing.
Hence write down the range of values of x for which V is decreasing.

(c)

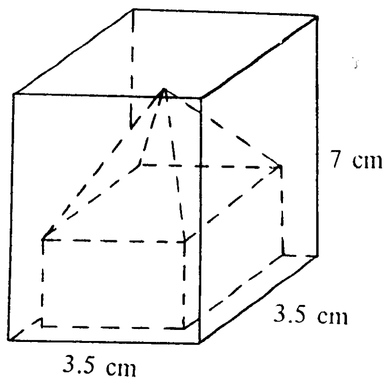


Figure 4 (b)

The solid is placed COMPLETELY inside a rectangular box as shown in Figure 4(b). The base of the box is a square of side 3.5 cm and the height of the box is 7 cm .

(i) Show that $3 \le x \le 3.5$.

(ii) Hence find, correct to one decimal place, the greatest volume of the solid.

(d) The side of the square base of the box in (c) is now changed to 4.7 cm and the height 5.5 cm . Find, correct to one decimal place, the greatest volume of the solid that can be placed COMPLETELY inside the box.

(1993-CE-A MATH 1 #09) (16 marks)

9.

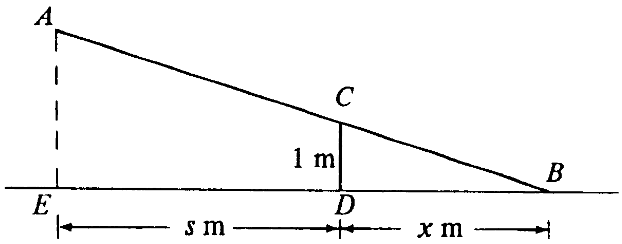


Figure 2

Figure 2 shows a straight rod AB of length 8 m resting on a vertical wall CD of height 1 m . The end B is free to slide along a horizontal rail such that AB is vertically above the rail. Let E be the projection of A on the rail, $DE = s$ m and $BD = x$ m , where $0 < x < 3\sqrt{7}$.

(a) Show that $s = \dfrac{8x}{\sqrt{1+x^2}} - x$.

(b) Find the maximum value of s .

(c) Let P m^2 be the area of the trapezium $CAED$.

(i) Show that $P = \dfrac{32x}{1+x^2} - \dfrac{x}{2}$.

(ii) Does P attain a maximum when s attains its maximum? Explain your answer.

(1994-AS-M & S #02) (5 marks)

2. The population size x of an endangered species of animals is modelled by the equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0,$$

where t denotes the time.

It is known that $x = 100e^{kt}$ where k is a negative constant. Determine the value of k .

(1994-CE-A MATH 1 #12) (16 marks)

12.

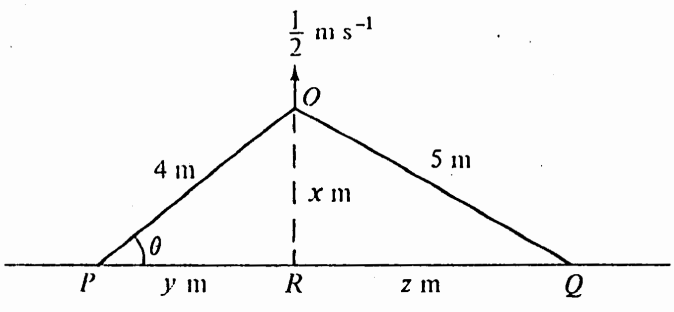


Figure 3

In Figure 3, two rods OP and OQ are hinged at O. The lengths of OP and OQ are 4 m and 5 m respectively. The end O is pushed upwards at a constant rate of $\frac{1}{2}$ ms^{-1} along a fixed vertical axis, and the ends P and Q move along a horizontal rail. R is the projection of O on the rail. At time t seconds, $OR = x$ m and $\angle OPQ = \theta$ where $0 < \theta < \frac{\pi}{2}$.

(a) Express x in terms of θ.
Hence find the rate of change of θ with respect to t in terms of θ.

(b) Let $PR = y$ m, $RQ = z$ m.
Express $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}z}{\mathrm{d}t}$ in terms of θ.
Hence find the rate of change of PQ with respect to t when $\theta = \frac{\pi}{6}$, giving your answer correct to 3 significant figures.

(c) Find the value of θ such that the area of ΔOPR is a maximum.
By considering the value of $\angle OQR$, find the value of θ such that the area of ΔORQ is a maximum, giving your answer correct to 3 significant figures.

(1995-CE-A MATH 1 #09) (16 marks)

9.

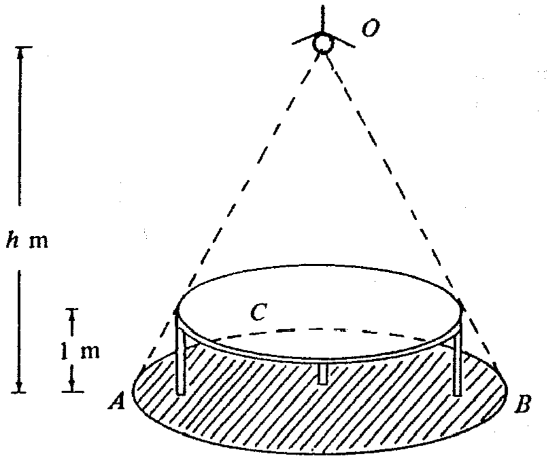


Figure 2

A small lamp O is placed h m above the ground, where $1 < h \leq 5$. Vertically below the lamp is the centre of a round table of radius 2 m and height 1 m . The lamp casts a shadow ABC of the table on the ground (see Figure 2). Let S m^2 be the area of the shadow.

(a) Show that $S = \dfrac{4\pi h^2}{(h-1)^2}$.

(b) If the lamp is lowered vertically at a constant rate of $\dfrac{1}{8}$ ms^{-1} , find the rate of change of S with respect to time when $h = 2$.

(c) Let V m^3 be the volume of the cone $OABC$.

(i) Show that $V = \dfrac{4\pi h^3}{3(h-1)^2}$.

(ii) Find the minimum value of V as h varies.
Does S attain a minimum when V attains its minimum? Explain your answer.

(1995-CE-A MATH 1 #12) (16 marks)

12.

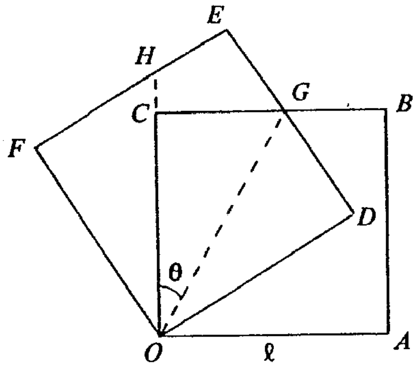


Figure 4

In Figure 4, $OABC$ is the position of a square of side ℓ . The square is rotated anticlockwise about O to a new position $ODEF$. BC cuts DE at G and OC produced cuts EF at H . Let $\angle COG = \theta$, where $\frac{\pi}{8} < \theta < \frac{\pi}{4}$.

(a) Name a triangle which is congruent to ΔOCG .

Hence show that the area of ΔOFH is $\dfrac{\ell^2}{2\tan 2\theta}$.

(b) Let S be the sum of the areas of ΔOFH and the quadrilateral $ODGC$.

(i) Show that $S = \dfrac{\ell^2}{2}\left(\dfrac{2-\cos 2\theta}{\sin 2\theta}\right)$.

