2000-CE
A MATH
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 2000

ADDITIONAL MATHEMATICS PAPER 1

8.30 am – 10.30 am (2 hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.

2. All working must be clearly shown.

3. Unless otherwise specified, numerical answers must be **exact**.

4. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.

5. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Solve $\frac{1}{x} > 1$.

(3 marks)

2. Find (a) $\frac{d}{dx}\sin^2 x$,

(b) $\frac{d}{dx}\sin^2(3x+1)$.

(4 marks)

3. (a) Show that $\frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}} = \frac{-\Delta x}{\sqrt{x}\,(\sqrt{x+\Delta x})\,(\sqrt{x}+\sqrt{x+\Delta x})}$.

(b) Find $\frac{d}{dx}(\frac{1}{\sqrt{x}})$ from first principles.

(5 marks)

4. $P(-1, 2)$ is a point on the curve $(x+2)(y+3) = 5$. Find

(a) the value of $\frac{dy}{dx}$ at P,

(b) the equation of the tangent to the curve at P.

(5 marks)

5. (a) Solve $|1-x| = 2$.

(b) By considering the cases $x \le 1$ and $x > 1$, or otherwise, solve $|1-x| = x-1$.

(5 marks)

6. Express the complex number $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$ in polar form.

Hence find the argument θ of $\left(\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^{2000}$, where θ is limited to the principal values $-\pi < \theta \le \pi$.

(6 marks)

7. α and β are the roots of the quadratic equation

$$x^2 + (p-2)x + p = 0,$$

where p is real.

(a) Express $\alpha + \beta$ and $\alpha\beta$ in terms of p.

(b) If α and β are real such that $\alpha^2 + \beta^2 = 11$, find the value(s) of p.

(7 marks)

8.

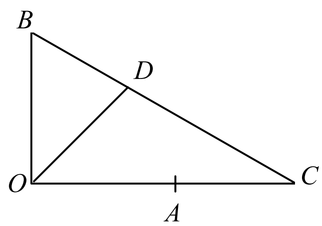

Figure 1

In Figure 1, $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$. C is a point on OA produced such that $AC = k$, where $k > 0$. D is a point on BC such that $BD : DC = 1 : 2$.

(a) Show that $\overrightarrow{OD} = \dfrac{1+k}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j}$.

(b) If $\overrightarrow{OD}$ is a unit vector, find

(i) k,

(ii) $\angle BOD$, giving your answer correct to the nearest degree.

(7 marks)

Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9.

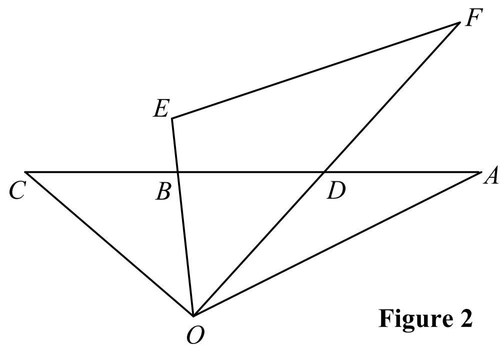

Figure 2

In Figure 2, OAC is a triangle. B and D are points on AC such that $AD = DB = BC$. F is a point on OD produced such that $OD = DF$. E is a point on OB produced such that $OE = k(OB)$, where $k > 1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) (i) Express $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) Show that $\overrightarrow{OC} = -\frac{1}{2}\mathbf{a} + \frac{3}{2}\mathbf{b}$.

(iii) Express $\overrightarrow{EF}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

(5 marks)

(b) It is given that $OA = 3$, $OB = 2$ and $\angle AOB = 60°$.

(i) Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{b} \cdot \mathbf{b}$.

(ii) Suppose that $\angle OEF = 90°$.

(1) Find the value of k.

(2) A student states that points C, E and F are collinear. Explain whether the student is correct.

(11 marks)

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10. Let $f(x)=\dfrac{7-4x}{x^2+2}$.

(a) (i) Find the x- and y-intercepts of the curve $y=f(x)$.

(ii) Find the range of values of x for which $f(x)$ is decreasing.

(iii) Show that the maximum and minimum values of $f(x)$ are 4 and $-\dfrac{1}{2}$ respectively.

(9 marks)

(b) In Figure 3, sketch the curve $y=f(x)$ for $-2\le x\le 5$.

(3 marks)

(c) Let $p=\dfrac{7-4\sin\theta}{\sin^2\theta+2}$, where θ is real.

