3. (a) Trigonometric Functions (Part 1)

(1980-CE-A MATH 1 #05) (6 marks)

5. Given that $\dfrac{\sin^2 A}{1+2\cos^2 A} = \dfrac{3}{19}$, where $\dfrac{\pi}{2} < A < \pi$, find the value of $\dfrac{\sin A}{1+2\cos A}$.

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(1980-CE-A MATH 1 #05) (6 marks)

5. $\dfrac{\sin A}{1 + 2\cos A} = -1$

3. (b) Trigonometric Functions (Part 2)

(1979-CE-A MATH 1 #05) (6 marks)

5. The quadratic equation in x

$$(2\sin A)x^2 - 2x - \cos A = 0$$

has two equal roots. Find A , where $0° \leq A < 360°$.

(1980-CE-A MATH 1 #09) (Modified) (20 marks)

9. (a) Show that $\tan 3\theta = \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$.

(b) Let $f(x) = 3x^3 + mx^2 - 9x + n$, where m and n are integers. When $f(x)$ is divided by $x - 1$, the remainder is -8 . When $f(x)$ is divided by $x - 2$, the remainder is -5 .

(i) Show that $m = -3$ and $n = 1$.

(ii) By putting $x = \tan\theta$ and using the result in (a), or otherwise, solve the equation

$$f(x) = 0 .$$

(Correct your answer to 2 decimal places.)

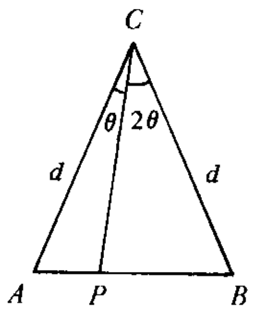
(1980-HL-GEN MATHS #03) (16 marks)

3. (a) Prove, by mathematical induction, or otherwise, that for any positive integer n ,

$$\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha = \frac{\sin\left(\frac{n+1}{2}\alpha\right)\sin\frac{n}{2}\alpha}{\sin\frac{\alpha}{2}},$$

where $\alpha \neq 2m\pi$ for any integer m .

(b) (Requires knowledge of Sine Law in the Compulsory Part)



In Figure 1, ΔABC is an isosceles triangle with $AC = BC = d$ and $AB = 1$. P is a point on AB such that $\angle ACP = \theta$ and $\angle BCP = 2\theta$. Using the sine law, show that

$$AP = \frac{1}{1 + 2\cos\theta} .$$

Hence, or otherwise, deduce that $\dfrac{1}{3} < AP < \dfrac{1}{2}$.

(1982-HL-GEN MATHS #04) (Modified) (16 marks)

4. (a) In ΔABC , $\angle BAC = 90°$, $BC = \ell$ and $\angle ACB = \theta$. D is a point on BC such that $AD \perp BC$ (see Figure 1).

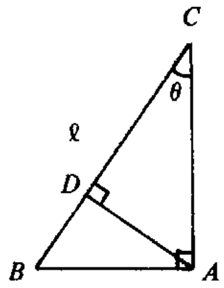


Figure 1

(i) Find the area of ΔABC in terms of ℓ and θ .

(ii) Find $\dfrac{\text{Area of } \Delta ACD}{\text{Area of } \Delta ABC}$ in terms of $\cos\theta$.

(iii) When $\dfrac{\text{Area of } \Delta ABD}{\text{Area of } \Delta ACD} = \dfrac{1}{3}$, find the value of θ .

(b) Let $\mathrm{f}(\theta) = 8\sin^4\dfrac{\theta}{2} + \cos 2\theta + 8\cos\theta$.

(i) Show that $\mathrm{f}(\theta) = 4\left(\cos\theta + \dfrac{1}{2}\right)^2$.

(ii) For $0 \le \theta \le 2\pi$,

(1) find the maximum value of $\mathrm{f}(\theta)$ and the value(s) of θ when $\mathrm{f}(\theta)$ attains its maximum value,

(2) find the minimum value of $\mathrm{f}(\theta)$ and the value(s) of θ when $\mathrm{f}(\theta)$ attains its minimum value.