(ii) Find the range of values of θ for which S is

(1) increasing,

(2) decreasing.

Hence find the minimum value of S .

(c) Find the maximum value of the area of the quadrilateral $CGEH$.

(1996-CE-A MATH 1 #11) (16 marks)

11.

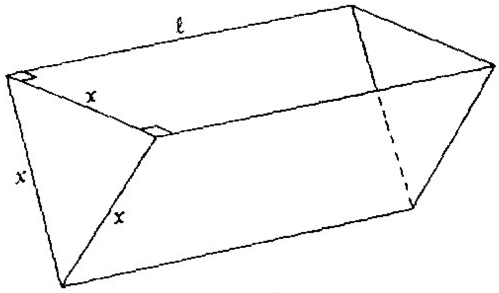


Figure 3(a)

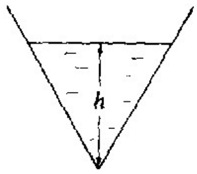


Figure 3(b)

Figure 3(a) shows a vessel with a capacity of 24 cubic units. The length of the vessel is ℓ and its vertical cross-section is an equilateral triangles of side x . The vessel is made of thin metal plates and has no lid. Let S be the total area of metal plates used to make the vessel.

(a) Show that $S = \dfrac{\sqrt{3}}{2}x^2 + \dfrac{64\sqrt{3}}{x}$.

(b) Find the values of x and ℓ such that the area of metal plates used to make the vessel is minimum.

(c) At time $t = 0$, the vessel described in part (b) is completely filled with water. Suppose the water evaporates at a rate proportional to the area of water surface at that instant such that $\dfrac{dV}{dt} = -\dfrac{1}{10}A$, where V and A are respectively the volume of water and the area of water surface at time t .

(i) Let h be the depth of water in the vessel at time t . (See Figure 3(b).) Show that $A = 4h$ and $V = 2h^2$. Hence, or otherwise, find $\dfrac{dh}{dt}$.

(ii) Find the time required for the water in the vessel to evaporate completely.

(1997-CE-A MATH 1 #04) (5 marks)

4.

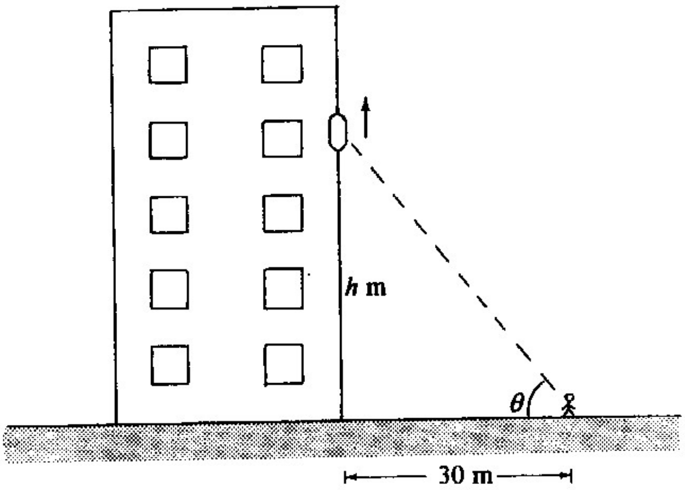


Figure 1

A man stands at a horizontal distance of 30 m from a sight-seeing elevator of a building as shown in Figure 1. The elevator is rising vertically with a uniform speed of $1.5\ \text{ms}^{-1}$. When the elevator is at a height h m above the ground, its angle of elevation from the man is θ. Find the rate of change of θ with respect to time when the elevator is at a height $30\sqrt{3}$ m above the ground. (Note: You may assume that the sizes of the elevator and the man are negligible.)

(1997-CE-A MATH 1 #12) (16 marks)

12.

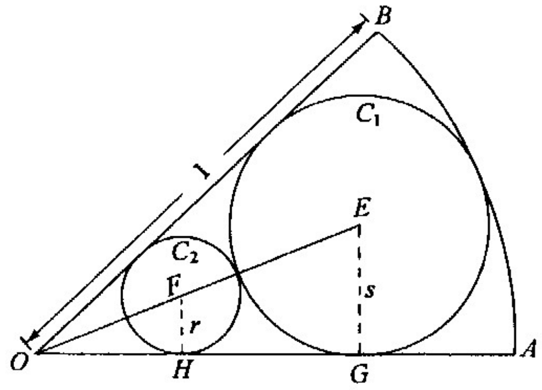


Figure 4

In Figure 4, OAB is a sector of unit radius and $\angle AOB = 2\theta$, where $0 < \theta < \dfrac{\pi}{2}$. C_1 is an inscribed circle of radius s in the sector. C_2 is another circle of radius r touching OA , OB and C_1 . Let E and F be the centres of C_1 and C_2 respectively. OA touches C_1 and C_2 at G and H respectively.

(a) Show that $s = \dfrac{\sin\theta}{1+\sin\theta}$.

Hence find $\dfrac{\mathrm{d}s}{\mathrm{d}\theta}$.

(b) By considering ΔOFH and ΔOEG , express r in terms of s .

Hence show that $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \dfrac{\cos\theta(1-3\sin\theta)}{(1+\sin\theta)^3}$.

(c) By considering the ranges of values of θ for which r is

(i) increasing, and

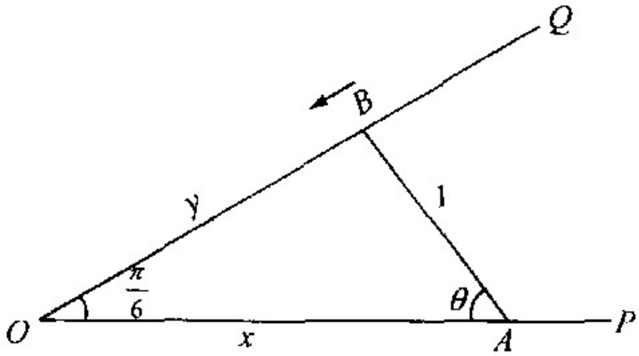
(ii) decreasing,

find the maximum area of circle C_2 . (Note : You may give your answers correct to three significant figures.)

(d) Does the area of circle C_1 attain a minimum when the area of the circle C_2 attains its maximum? Explain your answer.

(1998-CE-A MATH 1 #13) (16 marks)

13.



In Figure 5, POQ is a rail and $\angle POQ = \frac{\pi}{6}$. AB is a rod of length 1 m which is free to slide on the rail with end A on OP and end B on OQ . Initially, end A is at the point on OP such the $\angle OAB = \frac{4\pi}{9}$. End B is pushed towards O at a constant speed. After t seconds, $OA = x$ m , $OB = y$ m and $\angle OAB = \theta$, where $0 \le \theta \le \frac{4\pi}{9}$.

(a) Express x and y in terms of θ .

(b) Let S m^2 be the area of ΔOAB .

Show that $\frac{dS}{d\theta} = \sin\left(\frac{5\pi}{6} - 2\theta\right)$.

Hence find the value of θ such that S is a maximum.

(c) Using (a), show that $\frac{dx}{dt} = \frac{-\cos\left(\frac{5\pi}{6} - \theta\right)}{\cos\theta}\frac{dy}{dt}$.

(d) A student makes the following prediction regarding the motion of end A of the rod:

> As end B moves from its initial position to point O , end A will first move away from O and then it will change its direction and move towards O .

