99-CE
A MATHS
PAPER 2

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1999

ADDITIONAL MATHEMATICS PAPER 2

11.15 am – 1.15 pm (2 hours)

This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.
2. All working must be clearly shown.
3. Unless otherwise specified, numerical answers must be **exact**.
4. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$.

(3 marks)

2. Find $\int x(x+2)^{99} \, dx$.

(4 marks)

3.

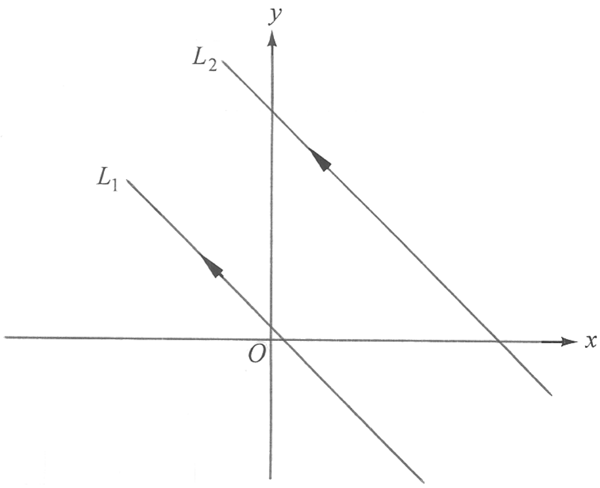

Figure 1

Figure 1 shows two parallel lines $L_1 : 2x + 2y - 1 = 0$ and $L_2 : 2x + 2y - 13 = 0$.

(a) Find the y-intercept of L_1.

(b) Find the distance between L_1 and L_2.

(c) L_3 is another line parallel to L_1. If the distance between L_1 and L_3 is equal to that between L_1 and L_2, find the equation of L_3.

(5 marks)

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4.

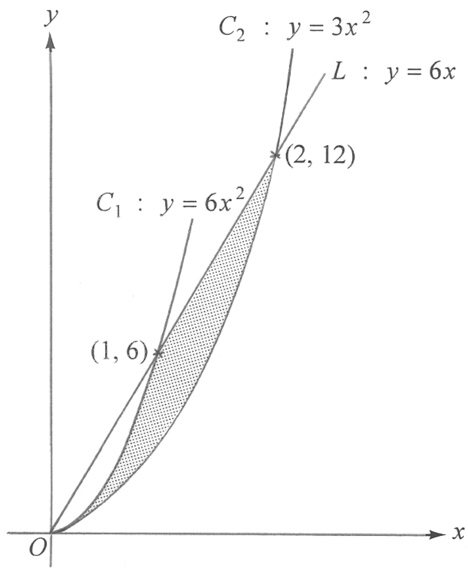

Figure 2

In Figure 2, the line $L : y = 6x$ and the curves $C_1 : y = 6x^2$ and $C_2 : y = 3x^2$ all pass through the origin. L also intersects C_1 and C_2 at the points $(1, 6)$ and $(2, 12)$ respectively. Find the area of the shaded region.

(5 marks)

5. A family of straight lines is given by the equation

$$y - 3 + k(x - y + 1) = 0 ,$$

where k is real.

(a) Find the equation of a line L_1 in the family whose x-intercept is 5.

(b) Find the equation of a line L_2 in the family which is parallel to the x-axis.

(c) Find the acute angle between L_1 and L_2.

(6 marks)

6. The slope at any point $(x,\ y)$ of a curve is given by

$$\frac{dy}{dx} = 3x^2 - 2x + k.$$

If the curve **touches** the x-axis at the point (2, 0), find

(a) the value of k,

(b) the equation of the curve. (6 marks)

7. (a) Expand $(1+2x)^n$ in ascending powers of x up to the term x^3, where n is a positive integer.

(b) In the expansion of $(x - \frac{3}{x})^2 (1+2x)^n$, the constant term is 210. Find the value of n.