(1983-HL-GEN MATHS #04) (Modified) (16 marks)

4. (a) Let $\mathrm{f}(x) = \dfrac{2(\sin^4 x - \cos^4 x - 2)}{4\cos^2 x + 5}$.

(i) Show that $\mathrm{f}(x) = \dfrac{3}{4\cos^2 x + 5} - 1$.

(ii) Find the maximum and minimum values of $\mathrm{f}(x)$.

(b) Solve

$$\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 0$$

for $0 \le \theta \le 2\pi$.

(1983-CE-A MATH 2 #07) (7 marks)

7. Show that $\sin^2 n\theta - \sin^2 m\theta = \sin(n+m)\theta \sin(n-m)\theta$.

Hence, or otherwise, solve the equation $\sin^2 3\theta - \sin^2 2\theta - \sin\theta = 0$ for $0 \le \theta \le \pi$.

(1984-HL GEN MATHS #05) (Modified) (16 marks)

5. (a) Express $\cot 4\theta$ in terms of $\cot\theta$.

Hence solve the equation $x^4 - 4x^3 - 6x^2 + 4x + 1 = 0$.

(Give your answers in terms of π.)

(b) (i) If $\cos\theta - \cos\phi = a$ and $\sin\theta - \sin\phi = b$ ($b \neq 0$),

show that

$$\frac{1}{2}(2 - a^2 - b^2) = \cos(\theta - \phi) \text{ and } \frac{-a}{b} = \tan\frac{\theta + \phi}{2} .$$

(ii) Solve the system of equations

$$\begin{cases} \cos\theta - \cos\phi = 1 \\ \sin\theta - \sin\phi = \sqrt{3} \end{cases}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq 2\pi$.

(1985-HL GEN MATHS #05) (8 marks)

5. (b) (i) Show that

$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - (\sin 2B + \sin 2C - \sin 2A).$$

(ii) If $A + B + C = 2\pi$, deduce, from (b) (i), that

$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - 4\sin A\cos B\cos C.$$

Furthermore, if $A = \frac{\pi}{2}$, find the greatest value of $\cos B\cos C$.

(1986-CE-A MATH 2 #05) (Modified) (16 marks) (Requires knowledge of Sine Law in the Compulsory Part)

5. In ΔABC , the lengths of the sides a , b and c form an arithmetic sequence, i.e.

$b - a = c - b$ where a is the length of the shortest side.

The difference between the greatest angle A and the smallest angle C is $90°$.

(a) (i) Using the sine law, or otherwise, show that

$$\sin B = \frac{1}{2}(\sin A + \sin C).$$

(ii) Using the relation $\angle A - \angle C = 90°$, show that

$$\sin A + \sin C = \sqrt{2}\cos\frac{B}{2} .$$

(iii) Hence deduce that $\sin\frac{B}{2} = \frac{\sqrt{2}}{4}$ and $\sin B = \frac{\sqrt{7}}{4}$.

(b) Show that $\angle B = 90° - 2\angle C$.

Hence deduce that $\sin C = \frac{\sqrt{7} - 1}{4}$.

(c) Show that the lengths of the sides of ΔABC are in the ratios

$\sqrt{7} + 1 : \sqrt{7} : \sqrt{7} - 1$.

(1987-HL-GEN MATHS #05) (Modified) (16 marks)

5. (a) Let ΔABC be an acute-angled triangle.

(i) Show that $\cos^2 A + \cos^2 B = \frac{1}{2}(\cos 2A + \cos 2B) + 1$.

(ii) Show that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.

(b) (i) Prove, by mathematical induction, that for any positive integers n ,

$$\cos\phi + \cos 3\phi + \cos 5\phi + \ldots + \cos(2n-1)\phi = \frac{\sin 2n\phi}{2\sin\phi} ,$$

where ϕ is not a multiple of π .