From the graph in (b), a student concludes that the greatest and least values of p are 4 and $-\dfrac{1}{2}$ respectively. Explain whether the student is correct. If not, what should be the greatest and least values of p ?

(4 marks)

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

If you attempt Question 10, fill in the first three boxes above and tie this sheet into your answer book.

10. (b) **(continued)**

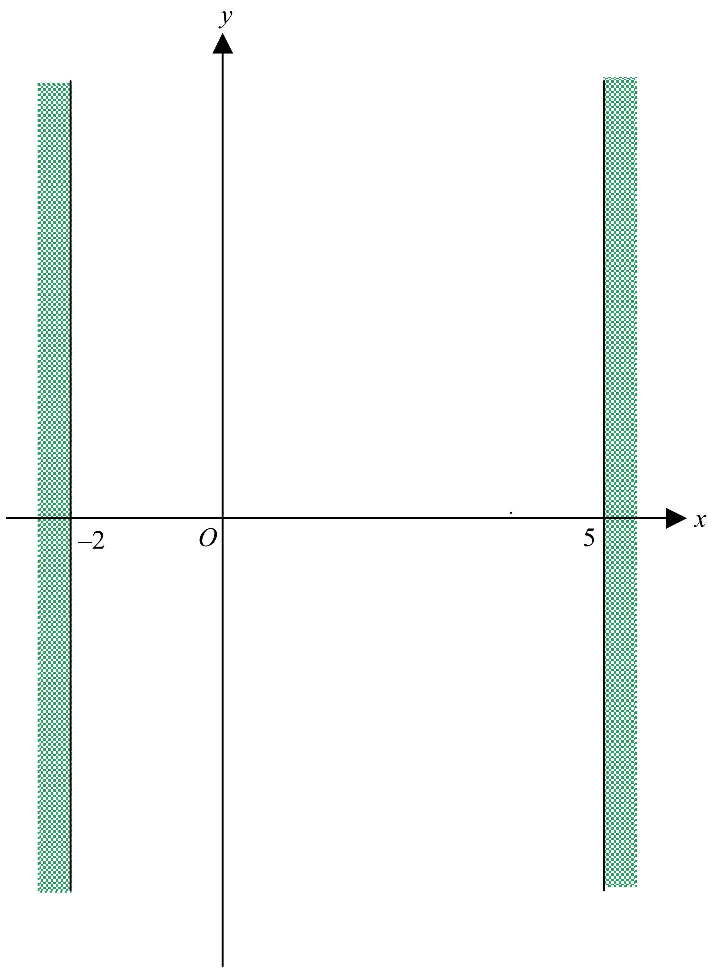

Figure 3

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11. (a) Let $w = \cos\theta + i\sin\theta$, where $0 < \theta < \pi$. It is given that the complex number $w^2 + \frac{5}{w} - 2$ is purely imaginary.

Show that $2\cos^2\theta + 5\cos\theta - 3 = 0$.

Hence, or otherwise, find w.

(8 marks)

(b) A and B are two points in an Argand diagram representing two distinct non-zero complex numbers z_1 and z_2 respectively. Suppose that $z_2 = wz_1$, where w is the complex number found in (a).

(i) Find $\left|\frac{z_2}{z_1}\right|$ and $\arg(\frac{z_2}{z_1})$.

(ii) Let O be the point representing the complex number 0. What type of triangle is ΔOAB ? Explain your answer.

(8 marks)

12. Consider the function $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$, where $m > \frac{1}{3}$.

(a) Show that the equation $f(x) = 0$ has distinct real roots.

(3 marks)

(b) Let α, β be the roots of the equation $f(x) = 0$, where $\alpha < \beta$.

(i) Express α and β in terms of m.

(ii) Furthermore, it is known that $4 < \beta < 5$.

(1) Show that $1 < m < \frac{6}{5}$.

(2) Figure 4 shows three sketches of the graph of $y = f(x)$ drawn by three students. Their teacher points out that the three sketches are all incorrect. Explain why each of the sketches is incorrect.