(6 marks)

8. (a) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(b) Find the general solution of the equation

$$\cos 6x + 4\cos 2x = 0.$$

(7 marks)

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Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9.

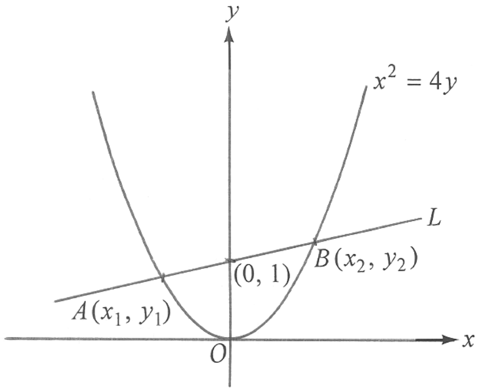

Figure 3

L is a straight line of slope m and passes through the point (0, 1). The line L cuts the parabola $x^2 = 4y$ at two points $A(x_1, y_1)$ and $B(x_2, y_2)$ as shown in Figure 3.

(a) Show that x_1 and x_2 are the roots of the equation

$$x^2 - 4mx - 4 = 0.$$ (3 marks)

(b) Find $(x_1 - x_2)^2$ in terms of m.

Hence, or otherwise, show that $AB = 4(1 + m^2)$. (6 marks)

(c) C is a circle with AB as a diameter.

(i) Find, in terms of m, the coordinates of the centre of C and its radius.

(ii) Find, in terms of m, the distance from the centre of C to the line $y + 1 = 0$.

State the geometrical relationship between C and the line $y + 1 = 0$. Explain your answer.

(7 marks)

10. $A(-3, 0)$ and $B(-1, 0)$ are two points and $P(x, y)$ is a variable point such that $PA = \sqrt{3}PB$. Let C be the locus of P.

(a) Show that the equation of C is $x^2 + y^2 = 3$. (3 marks)

(b) $T(a, b)$ is a point on C. Find the equation of the tangent to C at T. (2 marks)

(c) The tangent from A to C touches C at a point S in the second quadrant. Find the coordinates of S. (3 marks)

(d)

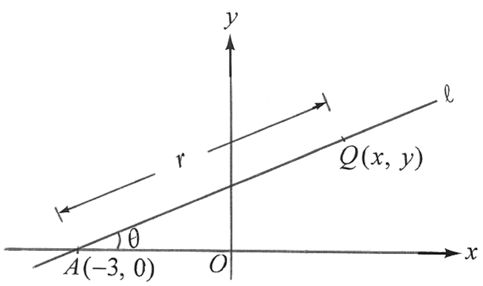

Figure 4

ℓ is a straight line which passes through point A and makes an angle θ with the positive x-axis, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. $Q(x, y)$ is a point on ℓ such that $AQ = r$. (See Figure 4.)

(i) Write down the coordinates of Q in terms of r and θ.

(ii) ℓ cuts C at two distinct points H and K. Let $AH = r_1$, $AK = r_2$.

(1) Show that r_1 and r_2 are the roots of the quadratic equation $r^2 - 6r\cos\theta + 6 = 0$.

(2) Find the range of possible values of θ, giving your answers correct to three significant figures. (8 marks)

11.