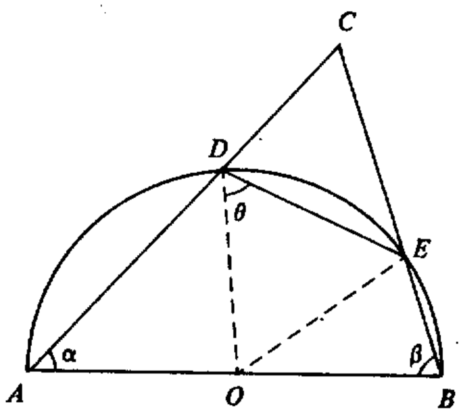
(1987-CE-A MATH 2 #06) (6 marks)

6. Express $\sin 3\theta$ in terms of $\sin\theta$. Hence find the three roots of the equation $8x^3 - 6x + 1 = 0$ to 2 significant figures.

(1988-HL-GEN MATHS #05) (16 marks)

5.

Figure 1



In Figure 1, $ADEB$ is a semi-circle with diameter AB and centre O . BE and AD are produced to meet at C . $AB = c$, $AC = b$, $BC = a$, $\angle A = \alpha$, $\angle B = \beta$ and $\angle ODE = \theta$.

(a) (i) By considering ΔDOE , find DE in terms of c and θ .

(ii) Show that $\angle C = \theta$ and ΔEDC is similar to ΔABC .
Hence express CD and CE in terms of a , b and θ .

(b) (i) Show that the area of $\Delta CED = \frac{1}{2}ab\cos^2\theta\sin\theta$

and hence the area of the quadrilateral $ADEB = \frac{1}{2}ab\sin^3\theta$.

(ii) If the area of ΔCED : the area of quadrilateral $ADEB$ = 1 : 3, find θ .
Suppose further that $ab = c^2$, show that ΔABC is equilateral.

(1988-CE-A MATH 2 #07) (7 marks)

7. (a) Without using calculators, show that $\frac{\pi}{10}$ is a root of $\cos 3\theta = \sin 2\theta$.

(b) Given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, find the value of $\sin\frac{\pi}{10}$, expressing the answer in surd form.

(1989-HL-GEN MATHS #05) (Modified) (16 marks)

5. (a) Find the solution of $\sin x - \sin 2x + \sin 3x = 0$ for $0 < \theta < 2\pi$.

(b) Let $f(\theta) = \sin 2\theta + \sin\theta + \cos\theta$.

(i) Express $f(\theta)$ in terms of p , where $p = \sin\theta + \cos\theta$.

(ii) Using (i) and the method of completing the square, find the smallest value of $f(\theta)$.
For $0 < \theta < \pi$, find also the value of θ such that $f(\theta)$ attains its smallest value.

(1989-CE-A MATH 2 #05) (5 marks)

5. Let $y = 5\sin\theta - 12\cos\theta + 7$.

(a) Express y in the form $r\sin(\theta - \alpha) + p$, where r , α and p are constants and $0° \leq \alpha \leq 90°$.

(b) Using the result in (a), find the least value of y .

(1990-CE-A MATH 2 #06) (5 marks)

6. (a) If $\cos\theta + \sqrt{3}\sin\theta = r\cos(\theta - \alpha)$, where $r > 0$ and $0° \leq \alpha \leq 90°$, find r and α .

(b) Let $x = \dfrac{1}{\cos\theta + \sqrt{3}\sin\theta + 5}$, find the range of values of x .

(1992-CE-A MATH 2 #12) (16 marks) (Requires knowledge of Cosine Law in the Compulsory Part)

12. (a) Using the identity $2\cos x\sin y = \sin(x+y) - \sin(x-y)$, show that

$$2\left[\cos\theta + \cos(\theta+2\alpha) + \cos(\theta+4\alpha) + \cos(\theta+6\alpha) + \cos(\theta+8\alpha)\right]\sin\alpha = \sin(\theta+9\alpha) - \sin(\theta-\alpha).$$