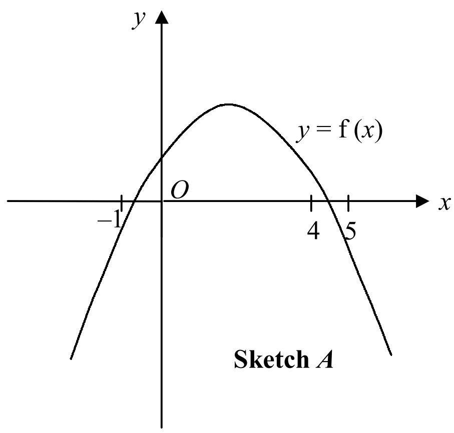
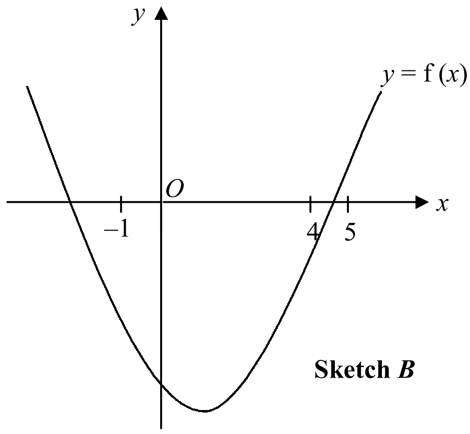
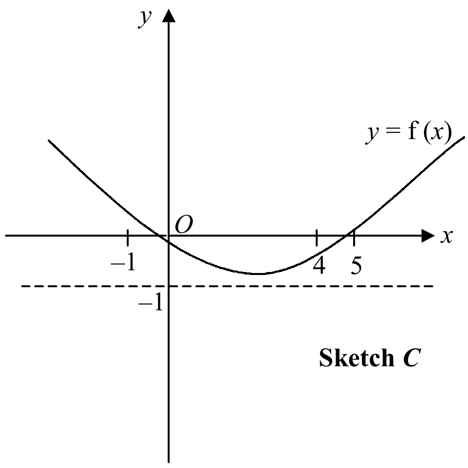


Figure 4

12. (b) (ii) (2) **(continued)**



Sketch *B*



Sketch *C*

Figure 4 (continued)

(13 marks)

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13.

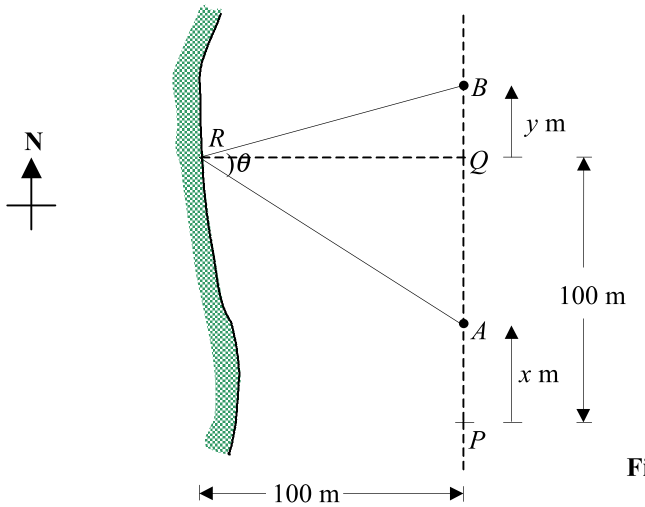


Figure 5

Two boats A and B are initially located at points P and Q in a lake respectively, where Q is at a distance 100 m due north of P. R is a point on the lakeside which is at a distance 100 m due west of Q. (See Figure 5.) Starting from time (in seconds) $t = 0$, boats A and B sail northwards. At time t, let the distances travelled by A and B be x m and y m respectively, where $0 \le x \le 100$. Let $\angle ARB = \theta$.

(a) Express $\tan \angle ARQ$ in terms of x.

Hence show that $\tan\theta = \dfrac{100\,(100 - x + y)}{10000 - 100y + xy}$. (4 marks)

(b) Suppose boat A sails with a constant speed of $2\ \text{m s}^{-1}$ and B adjusts its speed continuously so as to keep the value of $\angle ARB$ unchanged.

(i) Using (a), show that $y = \dfrac{100x}{200 - x}$.

(ii) Find the speed of boat B at $t = 40$.

(iii) Suppose the maximum speed of boat B is $3\ \text{m s}^{-1}$. Explain whether it is possible to keep the value of $\angle ARB$ unchanged before boat A reaches Q.