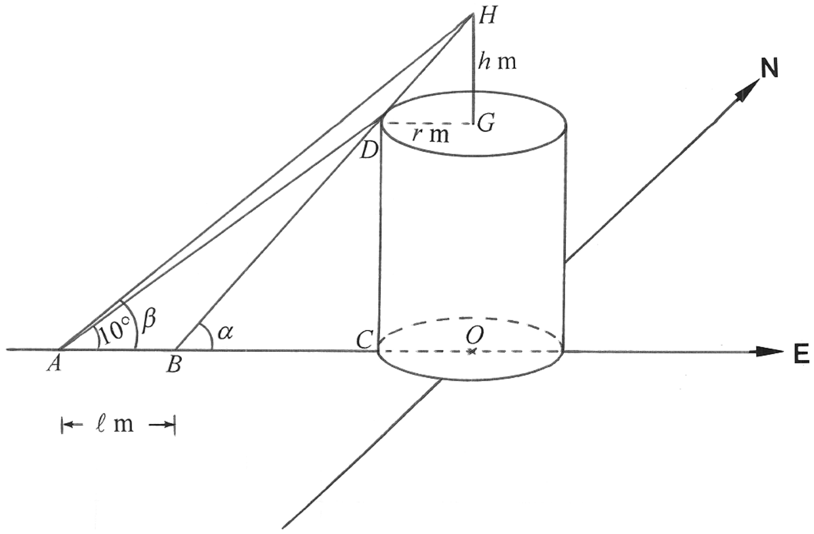

Figure 5

Figure 5 shows a right cylindrical tower with a radius of r m standing on horizontal ground. A vertical pole HG, h m in height, stands at the centre G of the roof of the tower. Let O be the centre of the base of the tower. C is a point on the circumference of the base of the tower due west of O and D is a point on the roof vertically above C. A man stands at a point A due west of O. The angles of elevation of D and H from A are $10°$ and β respectively. The man walks towards the east to a point B where he can just see the top of the pole H as shown in Figure 5. (Note : If he moves forward, he can no longer see the pole.) The angle of elevation of H from B is α. Let $AB = \ell$ m.

(a) Show that $AD = \dfrac{\ell \sin \alpha}{\sin(\alpha - 10°)}$ m.

Hence (i) express CD in terms of ℓ and α,

(ii) show that $h = \dfrac{\ell \sin^2 \alpha \sin(\beta - 10°)}{\sin(\alpha - 10°)\sin(\alpha - \beta)}$.

(Hint : You may consider ΔADH.)

(6 marks)

(b) **In this part, numerical answers should be given correct to two significant figures.**

Suppose $\alpha = 15°$, $\beta = 10.2°$ and $\ell = 97$.

(i) Find

(1) the height of the pole HG,

(2) the height and radius of the tower.

(ii) P is a point south-west of O. Another man standing at P can just see the top of the pole H. Find

(1) the distance of P from O,

(2) the bearing of B from P.

(10 marks)

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12. (a) Prove, by mathematical induction, that

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta},$$

where $\sin\theta \neq 0$, for all positive integers n.

(6 marks)

(b) Using (a) and the substitution $\theta = \frac{\pi}{2} - x$, or otherwise, show that

$$\sin x - \sin 3x + \sin 5x = \frac{\sin 6x}{2\cos x},$$

where $\cos x \neq 0$.

(2 marks)

(c) Using (a) and (b), evaluate

$$\int_{0.1}^{0.5} \left(\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} \right)^2 \mathrm{d}x,$$

giving your answer correct to two significant figures.

(4 marks)

(d) Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x)\,\mathrm{d}x 。$$

(4 marks)

13.

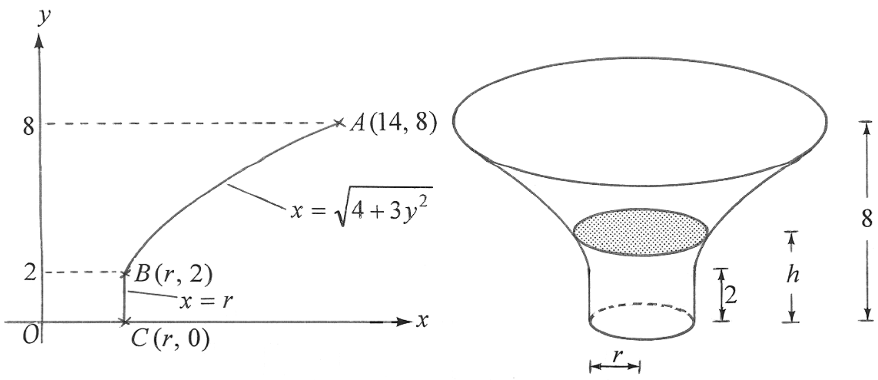

Figure 6(a) **Figure 6(b)**

A curve passes through three points $A(14, 8)$, $B(r, 2)$ and $C(r, 0)$ as shown in Figure 6(a). The curve consists of two parts. The equation of the part joining A and B is $x=\sqrt{4+3y^2}$ and the part joining B and C is the vertical line $x=r$.