Hence show that

$$\cos\theta + \cos\left(\theta+\frac{2\pi}{5}\right) + \cos\left(\theta+\frac{4\pi}{5}\right) + \cos\left(\theta+\frac{6\pi}{5}\right) + \cos\left(\theta+\frac{8\pi}{5}\right) = 0.$$

(b)

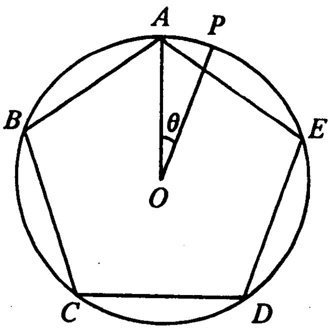


Figure 4

A , B , C , D and E are the vertices of a regular pentagon inscribed in a circle of radius r and centred at O . P is a point on the circumference of the circle such that $\angle POA = \theta$, as shown in Figure 4.

(i) By considering ΔOPD , show that

$$PD^2 = 2r^2 - 2r^2\cos\left(\theta+\frac{6\pi}{5}\right).$$

(ii) Show that $PA^2 + PB^2 + PC^2 + PD^2 + PE^2 = 10r^2$.

(iii) QP is a line perpendicular to the plane of the circle such that $QP = 2r$.
Find $QA^2 + QB^2 + QC^2 + QD^2 + QE^2$.

(1997-CE-A MATH 2 #01) (4 marks)

1. Show that $\dfrac{\sin 3\theta}{\sin\theta} + \dfrac{\cos 3\theta}{\cos\theta} = 4\cos 2\theta$.

(2002-CE-A MATH #08) (5 marks)

8. Given $0 < x < \dfrac{\pi}{2}$. Show that $\dfrac{\tan x - \sin^2 x}{\tan x + \sin^2 x} = \dfrac{4}{2+\sin 2x} - 1$.

Hence, or otherwise, find the least value of $\dfrac{\tan x - \sin^2 x}{\tan x + \sin^2 x}$.

(2003-CE-A MATH #10) (5 marks)

10. Given two acute angles α and β . Show that $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\left(\dfrac{\alpha+\beta}{2}\right)$.

If $3\sin\alpha - 4\cos\alpha = 4\cos\beta - 3\sin\beta$, find the value of $\tan(\alpha+\beta)$.

(2006-CE-A MATH #02) (3 marks)

2. Prove the identity $\cos^2 x - \cos^2 y = -\sin(x+y)\sin(x-y)$.

(2008-CE-A MATH #03) (4 marks)

3. Find the value of $\tan 22.5°$ in surd form.

(2008-CE-A MATH #09) (5 marks)

9. (a) Express $\sin x + \sqrt{3}\cos x$ in the form $r\sin(x+\alpha)$, where $r > 0$ and $0° < \alpha < 90°$.

(b) Using (a), find the least and the greatest values of $\sin x + \sqrt{3}\cos x$ for $0° \le x \le 90°$.

(2011-CE-A MATH #07) (6 marks)

7. Solve $\sin 5x + \sin x = \cos 2x$ for $0° \le x \le 90°$.

(SP-DSE-MATH-EP(M2) #05) (4 marks)

5. By considering $\sin\frac{\pi}{7}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$, find the value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$.

(PP-DSE-MATH-EP(M2) #04) (5 marks)

4. (a) Let $x = \tan\theta$, show that $\frac{2x}{1+x^2} = \sin 2\theta$.

(b) Using (a), find the greatest value of $\frac{(1+x)^2}{1+x^2}$, where x is real.

(2012-DSE-MATH-EP(M2) #10) (6 marks) (Requires knowledge of Sine Law)

10.

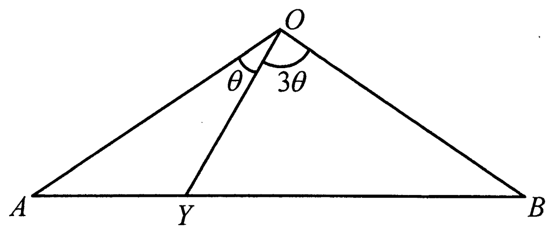


Figure 5

In Figure 5, OAB is an isosceles triangle with $OA = OB$, $AB = 1$, $AY = y$, $\angle AOY = \theta$ and $\angle BOY = 3\theta$.