Is the student's prediction correct? Explain your answer.

(1999-CE-A MATH 1 #08) (7 marks)

8.

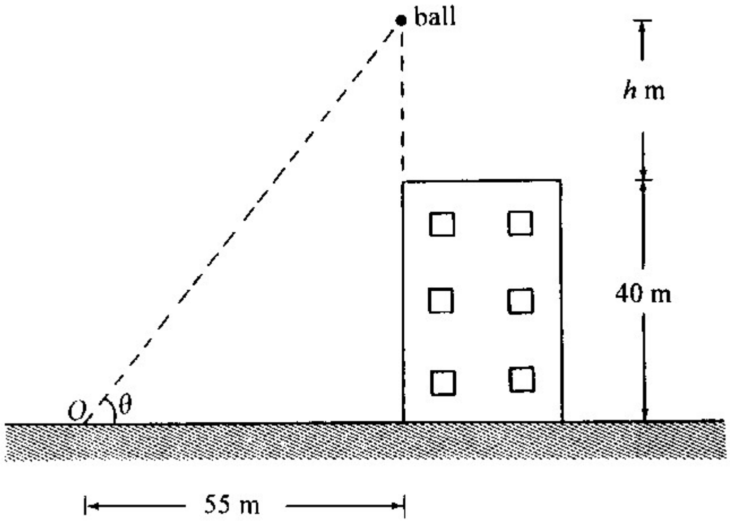


Figure 1

A ball is thrown vertically upwards from the roof of a building 40 m in height. After t seconds, the height of the ball above the roof is h m, where $h = 20t - 5t^2$. At this instant, the angle of elevation of the ball from a point O, which is at a horizontal distance of 55 m from the building, is θ. (See Figure 1.)

(a) Find

(i) $\tan\theta$ in terms of t.

(ii) the value of θ when $t = 3$.

(b) Find the rate of change of θ with respect to time when $t = 3$.

(1999-CE-A MATH 1 #12) (16 marks)

12.

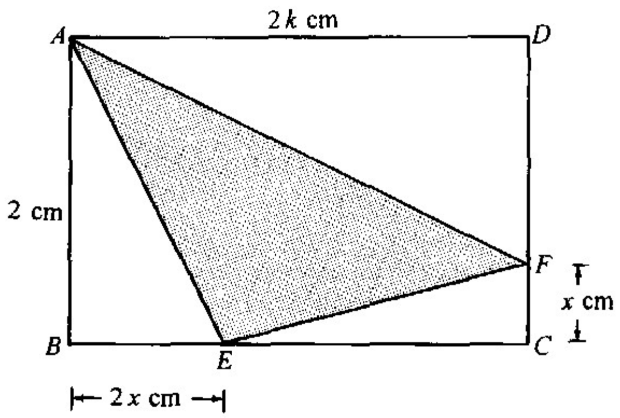


Figure 5

Figure 5 shows a rectangle $ABCD$ with $AB = 2$ cm and $AD = 2k$ cm , where k is a positive number. E and F are two variable points on the sides BC and CD respectively such that $CF = x$ cm and $BE = 2x$ cm , where x is a non-negative number. Let S cm^2 denote the area of ΔAEF .

(a) Show that $S = x^2 - 2x + 2k$.

(b) Suppose $k = \dfrac{3}{2}$.

(i) By considering that points E and F lie on the sides BC and CD respectively, show that $0 \le x \le \dfrac{3}{2}$.

(ii) Find the least value of S and the corresponding value of x .

(iii) Find the greatest value of S .

(c) Suppose $k = \dfrac{3}{8}$. A student says that S is least when $x = 1$.

(i) Explain whether the student is correct.

(ii) Find the least value of S .

(1999-CE-A MATH 1 #13) (16 marks)

13.

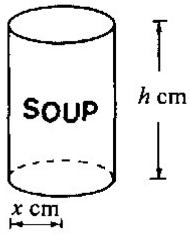


Figure 6

A food company produces cans of instant soup. Each can is in the form of a right cylinder with a base radius of x cm and a height of h cm (see Figure 6) and its capacity is $V\text{ cm}^3$, where V is constant. The cans are made of thin metal sheets. The cost of the curved surface of the can is 1 cent per cm^2 and the cost of the plane surfaces is k cents per cm^2. Let C cents be the production cost of one can. For economic reasons, the value of C is minimized.

(a) Express h in terms of π, x and V.

Hence show that $C = \dfrac{2V}{x} + 2\pi k x^2$.

(b) If $\dfrac{dC}{dx} = 0$, express x^3 in terms of π, k and V.

Hence show that C is minimum when $\dfrac{x}{h} = \dfrac{1}{2k}$.

(c) Suppose $k = 2$ and $V = 256\pi$.

(i) Find the values of x and h.

(ii) If the value of k increases, how would the dimensions of the can be affected? Explain your answer.

(d) The company intends to produce a bigger can of capacity $2V\text{ cm}^3$, which is also in the form of a right cylinder. Suppose the costs of the curved surface and plane surfaces of the bigger can are maintained at 1 cent and k cents per cm^2 respectively. A worker suggests that the ratio of base radius to height of the bigger can should be twice that of the smaller can in order to minimize the production cost. Explain whether the worker is correct.

(2000-CE-A MATH 1 #13) (16 marks)

13.

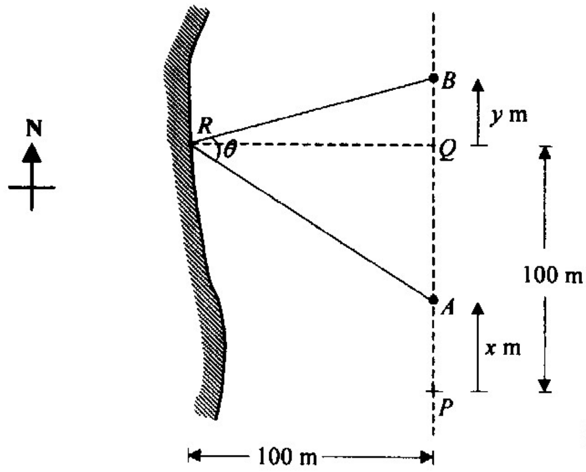


Figure 5

Two boats A and B are initially located at points P and Q in a lake respectively, where Q is at a distance 100 m due north of P. R is a point on the lakeside which is at a distance 100 m due west of Q. (See Figure 5.) Starting from time (in seconds) $t = 0$, boats A and B sail northwards. At time t, let the distances traveled by A and B be x m and y m respectively, where $0 \le x \le 100$. Let $\angle ARB = \theta$.

(a) Express $\tan\angle ARQ$ in terms of x.

Hence show that $\tan\theta = \dfrac{100(100 - x + y)}{10000 - 100y + xy}$.

(b) Suppose boat A sails with a constant speed of $2\ \text{ms}^{-1}$ and B adjust its speed continuously so as to keep the value of $\angle ARB$ unchanged.

(i) Using (a), show that $y = \dfrac{100x}{200 - x}$.

(ii) Find the speed of boat B at $t = 40$.

(iii) Suppose the maximum speed of boat B is $3\ \text{ms}^{-1}$. Explain whether it is possible to keep the value of $\angle ARB$ unchanged before boat A reaches Q.

(2002-CE-A MATH #14) (12 marks)

14.