(a) Find the value of r.

(2 marks)

(b) A pot, 8 units in height, is formed by revolving the curve and the line segment OC about the y-axis, where O is the origin. (See Figure 6(b).) If the pot contains water to a depth of h units, where $h>2$, show that the volume of water V in the pot is $(h^3+4h+16)\pi$ cubic units.

(7 marks)

(c) Initially, the pot in (b) contains water to a depth greater than 3 units. The water is now pumped out at a constant rate of 2π cubic units per second. Find the rate of change of the depth of the water in the pot with respect to time when

(i) the depth of the water is 3 units, and

(ii) the depth of the water is 1 unit.

(7 marks)

END OF PAPER

Outlines of Solutions

1999 Additional Mathematics

Paper 2

Section A

1. $\frac{\pi}{4}$

2. $\frac{(x+2)^{101}}{101} - \frac{(x+2)^{100}}{50} + c$, where c is a constant

3. (a) $\frac{1}{2}$

   (b) $3\sqrt{2}$

   (c) $2x + 2y + 11 = 0$

4. 3

5. (a) $x + y - 5 = 0$

   (b) $y - 3 = 0$

   (c) $\frac{\pi}{4}$

6. (a) –8

   (b) $y = x^3 - x^2 - 8x + 12$

7. (a) $1 + 2{}_nC_1x + 4{}_nC_2x^2 + 8{}_nC_3x^3 + \ldots$

   (b) 4

8. (b) $k\pi \pm \frac{\pi}{4}$, where k is an integer

Section B

Q.9 (a) The equation of L is $y = mx + 1$.

Substitute $y = mx + 1$ into $x^2 = 4y$:

$x^2 = 4(mx + 1)$

$x^2 - 4mx - 4 = 0$

$\therefore x_1, x_2$ are the roots of the equation $x^2 - 4mx - 4 = 0$.

(b) $\begin{cases} x_1 + x_2 = 4m \\ x_1 x_2 = -4 \end{cases}$

$$\begin{aligned}(x_1 - x_2)^2 &= (x_1 + x_2)^2 - 4x_1x_2 \\ &= (4m)^2 - 4(-4) \\ &= 16(m^2 + 1)\end{aligned}$$

$$\begin{aligned}AB^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (x_1 - x_2)^2 + (mx_1 + 1 - mx_2 - 1)^2 \\ &= (x_1 - x_2)^2 + (mx_1 - mx_2)^2 \\ &= (1 + m^2)[16(m^2 + 1)]\end{aligned}$$

$AB = 4(1 + m^2)$

(c) (i) x-coordinate of centre of $C = \dfrac{x_1 + x_2}{2} = 2m$

y-coordinate of centre of $C = \dfrac{y_1 + y_2}{2} = \dfrac{mx_1 + 1 + mx_2 + 1}{2}$

$= \dfrac{m}{2}(4m) + 1 = 2m^2 + 1$

$\therefore$ the coordinates of the centre are $(2m, 2m^2 + 1)$.

Radius of $C = \dfrac{AB}{2} = 2(1 + m^2)$

(ii) Distance from centre of C to $y + 1 = 0$

$= |2m^2 + 1 - (-1)|$

$= 2(m^2 + 1)$

As the distance from centre of C to $y + 1 = 0$ is equal to the radius C, the line $y + 1 = 0$ is a tangent to C.