(a) Show that $y = \frac{1}{4}\sec^2\theta$.

(b) Find the range of values of y .

(Hint: you may use the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)

(2013-DSE-MATH-EP(M2) #07) (5 marks)

7. (a) Prove the identity $\tan x = \dfrac{\sin 2x}{1+\cos 2x}$.

(b) Using (a), prove the identity $\tan y = \dfrac{\sin 8y \cos 4y \cos 2y}{(1+\cos 8y)(1+\cos 4y)(1+\cos 2y)}$.

(2015-DSE-MATH-EP(M2) #07) (7 marks)

7. (a) Prove that $\sin^2 x \cos^2 x = \dfrac{1-\cos 4x}{8}$.

(b) Let $\mathrm{f}(x) = \sin^4 x + \cos^4 x$.

(i) Express $\mathrm{f}(x)$ in the form $A \cos Bx + C$, where A , B and C are constants.

(ii) Solve the equation $8\mathrm{f}(x) = 7$, where $0 \le x \le \dfrac{\pi}{2}$.

(2015-DSE-MATH-EP(M2) #08) (8 marks)

8. (a) Using mathematical induction, prove that $\sin \dfrac{x}{2} \displaystyle\sum_{k=1}^{n} \cos kx = \sin \dfrac{nx}{2} \cos \dfrac{(n+1)x}{2}$ for all positive integers n .

(b) Using (a), evaluate $\displaystyle\sum_{k=1}^{567} \cos \dfrac{k\pi}{7}$.

(2016-DSE-MATH-EP(M2) #06) (6 marks)

6. (a) Prove that $x+1$ is a factor of $4x^3+2x^2-3x-1$.

(b) Express $\cos 3\theta$ in terms of $\cos\theta$.

(c) Using the results of (a) and (b), prove that $\cos \dfrac{3\pi}{5} = \dfrac{1-\sqrt{5}}{4}$.

(2017-DSE-MATH-EP(M2) #07) (8 marks)

7. (a) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

(b) Let $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$.

(i) Prove that $\dfrac{\sin 3\left(x-\dfrac{\pi}{4}\right)}{\sin\left(x-\dfrac{\pi}{4}\right)} = \dfrac{\cos 3x + \sin 3x}{\cos x - \sin x}$.

(ii) Solve the equation $\dfrac{\cos 3x + \sin 3x}{\cos x - \sin x} = 2$.

(2018-DSE-MATH-EP(M2) #03) (5 marks)

3. (a) If $\cot A = 3\cot B$, prove that $\sin(A+B) = 2\sin(B-A)$.

   (b) Using (a), solve the equation $\cot\left(x+\dfrac{4\pi}{9}\right) = 3\cot\left(x+\dfrac{5\pi}{18}\right)$, where $0 \le x \le \dfrac{\pi}{2}$.

(2020-DSE-MATH-EP(M2) #03) (6 marks)

3. (a) Let x be an angle which is not a multiple of $30°$. Prove that

   (i) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$,

   (ii) $\tan x\ \tan(60° - x)\ \tan(60° + x) = \tan 3x$.

   (b) Using (a) (ii), prove that $\tan 55°\ \tan 65°\ \tan 75° = \tan 85°$.

(2021-DSE-MATH-EP(M2) #04) (6 marks)

4. (a) Prove that $\cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x - 1$.

   (b) Solve the equation $\cos 4\theta + \cos 8\theta + \cos 12\theta = -1$, where $0 \le \theta \le \dfrac{\pi}{2}$.