(12 marks)

END OF PAPER

2000

Additional Mathematics

Paper 1

Section A

1. $0 < x < 1$

2. (a) $2\sin x\cos x$

   (b) $6\sin(3x+1)\cos(3x+1)$

3. (b) $\dfrac{-\sqrt{x}}{2x^2}$

4. (a) -5

   (b) $5x + y + 3 = 0$

5. (a) $x = -1$ or 3

   (b) $x \geq 1$

6. $\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}, -\dfrac{2\pi}{3}$

7. (a) $2 - p$, p

   (b) -1

8. (b) (i) $\sqrt{5} - 1$

   (ii) $48°$

Section B

Q.9 (a) (i) $\overrightarrow{OD} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$

(ii) $\overrightarrow{OB} = \dfrac{\overrightarrow{OA}+2\overrightarrow{OC}}{1+2}$

$\mathbf{b} = \dfrac{\mathbf{a}+2\overrightarrow{OC}}{3}$

$\overrightarrow{OC} = -\dfrac{1}{2}\mathbf{a}+\dfrac{3}{2}\mathbf{b}$

(iii) $\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$

$= 2\overrightarrow{OD} - k\overrightarrow{OB}$

$= 2(\dfrac{\mathbf{a}+\mathbf{b}}{2}) - k\mathbf{b}$

$= \mathbf{a} + (1-k)\,\mathbf{b}$

(b) (i) $\mathbf{a}\cdot\mathbf{b} = |\,\mathbf{a}\,||\,\mathbf{b}\,|\cos\angle AOB$

$= 3(2)\cos 60^\circ$

$= 3$

$\mathbf{b}\cdot\mathbf{b} = |\,\mathbf{b}\,|^2 = 4$

(ii) (1) $\overrightarrow{OE}\cdot\overrightarrow{EF} = 0$

$k\mathbf{b}\cdot[\,\mathbf{a}+(1-k)\,\mathbf{b}\,] = 0$

$k\mathbf{a}\cdot\mathbf{b} + k(1-k)\,\mathbf{b}\cdot\mathbf{b} = 0$

$3k + 4k(1-k) = 0$

$7k - 4k^2 = 0$

$k = 0$ (rejected) or $k = \dfrac{7}{4}$

$\therefore\ k = \dfrac{7}{4}$.

(2) Put $k=\frac{7}{4}$:

$\overrightarrow{EF}=\mathbf{a}+(1-\frac{7}{4})\mathbf{b}=\mathbf{a}-\frac{3}{4}\mathbf{b}$

$\overrightarrow{CE}=\overrightarrow{OE}-\overrightarrow{OC}$

$=\frac{7}{4}\mathbf{b}-(-\frac{1}{2}\mathbf{a}+\frac{3}{2}\mathbf{b})$

$=\frac{1}{2}\mathbf{a}+\frac{1}{4}\mathbf{b}$

Since $\overrightarrow{CE}\neq\mu\,\overrightarrow{EF}$, C, E, F are not collinear. The student is incorrect.

Q.10 (a) (i) Put $x=0$, $y=\frac{7}{2}$ $\therefore$ the y-intercept is $\frac{7}{2}$.

Put $y=0$, $x=\frac{7}{4}$ $\therefore$ the x-intercept is $\frac{7}{4}$.

(ii) $f(x)$ is decreasing when $f'(x)\le 0$.

$$f'(x)=\frac{-4(x^2+2)-(7-4x)2x}{(x^2+2)^2}$$

$$=\frac{4x^2-14x-8}{(x^2+2)^2}$$

$$\frac{4x^2-14x-8}{(x^2+2)^2}\le 0$$

$$(2x+1)(x-4)\le 0$$

$$-\frac{1}{2}\le x\le 4$$

(iii) $f(x)$ is increasing when $f'(x)\ge 0$,

i.e. $x\ge 4$ or $x\le -\frac{1}{2}$.

$f'(x)=0$ when $x=4$ or $-\frac{1}{2}$.

As $f'(x)$ changes from positive to negative as

x increases through $-\frac{1}{2}$, so $f(x)$ attains a

maximum at $x=-\frac{1}{2}$.

At $x=-\frac{1}{2}$, $y=4$

$\therefore$ the maximum value of $f(x)$ is 4.