Q.10 (a) $PA = \sqrt{3}PB$

$\sqrt{(x+3)^2 + y^2} = \sqrt{3}\sqrt{(x+1)^2 + y^2}$

$x^2 + 6x + 9 + y^2 = 3(x^2 + 2x + 1 + y^2)$

$x^2 + y^2 = 3$

(b) Differentiate $x^2 + y^2 = 3$ with respect to x :

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

Equation of tangent at $T(a, b)$ is

$\frac{y-b}{x-a} = \frac{-a}{b}$

$ax + by = a^2 + b^2$

(c) Substitute $A(-3, 0)$ into the equation of tangent :

$a(-3) + b(0) = 3$

$a = -1$

$b = \sqrt{3-(-1)^2}$ (Θ S lies in the 2nd quadrant.)

$= \sqrt{2}$

$\therefore$ the coordinates of S are $(-1, \sqrt{2})$.

(d) (i) The coordinates of Q are $(-3 + r\cos\theta, r\sin\theta)$.

(ii) (1) Substitute $(-3 + r\cos\theta, r\sin\theta)$ into C :

$(-3 + r\cos\theta)^2 + (r\sin\theta)^2 = 3$

$9 - 6r\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta = 3$

$r^2 - 6r\cos\theta + 6 = 0$ ---- (*)

Since $AH = r_1$, $AK = r_2$, r_1 and r_2 are the roots of (*).

(2) Since ℓ cuts C at two distinct points, (*) has two distinct real roots.

$$(6\cos\theta)^2 - 4(6) > 0$$

$$\cos^2\theta > \frac{2}{3}$$

$$\cos\theta > \sqrt{\frac{2}{3}} \quad \text{or} \quad \cos\theta < -\sqrt{\frac{2}{3}} \quad \text{(rejected)}$$

$\therefore -0.615 < \theta < 0.615$ (correct to 3 sig. figures)

Q.11 (a) Consider ΔABD :

By Sine Law,

$$\frac{AD}{\sin\angle ABD}=\frac{\ell}{\sin\angle ADB}$$

$$\frac{AD}{\sin(180°-\alpha)}=\frac{\ell}{\sin(\alpha-10°)}$$

$$AD=\frac{\ell\sin\alpha}{\sin(\alpha-10°)}\text{ m}$$

(i) Consider ΔACD :

$$CD=AD\sin 10°$$
$$=\frac{\ell\sin\alpha\sin 10°}{\sin(\alpha-10°)}\text{ m}$$

(ii) Consider ΔADH :

$$\frac{AD}{\sin(\alpha-\beta)}=\frac{DH}{\sin(\beta-10°)}$$

$$DH=AD\frac{\sin(\beta-10°)}{\sin(\alpha-\beta)}=\frac{\ell\sin\alpha\sin(\beta-10°)}{\sin(\alpha-10°)\sin(\alpha-\beta)}$$

Consider ΔDHG :

$$h=DH\sin\alpha$$
$$=\frac{\ell\sin^2\alpha\sin(\beta-10°)}{\sin(\alpha-10°)\sin(\alpha-\beta)}$$

(b) (i) (1) Using (a) (ii) :

$$\text{height of pole }=\frac{97\sin^2 15°\sin(10.2°-10°)}{\sin(15°-10°)\sin(15°-10.2°)}$$
$$=3.1100=3.1\text{ m (correct to 2 sig. fig.)}$$

(2) Using (a) (i) :

$$\text{height of tower }\ CD=\frac{97\sin 15°\sin 10°}{\sin(15°-10°)}$$
$$=50.020=50\text{ m (correct to 2 sig. fig.)}$$

$$\text{radius of tower}=\frac{h}{\tan 15°}$$
$$=\frac{3.1100}{\tan 15°}$$
$$=11.607=12\text{ m (correct to 2 sig. fig.)}$$

(ii) (1) Consider ΔHPO :

$$\tan \angle HPO = \frac{OH}{OP}$$

$$OP = \frac{OH}{\tan 15°}$$

$$= \frac{3.1100 + 50.020}{\tan 15°}$$

$$= 198.28$$

= 200 m (correct to 2 sig. fig.)