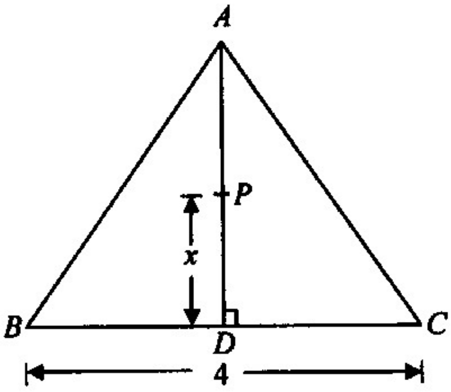


Figure 5 shows an isosceles triangle ABC with $AB = AC$ and $BC = 4$. D is the foot of perpendicular from A to BC and P is a point on AD . Let $PD = x$ and $r = PA + PB + PC$, where $0 \leq x \leq AD$.

(a) Suppose that $AD = 3$.

(i) Show that $\dfrac{\mathrm{d}r}{\mathrm{d}x} = \dfrac{2x}{\sqrt{x^2+4}} - 1$.

(ii) Find the range of values of x for which

(1) r is increasing.

(2) r is decreasing.

Hence, or otherwise, find the least value of r .

(iii) Find the greatest value of r .

(b) Suppose that $AD = 1$. Find the least value of r .

(2003-AL-P MATH 2 #02)

2. (a) Let $\mathrm{f}(x) = x^{\frac{1}{x}}$ for all $x \geq 1$. Find the greatest value of $\mathrm{f}(x)$.

(b) Using (a) or otherwise, find a positive integer m , such that $m^{\frac{1}{m}} \geq n^{\frac{1}{n}}$ for all positive integers n .

(2004-CE-A MATH #16) (12 marks)

16.

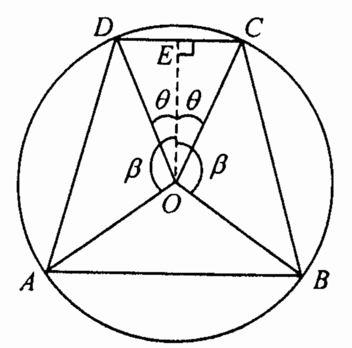


Figure 9

In Figure 9, $ABCD$ is a quadrilateral inscribed in a circle centred at O and with radius r, such that $AB \parallel DC$ and O lies inside the quadrilateral. Let $\angle COD = 2\theta$ and reflex $\angle AOB = 2\beta$, where $0 < \theta < \frac{\pi}{2} < \beta < \pi$. Point E denotes the foot of perpendicular from O to DC. Let S be the area of $ABCD$.

(a) Show that $S = \frac{r^2}{2}\left[\sin 2\theta - \sin 2\beta + 2\sin(\beta - \theta)\right]$.

(b) Suppose β is fixed. Let S_β be the greatest value of S as θ varies.

Show that $S_\beta = 2r^2\sin^3\left(\frac{2\beta}{3}\right)$ and the corresponding value of θ is $\frac{\beta}{3}$.

(Hint: You may use the identity $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$.)

(c) A student says:

Among all possible values of β, the quadrilateral $ABCD$ becomes a square when S_β in (b) attains its greatest value.

Determine whether the student is correct or not.

(2005-CE-A MATH #18) (12 marks)

18.

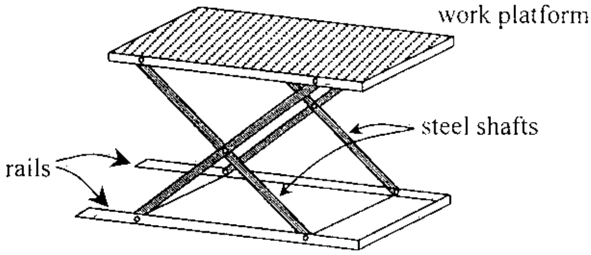


Figure 10

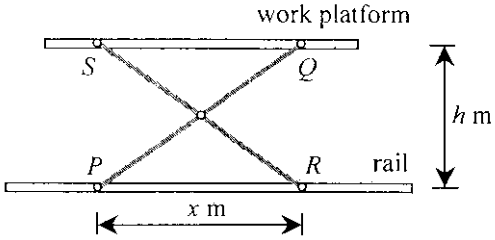


Figure 11

Figure 10 shows an elevating platform for lifting workers to work at different heights. The horizontal work platform is supported by two identical pairs of steel shafts. Figure 11 shows a cross-section of the elevating platform in a vertical plane containing one pair of shafts PQ and RS . The two shafts, each of length 4 m, are hinged at their mid-points. The ends P and R of the shafts can move along a straight horizontal rail with identical uniform speed and in opposite directions. Suppose that the elevating platform is operated under the following conditions:

(*) Initially, $PR = 3.6$ m. The work platform is the lifted upward by moving the ends P and R of the shafts towards each other such that both PR and SQ decrease at a uniform rate of $\frac{1}{2}$ ms^{-1}. Let $PR = x$ m at time t s . It is given that $0.8 \le x \le 3.6$.

In this question, numerical answers should be correct to three significant figures.

(a) Let h m be the height of the work platform above the rail at time t s .

(i) Find the range of possible values of h .

(ii) Show that $\frac{dh}{dt} = \frac{x}{2\sqrt{16 - x^2}}$.

(b) Suppose that the operation of elevating platform has to comply with the following safety regulation:

At any instant, the elevating speed of work platforms should not exceed 2 ms^{-1}.

(i) Determine whether the operation of the above elevating platform under the conditions (*) will comply with this regulation.

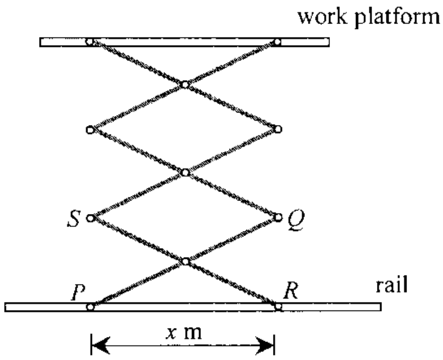


Figure 12

(ii) Figure 12 shows a vertical cross-section of a scissors-type elevating platform which can bring workers to a greater height. Two more identical pairs of shafts are installed on each side of the elevating platform as shown. Suppose that this elevating platform is operated under the same conditions (*) as described above. Do you think the operation of this elevating platform will comply with the safety regulation?

If "Yes", state your reasoning.

If "No", find the range of possible values of $\frac{\mathrm{d}x}{\mathrm{d}t}$ in order for the operation of this elevating platform to comply with the safety regulation.

(2006-CE-A MATH #15) (12 marks)

15.