As $f'(x)$ changes from negative to positive as

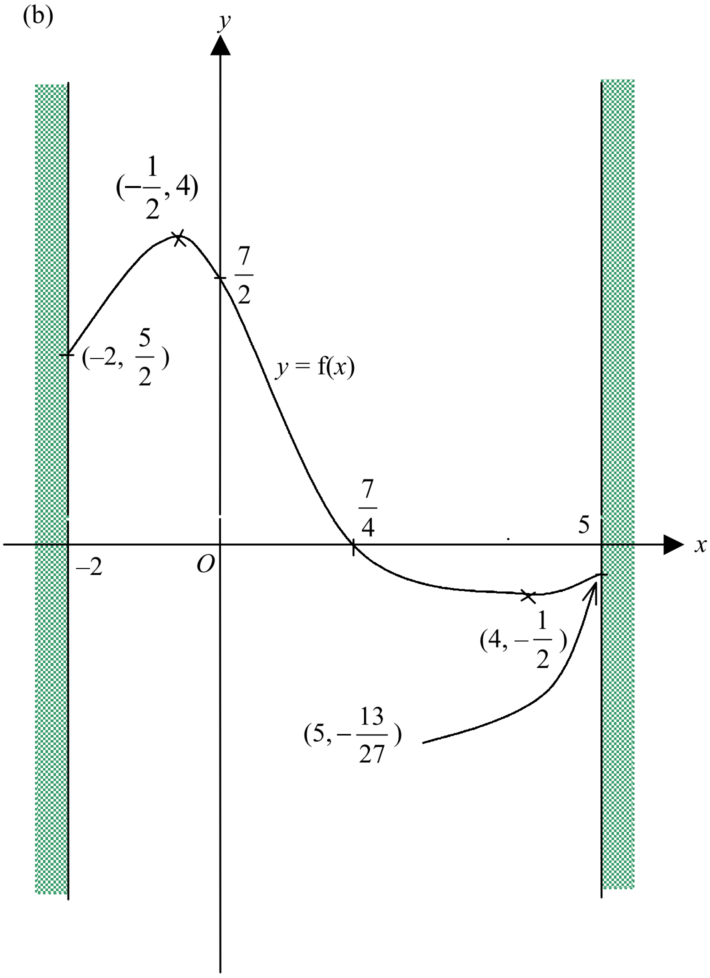
x increases through 4, so $f(x)$ attains a

minimum at $x=4$.

At $x=4$, $y=-\frac{1}{2}$

$\therefore$ the minimum value of $f(x)$ is $-\frac{1}{2}$.

(b)



(c) Put $x = \sin\theta$, $f(\sin\theta) = \dfrac{7 - 4\sin\theta}{\sin^2\theta + 2} = p$.

The range of possible value of $\sin\theta$ is $-1 \le \sin\theta \le 1$.
From the graph in (b), the greatest value of f(x) in the range $-1 \le x \le 1$ is 4.

$\therefore$ the greatest value of p is 4 and the student is correct.

From the graph in (b), f(x) attains its least value at one of the end-points.

$f(1) = 1$, $f(-1) = \dfrac{11}{3}$.

$\therefore$ the least value of p is 1 and the student is incorrect.

Q.11 (a)

$w = \cos\theta + i\sin\theta$

$w^2 = \cos 2\theta + i\sin 2\theta$

$\frac{1}{w} = \frac{1}{\cos\theta + i\sin\theta}$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

$w^2 + \frac{5}{w} - 2$

$= \cos 2\theta + i\sin 2\theta + 5(\cos\theta - i\sin\theta) - 2$

$= \cos 2\theta + 5\cos\theta - 2 + i(\sin 2\theta - 5\sin\theta)$

Since $w^2 + \frac{5}{w} - 2$ is purely imaginary,

$\cos 2\theta + 5\cos\theta - 2 = 0$

$(2\cos^2\theta - 1) + 5\cos\theta - 2 = 0$

$2\cos^2\theta + 5\cos\theta - 3 = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -3$ (rejected)

$\theta = \frac{\pi}{3}$ $(\because\ 0 < \theta < \pi)$

Imaginary part

$= \sin\frac{2\pi}{3} - 5\sin\frac{\pi}{3} \neq 0$

$\therefore w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

(b) (i) $\left|\frac{z_2}{z_1}\right| = |w|$

$= 1$

$\arg(\frac{z_2}{z_1}) = \arg(w)$

$= \frac{\pi}{3}$

(ii) $\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} = 1$

$\therefore |z_2| = |z_1|$

i.e. $OA = OB$.

$\angle AOB = \arg(z_2) - \arg(z_1)$

$= \arg(\frac{z_2}{z_1})$

$= \frac{\pi}{3}$

Since $OA = OB$, ΔOAB is isosceles.