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(1979-CE-A MATH 1 #05)

5. $A = 135°$ or $315°$

(1980-CE-A MATH 1 #09)

9. (b) (ii) $x = 0.11, 2.26, -1.37$

(1982-HL-GEN MATHS #04) (Modified)

4. (a) (i) $\frac{1}{2}\ell^2 \sin\theta \cos\theta$

(ii) $\cos^2\theta$

(iii) $\frac{\pi}{6}$

(b) (ii) (1) Maximum value = 9

$\theta = 0, 2\pi$

(1) Minimum value = 0

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

(1983-HL-GEN MATHS #04) (Modified)

4. (a) (ii) Maximum value $= \frac{-2}{5}$

Minimum value $= \frac{-2}{3}$

(b) $x = \frac{\pi}{6}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{4\pi}{5}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{6\pi}{5}, \frac{3\pi}{2}, \frac{8\pi}{5}, \frac{11\pi}{6}$

(1983-CE-A MATH 2 #07)

7. $\theta = 0, \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$ or π

(1984-HL GEN MATHS #05)

5. (a) $\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$

$x = \cot\frac{\pi}{16}, \cot\frac{5\pi}{16}, \cot\frac{9\pi}{16}, \cot\frac{13\pi}{16}$

(b) (ii) $\theta = \frac{\pi}{3}, \phi = \frac{4\pi}{3}$

(1985-HL GEN MATHS #05)

5. (b) (ii) $\frac{1}{2}$

(1987-CE-A MATH 2 #06)

6. $x = 0.17\ ,\ 0.77\ ,\ -0.94$

(1988-HL-GEN MATHS #05)

5. (a) (i)

(ii)

(b) (ii)

(1988-CE-A MATH 2 #07)

7. (b) $\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4}$

(1989-HL-GEN MATHS #05) (Modified)

5. (a)

(b) (i)

(ii)

(1989-CE-A MATH 2 #05)

5. (a) $y = 13\sin(\theta - 67.4°) + 7$

(b) The least value of $y = -6$

(1990-CE-A MATH 2 #06)

6. (a) $r = 2, \alpha = 60°$

(b) $\frac{1}{7} \le x \le \frac{1}{3}$

(1992-CE-A MATH 2 #12)

12. (b)

(iii) $QA^2 + QB^2 + QC^2 + QD^2 + QE^2 = 30r^2$

(2002-CE-A MATH #08)

8. The least value $= \frac{1}{3}$

(2003-CE-A MATH #10)

10. $\tan(\alpha + \beta) = \frac{-24}{7}$

(2008-CE-A MATH #03)

3. $\tan 22.5° = \sqrt{2} - 1$

(2008-CE-A MATH #09)

9. (a) $\sin x + \sqrt{3}\cos x = 2\sin(x + 60°)$

   (b) Least value = 1, Greatest value = 2

(2011-CE-A MATH #07)

7. $x = 10°$ or $45°$ or $50°$

(SP-DSE-MATH-EP(M2) #05)

5. $\dfrac{1}{8}$

(PP-DSE-MATH-EP(M2) #04)

4. (b) The greatest value = 2

(2012-DSE-MATH-EP(M2) #10)

10. (b) $\dfrac{1}{4} < y < \dfrac{1}{2}$

(2015-DSE-MATH-EP(M2) #07)

7. (b) (i) $f(x) = \dfrac{1}{4}\cos 4x + \dfrac{3}{4}$

   (ii) $x = \dfrac{\pi}{12}$ or $\dfrac{5\pi}{12}$

(2015-DSE-MATH-EP(M2) #08)

8. (b) -1

(2016-DSE-MATH-EP(M2) #06)

6. (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(2017-DSE-MATH-EP(M2) #07)

7. (b) (ii) $x = \dfrac{5\pi}{12}$

(2018-DSE-MATH-EP(M2) #03)

3. (b) $\dfrac{7\pi}{18}$

(2021-DSE-MATH-EP(M2) #04)

4. (b) $\theta = \dfrac{\pi}{12}$, $\theta = \dfrac{\pi}{8}$, $\theta = \dfrac{\pi}{4}$,

   $\theta = \dfrac{3\pi}{8}$ or $\theta = \dfrac{5\pi}{12}$