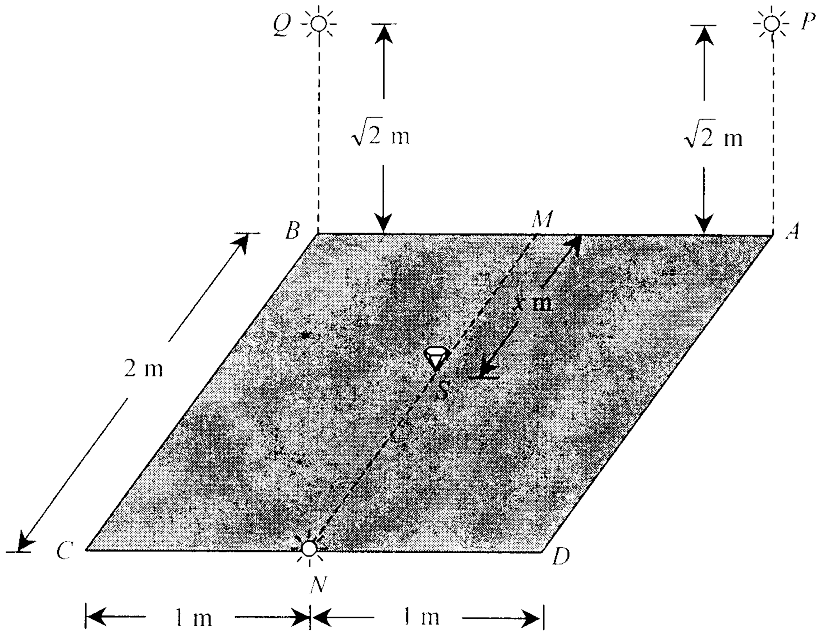


Figure 5

In Figure 5, $ABCD$ is a horizontal square board of side 2 m for displaying diamonds. Let M, N be the mid-points of BA and CD respectively. Three identical small bulbs are located at points N, P and Q respectively for illumination purpose, where P and Q are at a height $\sqrt{2}$ m vertically above A and B respectively. A diamond is placed at a point S along MN and $MS = x$ cm, where $0 \le x \le \frac{3}{2}$. Let $PS + QS + NS = \ell$ m.

(a) Express ℓ in terms of x.

Hence show that $\frac{\mathrm{d}\ell}{\mathrm{d}x} = \frac{2x}{\sqrt{x^2+3}} - 1$.

(b) Find the values of x at which ℓ attains

(i) the least value, and

(ii) the greatest value.

(c) Suppose that the intensity of light entry received by the diamond from each bulb varies inversely as the square of the distance of the bulb from the diamond, with k (> 0, in suitable unit) being the variation constant. Let E (in suitable unit) be the total intensity of light energy received by the diamond from the three bulbs.

(i) Express E in terms of k and x.

(ii) A student guesses that when ℓ attains its least value, E will attain its greatest value.

Explain whether the student's guess is correct or not.

(2007-CE-A MATH #09) (5 marks)

9.

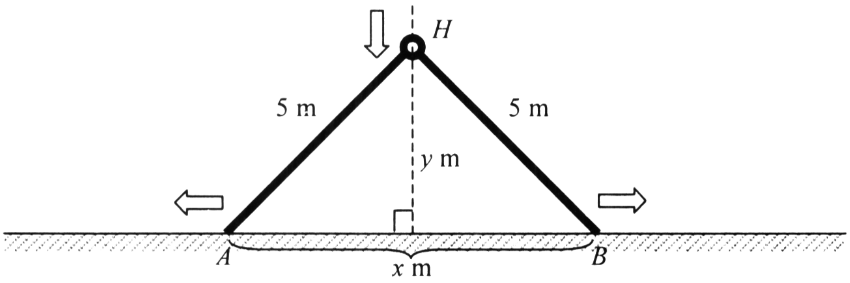


Figure 3

Two rods HA and HB , each of length 5 m , are hinged at H . The rods slide such that A , B , H are on the same vertical plane and A , B move in opposite directions on the horizontal floor, as shown in Figure 3. Let AB be x m and the distance of H from the floor be y m .

(a) Write down an equation connecting x and y .

(b) When H is 3 m from the ground, its falling speed is $2\ \text{ms}^{-1}$. Find the rate of change of the distance between A and B with respect to time at that moment.

(2007-CE-A MATH #16) (12 marks)

16.

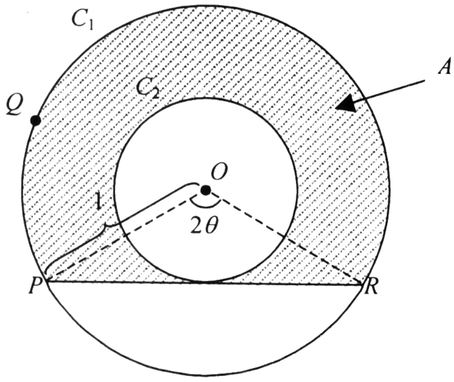


Figure 9

C_1 is a circle with centre O and radius 1 . PR is a variable chord which subtends an angle 2θ at O , where $0 < \theta < \dfrac{\pi}{2}$. C_2 is a circle with centre O and touches PR . Let the area of the shaded region bounded by C_1 , C_2 and PR be A (see Figure 9).

(a) Show that

(i) $A = \pi \sin^2\theta - \theta + \dfrac{1}{2}\sin 2\theta$,

(ii) $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = (\pi - \tan\theta)\sin 2\theta$.

(b) When A attains its greatest value, find the value of $\tan\theta$.

(c) A student guesses that when A attains its greatest value, the perimeter of the shaded region will also attain its greatest value. Explain whether the student's guess is correct or not.

(Note: the perimeter of the shaded region $= \overset{\frown}{PQR} + PR +$circumference of C_2 .)

(2008-CE-A MATH #18) (12 marks)

18.

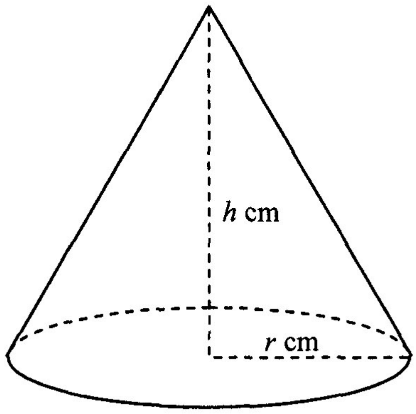


Figure 7

In a Winter Carnival, a display item is in the shape of a right circular cone. It is made of ice and a stabilizer so that the display remains in the shape of a right circular cone with the volume remaining constant. Within the duration of the Carnival, the height of the cone decreased at a constant rate of 2 cm per day. At time t days after the beginning of the Carnival, the base radius and height of the cone are r cm and h cm respectively (see Figure 7).

(a) Show that $\dfrac{dr}{dt} = \dfrac{r}{h}$.

(b) Let S cm^2 be the curved surface area of the cone.

(i) Show that $\dfrac{d}{dt}(S^2) = \dfrac{2\pi^2 r^2}{h}(2r^2 - h^2)$.

(ii) At the beginning of the Carnival, the height of the cone is 1.2 times the base radius. The gatekeeper of the Carnival claims that the curved surface area of the display increases during the whole period of the Carnival. Do you agree with the gatekeeper? Explain your answer.

(2009-CE-A MATH #16) (12 marks)

16. (a) Let $f(x) = (14 - x)(x^2 + 9)$.

(i) Find the coordinates of all the maximum and minimum points of the curve $y = f(x)$.

(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 14$ in the answer book.

(Note: the suggested range of values of y is $0 \le y \le 500$.)

(b)

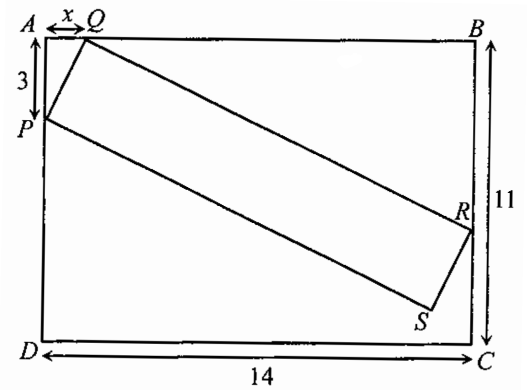


Figure 8

Figure 8 shows a rectangular cardboard $ABCD$ with $BC = 11$ and $DC = 14$. A variable rectangle $PQRS$ is cut from the cardboard according to the following rules:

(1) P is a fixed point on AD such that $AP = 3$,

(2) Q and R are points on AB and BC respectively.