$\angle OAB = \angle OBA = \frac{1}{2}(\pi - \frac{\pi}{3}) = \frac{\pi}{3}$

$\therefore$ ΔOAB is equilateral.

Q.12 (a) $f(x) = x^2 - 4mx - (5m^2 - 6m + 1)$

Discriminant $\Delta = (-4m)^2 + 4(5m^2 - 6m + 1)$

$= 36m^2 - 24m + 4$

$= 4(9m^2 - 6m + 1)$

$= 4(3m-1)^2 > 0 \quad (\because\ m > \frac{1}{3})$

$\therefore$ the equation $f(x) = 0$ has distinct real roots.

(b) (i) $x = \dfrac{4m \pm \sqrt{\Delta}}{2}$

$= 2m \pm (3m - 1)$

Since $\alpha < \beta$,

$\alpha = 2m - (3m-1) = -m + 1$

$\beta = 2m + (3m-1) = 5m - 1$

(ii) (1) Since $4 < \beta < 5$,

$4 < 5m - 1 < 5$

$5 < 5m < 6$

$1 < m < \frac{6}{5}$

(2) Sketch *A* :

Since the coefficient of x^2 in f(x) is positive, the graph of y = f(x) should open upwards. However, the graph in sketch *A* opens downwards, so sketch *A* is incorrect.

Sketch *B* :

Since $\alpha = 1 - m$ and $1 < m < \frac{6}{5}$,

$1 - 1 > 1 - m > 1 - \frac{6}{5}$

$0 > \alpha > -\frac{1}{5}$

In sketch *B*, α is less than -1, so sketch *B* is incorrect.

Sketch C :

$$\begin{cases} y = x^2 - 4mx - (5m^2 - 6m + 1) \\ y = -1 \end{cases}$$

$-1 = x^2 - 4mx - (5m^2 - 6m + 1)$

$x^2 - 4mx - (5m^2 - 6m) = 0$ ---- (*)

Discriminant $\Delta = (-4m)^2 + 4(5m^2 - 6m)$

$= 36m^2 - 24m$

$= 12m(3m - 2)$

Since $1 < m < \frac{6}{5}$, $\Delta > 0$.

As $\Delta > 0$, equation (*) has real roots,
i.e. $y = f(x)$ and $y = -1$ always have intersecting points. However, the line and the graph in sketch C do not intersect, so sketch C is incorrect.

Q.13 (a) $\tan \angle ARQ = \dfrac{100-x}{100}$

$\tan\theta = \tan(\angle ARQ + \angle QRB)$

$= \dfrac{\tan \angle ARQ + \tan \angle QRB}{1-(\tan \angle ARQ)(\tan \angle QRB)}$

$= \dfrac{\dfrac{100-x}{100} + \dfrac{y}{100}}{1-(\dfrac{100-x}{100})(\dfrac{y}{100})}$

$= \dfrac{100(100-x+y)}{10000-100y+xy}$

(b) (i) At $t=0$, $\tan\theta = \dfrac{PQ}{RQ}$

$= \dfrac{100}{100} = 1.$

Since $\angle ARB$ remains unchanged,

$\dfrac{100(100-x+y)}{10000-100y+xy} = 1$

$10000-100x+100y = 10000-100y+xy$

$200y - xy = 100x$

$y = \dfrac{100x}{200-x}$

(ii) $\dfrac{dy}{dt} = \dfrac{(200-x)(100)-100x(-1)}{(200-x)^2}\dfrac{dx}{dt}$

$= \dfrac{20000}{(200-x)^2}\dfrac{dx}{dt}$

$= \dfrac{40000}{(200-x)^2}$

At $t=40$, $x = 40\times 2 = 80$.

$\dfrac{dy}{dt} = \dfrac{40000}{(200-80)^2}$

$= \dfrac{25}{9}$

$\therefore$ the speed of boat B at $t=40$ is $\dfrac{25}{9}$ m s^{-1}.

(iii) From (ii), $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{40000}{(200-x)^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le 3$$

$$\frac{40000}{(200-x)^2} \le 3$$

$$200 - x \ge \frac{200}{\sqrt{3}}$$

$$x \le 200(1 - \frac{1}{\sqrt{3}})$$

When $x > 200(1 - \frac{1}{\sqrt{3}})$, $\frac{\mathrm{d}y}{\mathrm{d}t} > 3$.

So it is impossible to keep $\angle ARB$ unchanged before boat A reaches Q.