Let x be the length of AQ and $g(x)$ be the area of the rectangle $PQRS$.

(i) By considering ΔAPQ and ΔBQR, express BR in terms of x.

Hence show that $g(x) = \dfrac{(14 - x)(9 + x^2)}{3}$.

(ii) By considering the fact that point S lies inside the cardboard $ABCD$, show that the range of values of x is given by

$$0 \le x \le 2 \text{ or } 12 \le x \le 14 .$$

(iii) Using (a)(ii), find the greatest value of $g(x)$ in the range shown in (b)(ii).

(2010-CE-A MATH #13) (12 marks)

13.

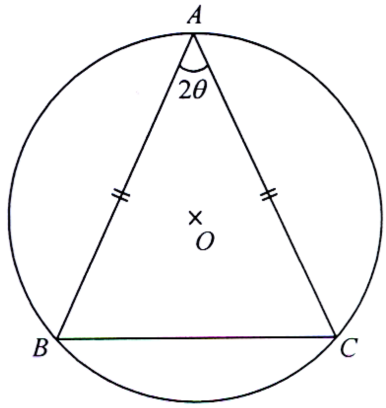


Figure 3

Figure 3 shows a circle with centre O and radius 1. A triangle ABC is inscribed in the circle with $AB = AC$. Let $\angle BAC = 2\theta$, where $0 < \theta < \frac{\pi}{4}$.

(a) Let S be the area of ΔABC .

(i) Show that $S = \frac{\sin 4\theta}{2} + \sin 2\theta$.

(ii) Find the maximum area of ΔABC .

(b)

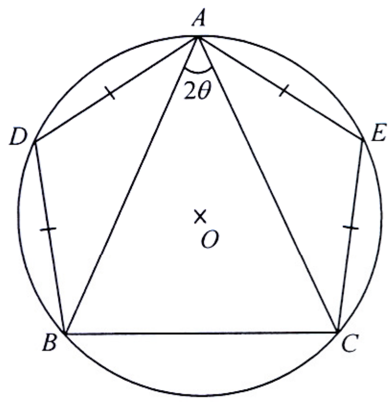


Figure 4

Two points D and E are added on the circle in Figure 3 such that $AD = BD = AE = CE$ (see Figure 4). When the area of ΔABC attains its maximum, will the area of pentagon $ADBCE$ also attain the maximum? Explain your answer.

(2011-CE-A MATH #15) (12 marks)

15. (a)

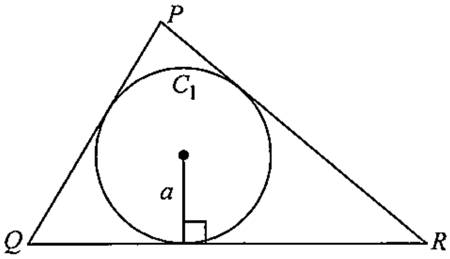


Figure 8

Figure 8 shows a triangle PQR with perimeter $2s$ and area A . A circle C_1 of radius a is inscribed in the triangle. Show that $a = \dfrac{A}{s}$.

(b)

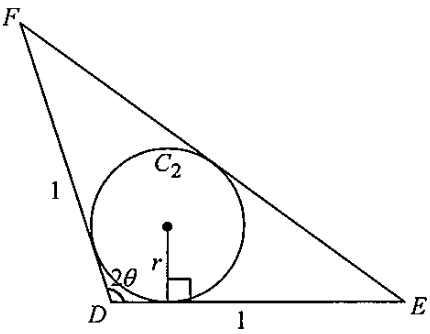


Figure 9

Figure 9 shows an isosceles triangle DEF with $DE = DF = 1$ and $\angle EDF = 2\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

A circle C_2 of radius r is inscribed in the triangle.

(i) Using (a), show that $r = \cos\theta - \dfrac{\cos\theta}{1+\sin\theta}$.

(ii) Find θ , correct to 3 decimal places, which maximizes the area of C_2 .

(iii) Frankie studies the relationship between the area of C_2 and the perimeter of ΔDEF when $\dfrac{\pi}{12} \le \theta \le \dfrac{5\pi}{12}$. Frankie claims that:

"When the perimeter of ΔDEF is the least, the area of the inscribed circle is also the least."

Do you agree with Frankie? Explain your answer.

(SP-DSE-MATH-EP(M2) #02) (4 marks)

2. A snowball in a shape of sphere is melting with its volume decreasing at a constant rate of $4\text{ cm}^3\text{s}^{-1}$. When its radius is 5 cm, find the rate of change of its radius.

(2012-DSE-MATH-EP(M2) #06) (6 marks)

6.

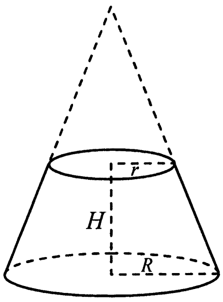


Figure 1

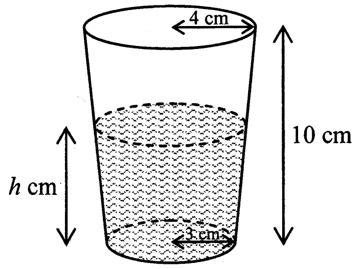


Figure 2

A frustum of height H is made by cutting off a right circular cone of base radius r from a right circular cone of base radius R (see Figure 1). It is given that the volume of the frustum is $\frac{\pi}{3}H\left(r^2 + rR + R^2\right)$. An empty glass is in the form of an inverted frustum described above with height 10 cm, the radii of the rim and the base 4 cm and 3 cm respectively. Water is being poured into the glass. Let $h\text{ cm}$ $(0 \le h \le 10)$ be the depth of the water inside the glass at time $t\text{ s}$ (see Figure 2).

(a) Show that the volume $V\text{ cm}^3$ of water inside the glass at time $t\text{ s}$ is given by

$$V = \frac{\pi}{300}(h^3 + 90h^2 + 2700h) .$$

(b) If the volume of water in the glass is increasing at the rate $7\pi\text{ cm}^3\text{s}^{-1}$, find the rate of increase of depth of water at the instant when $h = 5$.

(2013-DSE-MATH-EP(M2) #12) (13 marks)

12.

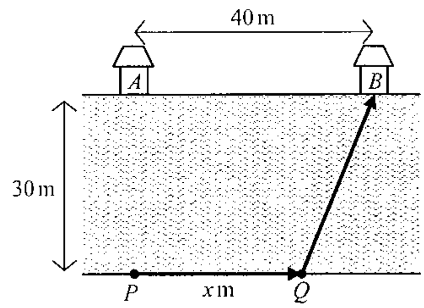


Figure 3

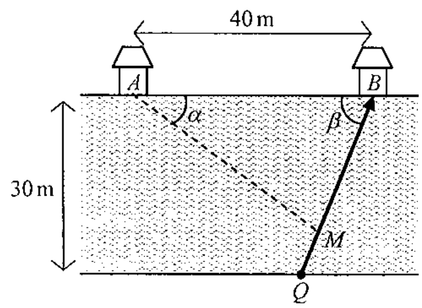


Figure 4

In Figure 3, the distance between two houses A and B lying on a straight river bank is 40 m. The width of the river is always 30 m. In the beginning , Mike stands at the starting point P in the opposite bank which is 30 m from A. Mike's wife, situated at A, is watching him running along the bank of $x\text{ m}$ at a constant speed of 7 ms^{-1} to point Q then swimming at a constant speed of 1.4 ms^{-1} along a straight path to teach B.

(a) Let T seconds be the time that Mike travels from P to B.

(i) Express T in terms of x.

(ii) When T is minimum, show that x satisfies the equation $2x^2 - 160x + 3125 = 0$.

Hence show that $QB = \dfrac{25\sqrt{6}}{2}\text{ m}$.

(b) In Figure 4, Mike is swimming from Q to B with QB equals to the value mentioned in (a)(ii). Let $\angle MAB = \alpha$ and $\angle ABM = \beta$, where M is the position of Mike.

(i) By finding $\sin\beta$ and $\cos\beta$, show that $MB = \dfrac{200\tan\alpha}{\tan\alpha + 2\sqrt{6}}$.

(ii) Find the rate of change of α when $\alpha = 0.2$ radian. Correct your answer to 4 decimal places.

(2014-DSE-MATH-EP(M2) #10) (12 marks)

10.

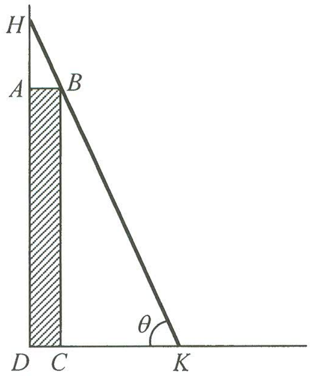


Figure 2

Thomas has a bookcase of dimensions 100 cm × 24 cm × 192 cm at the corner in his room. He want to hang a decoration on the wall above the bookcase. Therefore, he find a ladder to climb up. Initially, the ladder touches the wall, the edge of the top of the bookcase and the floor at the same time. Let rectangle $ABCD$ be the side-view of the bookcase and HK be the side-view of the ladder, so that $AB = 24$ cm and $BC = 192$ cm (see Figure 2). Let $\angle HKD = \theta$.

(a) Find the length of HK in terms of θ .

(b) Prove that the shortest length of the ladder is $120\sqrt{5}$ cm .

(c)

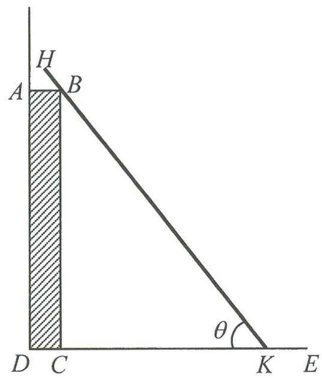


Figure 3

Suppose the length of the ladder is 270 cm . Suddenly, the ladder slides down so that the end of the ladder, K , moves towards E (see Figure 3). The ladder touches the edge of the top of the bookcase and the floor at the same time. Let x cm and y cm be the horizontal distances from H and K respectively to the wall.

(i) When $CK = 160$ cm , the rate of change of θ is -0.1 rad s^{-1}. Find the rate of change of x at this moment, correct to 4 significant figures.

(ii) Thomas claims that K is moving towards E at a speed faster than the horizontal speed H is leaving the wall. Do you agree? Explain your answer.

(2016-DSE-MATH-EP(M2) #03) (5 marks)

3. Consider the curve $C : y = 2e^x$, where $x > 0$. It is given that P is a point lying on C . The horizontal line which passes through P cuts the y-axis at the point Q . Let O be the origin. Denote the x-coordinate of P by u .

   (a) Express the area of ΔOPQ in term of u .

   (b) If P moves along C such that OQ increases at a constant rate of 6 units per second, find the rate of change of the area of ΔOPQ when $u = 4$.

(2017-DSE-MATH-EP(M2) #06) (7 marks)

6. A container in the form of an inverted right circular cone is held vertically. The height and the base radius of the container are 20 cm and 15 cm respectively. Water is now poured into the container.

   (a) Let A cm² be the wet curved surface area of the container and h cm be the depth of water in the container. Prove that $A = \dfrac{15}{16}\pi h^2$.

   (b) The depth of water in the container increases at a constant rate of $\dfrac{3}{\pi}$ cm/s . Find the rate of change of the wet curved surface area of the container when the volume of water in the container is 96π cm³ .

(2018-DSE-MATH-EP(M2) #09) (12 marks)

9. Consider the curve $C : y = \ln\sqrt{x}$, where $x > 1$. Let P be a moving point lying on C . The normal to C at P cuts the x-axis at the point Q while the vertical line passing through P cuts the x-axis at the point R .

   (a) Denote the x-coordinate of P by r . Prove that the x-coordinate of Q is $\dfrac{4r^2 + \ln r}{4r}$.

   (b) Find the greatest area of ΔPQR .

   (c) Let O be the origin. It is given that OP increases at a rate not exceeding $32e^2$ units per minute. Someone claims that the area of ΔPQR increases at a rate lower than 2 square units per minute when the x-coordinate of P is e . Is the claim correct? Explain your answer.

(2020-DSE-MATH-EP(M2) #06) (7 marks)

6. Consider the curve $C_1 : y = 2^{x-1}$, where $x > 0$. Denote the origin by O . Let $P(u, v)$ be a moving point on C_1 such that the area of the circle with OP as a diameter increases at a constant rate of 5π square units per second.

   (a) Define $S = u^2 + v^2$. Does S increases at a constant rate? Explain your answer.

   (b) Let C_2 be the curve $y = 2^x$, where $x > 0$. The vertical line passing through P cuts C_2 at the point Q . Find the rate of change of the area of ΔOPQ when $u = 2$.

(2021-DSE-MATH-EP(M2) #10) (13 marks)

10. Denote the graph of $y = \sqrt{x^2 + 36}$ and the graph of $y = -\sqrt{(20 - x)^2 + 16}$ by F and G respectively, where $0 < x < 20$. Let P be a moving point on F . The vertical line passing through P cuts G at the point Q . Denote the x-coordinate of P by u . It is given that the length of PQ attains its minimum value when $u = a$.

(a) Find a .

(b) The horizontal line passing through P cuts the y-axis at the point R while the horizontal line passing through Q cuts the y-axis at the point S .

(i) Someone claims that the area of the rectangle $PQSR$ attains its minimum value when $u = a$. Do you agree? Explain your answer.

(ii) The length of OP increases at a constant rate of 28 units per minute. Find the rate of change of the perimeter of the rectangle $PQSR$ when $u = a$.

ANSWERS

(1991-CE-A MATH 1 #11) (16 marks)

11. (c) (i) $h > 9$

(ii)

(d) (ii)

(1991-CE-A MATH 1 #12) (16 marks)

12. (a) (i) $CP = \cos\theta$

(b) (i) $\dfrac{d\phi}{d\theta} = \dfrac{\sin\theta}{2\sin\phi}$

(ii) $\dfrac{dS}{d\theta} = 4\sin\theta\sqrt{1-\dfrac{1}{4}\cos^2\theta}$

(c) $\dfrac{\sqrt{5}}{20}$ per second

(1992-CE-A MATH 1 #07) (7 marks)

7. (a) $V = \dfrac{\pi}{9}h^3$

(b) $\dfrac{3}{16}$ cm/s

(1992-CE-A MATH 1 #11) (16 marks)

11. (b) (ii) V is increasing when $0 < x \le 4$

V is decreasing when $4 \le x < 5$

(c) (ii) 51.0 cm^3

(d) 50.6 cm^3

(1993-CE-A MATH 1 #09) (16 marks)

9. (b) $3\sqrt{3}$

(c) (ii) No

(1994-AS-M & S #02) (5 marks)

2. $k = -1$

(1994-CE-A MATH 1 #12) (16 marks)

12. (a) $x = 4\sin\theta$

$\dfrac{d\theta}{dt} = \dfrac{1}{8\cos\theta}$

(b) $\dfrac{dy}{dt} = \dfrac{-\tan\theta}{2}$, $\dfrac{dz}{dt} = \dfrac{-2\sin\theta}{\sqrt{25-16\sin^2\theta}}$

Rate $= -0.507$ m/s

(c) Area of ΔOPR is maximum when $\theta = \dfrac{\pi}{4}$

Area of ΔORQ is maximum when $\theta = 1.08$

(1995-CE-A MATH 1 #09) (16 marks)

9. (b) $\dfrac{dS}{dt} = 2\pi$ per second

(c) (ii) Minimum value of $V = 9\pi$

S does not attain a minimum when V attains its minimum.

(1995-CE-A MATH 1 #12) (16 marks)

12. (a) $\Delta ODG \cong \Delta OCG$

(b) (ii) (1) $\dfrac{\pi}{6} < \theta < \dfrac{\pi}{4}$

(2) $\dfrac{\pi}{8} < \theta < \dfrac{\pi}{6}$

Minimum value of $S = \dfrac{\sqrt{3}}{2}\ell^2$

(c) $\left(1-\dfrac{\sqrt{3}}{2}\right)\ell^2$

(1996-CE-A MATH 1 #11) (16 marks)

11. (b) $x = 4$, $\ell = 2\sqrt{3}$

(c) (i) $\dfrac{dh}{dt} = \dfrac{-1}{10}$

(ii) $20\sqrt{3}$

(1997-CE-A MATH 1 #04) (5 marks)

4. $\dfrac{1}{80}$ per second

(1997-CE-A MATH 1 #12) (16 marks)

12. (a) $\frac{ds}{d\theta} = \frac{\cos\theta}{(1+\sin\theta)^2}$

(c) (i) $0 < \theta \le 0.340$

(ii) $0.340 \le \theta < \frac{\pi}{2}$

Maximum area of circle $C_2 = \frac{\pi}{64}$

(d) No

(1998-CE-A MATH 1 #13) (16 marks)

13. (a) $x = 2\sin\left(\frac{5\pi}{6} - \theta\right)$, $y = 2\sin\theta$

(b) $\theta = \frac{5\pi}{12}$

(d) Yes

(1999-CE-A MATH 1 #08) (7 marks)

8. (a) (i) $\tan\theta = \frac{4t - t^2 + 8}{11}$

(ii) $\frac{\pi}{4}$

(b) $\frac{1}{11}$ per second

(1999-CE-A MATH 1 #12) (16 marks)

12. (b) (ii) Least value of $S = 2$

Corresponding value of $x = 1$

(iii) 3

(c) (i) Incorrect

(ii) $\frac{9}{64}$

(1999-CE-A MATH 1 #13) (16 marks)

13. (a) $h = \frac{V}{\pi x^2}$

(b) $x^3 = \frac{V}{2\pi k}$

(c) (i) $x = 4$, $h = 16$

(ii) The base radius decreases,

the height increases.

(d) Incorrect

(2000-CE-A MATH 1 #13) (16 marks)

13. (a) $\tan\angle ARQ = \frac{100 - x}{100}$

(b) (ii) $\frac{25}{9}$ m/s

(iii) Impossible

(2002-AS-M & S #02) (5 marks)

2. (a) 173.35 m^3/h

(b) 920.49 m^3

(2002-CE-A MATH #14) (12 marks)

14. (a) (ii) (1) $\frac{2}{\sqrt{3}} \le x \le 3$

(2) $0 \le x \le \frac{2}{\sqrt{3}}$

Least value of $r = 2\sqrt{3} + 3$

(iii) $2\sqrt{13}$

(b) $2\sqrt{5}$

(2003-AL-P MATH 2 #02)

2. (a) $e^{\frac{1}{e}}$

(b) $m = 3$

(2004-CE-A MATH #16) (12 marks)

16. (c) Correct

(2005-CE-A MATH #18) (12 marks)

18. (a) (i) $1.74 \le h \le 3.92$

(b) (i) Yes

(ii) No

(2006-CE-A MATH #15) (12 marks)

15. (a) $\ell = 2\sqrt{x^2 + 3} + 2 - x$

(b) (i) 1

(ii) 0

(c) (i) $E = \frac{k}{(2-x)^2} + \frac{2k}{x^2+3}$

(ii) No

(2007-CE-A MATH #09) (5 marks)

9. (a) $\frac{x^2}{4} + y^2 = 25$

(b) 3 m/s

(2007-CE-A MATH #16) (12 marks)

16. (b) π

(c) No

(2008-CE-A MATH #18) (12 marks)

18. (b) (ii) Agreed

(2009-CE-A MATH #16) (12 marks)

16. (a) (i) Minimum point $= \left(\frac{1}{3}, \frac{3362}{27}\right)$

Maximum point $= (9,450)$

(b) (i) $BR = \frac{x(14-x)}{3}$

(iii) 102

(2010-CE-A MATH #13) (12 marks)

13. (a) (ii) $\frac{3\sqrt{3}}{4}$

(b) No

(2011-CE-A MATH #15) (12 marks)

15. (b) (ii) 0.666 rad

(iii) No

(SP-DSE-MATH-EP(M2) #02) (4 marks)

2. $\frac{-1}{25\pi}$ cm/s

(2012-DSE-MATH-EP(M2) #06) (6 marks)

6. (b) $\frac{4}{7}$ cm/s

(2013-DSE-MATH-EP(M2) #12) (13 marks)

12. (a) (i) $T = \frac{x + 5\sqrt{x^2 - 80x + 2500}}{7}$

(b) (i) $\sin\beta = \frac{2\sqrt{6}}{5}$, $\cos\beta = \frac{1}{5}$

(ii) -0.0357 rad/s

(2014-DSE-MATH-EP(M2) #10) (12 marks)

10. (a) $HK = \left(\frac{24}{\cos\theta} + \frac{192}{\sin\theta}\right)$ cm

(c) (i) 11.79 cm/s

(ii) Agreed

(2016-DSE-MATH-EP(M2) #03) (5 marks)

3. (a) ue^u

(b) 15 sq. unit per second

(2017-DSE-MATH-EP(M2) #06) (7 marks)

6. (b) 45 cm^2/s

(2018-DSE-MATH-EP(M2) #09) (12 marks)

9. (b) $\frac{1}{4e^2}$ square units

(c) $\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=e} \le \frac{4e}{\sqrt{4e^2+1}} < 2$

(2020-DSE-MATH-EP(M2) #06) (7 marks)

6. (a) ... Yes

(b) 5 square units per second

(2021-DSE-MATH-EP(M2) #10) (13 marks)

10. (a) $a = 12$

(b) (i) The claim is disagreed.

(ii) 42 units per minutes