(2) $\angle BPO = \frac{1}{2}(180° - 45°) \quad = 67.5°$

Bearing of B from P is N(67. 5° – 45°)W, i.e. N22. 5° W.

Q.12 (a) For $n=1$, LHS $=\cos\theta$.

$$\text{RHS}=\frac{\sin 2\theta}{2\sin\theta}$$
$$=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta=\text{LHS}.$$

$\therefore$ the statement is true for $n=1$.

Assume $\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2k-1)\theta=\dfrac{\sin 2k\theta}{2\sin\theta}$

for some positive integer k.

Then $\cos\theta+\cos 3\theta+\cos 5\theta+\cdots+\cos(2k-1)\theta+\cos[2(k+1)-1]\theta$

$$=\frac{\sin 2k\theta}{2\sin\theta}+\cos(2k+1)\theta$$
$$=\frac{\sin 2k\theta+2\sin\theta\cos(2k+1)\theta}{2\sin\theta}$$
$$=\frac{\sin 2k\theta+\sin(2k+2)\theta-\sin 2k\theta}{2\sin\theta}$$
$$=\frac{\sin 2(k+1)\theta}{2\sin\theta}$$

The statement is also true for $n=k+1$ if it is true for $n=k$.

By the principle of mathematical induction, the statement is true for all positive integers n.

(b) Using (a) : $\cos\theta+\cos 3\theta+\cos 5\theta=\dfrac{\sin 6\theta}{2\sin\theta}$, where $\sin\theta\neq 0$.

Put $\theta=\dfrac{\pi}{2}-x$:

$$\cos(\frac{\pi}{2}-x)+\cos 3(\frac{\pi}{2}-x)+\cos 5(\frac{\pi}{2}-x)=\frac{\sin 6(\frac{\pi}{2}-x)}{2\sin(\frac{\pi}{2}-x)}$$

$$\cos(\frac{\pi}{2}-x)+\cos(\frac{3\pi}{2}-3x)+\cos(\frac{5\pi}{2}-5x)=\frac{\sin(3\pi-6x)}{2\sin(\frac{\pi}{2}-x)}$$

$$\sin x-\sin 3x+\sin 5x=\frac{\sin 6x}{2\cos x}\text{, where }\sin(\frac{\pi}{2}-x)=\cos x\neq 0.$$

(c) $\int_{0.1}^{0.5} (\frac{\sin x - \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x})^2 \, dx$

$= \int_{0.1}^{0.5} [\frac{\sin 6x}{2\cos x} / \frac{\sin 6x}{2\sin x}]^2 \, dx$

$= \int_{0.1}^{0.5} \tan^2 x \, dx$

$= \int_{0.1}^{0.5} (\sec^2 x - 1) \, dx$

$= [\tan x - x]_{0.1}^{0.5}$

$= 0.046$ (correct to 2 sig. fig.)

(d) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin x + 3\sin 3x + 5\sin 5x + 7\sin 7x + \cdots + 1999\sin 1999x) dx$

$= [-\cos x - \cos 3x - \cos 5x - \cos 7x - \cdots - \cos 1999x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= -\frac{1}{2}[\frac{\sin 2000x}{\sin x}]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{1}{2}$

Q.13 (a) Substitute $(r, 2)$ into $x=\sqrt{4+3y^2}$:

$r=\sqrt{4+3(2)^2}=4$

(b) V = Volume of lower cylindrical part + volume of upper part

Volume of lower cylindrical part $=\pi r^2 h$

$=\pi(4)^2(2)$

$=32\pi$

Volume of upper part $=\pi\int_2^h x^2\,\mathrm{d}y$

$=\pi\int_2^h (4+3y^2)\,\mathrm{d}y$

$=\pi[4y+y^3]_2^h$

$=(h^3+4h-16)\pi$

$\therefore\ V=32\pi+(h^3+4h-16)\pi$

$=(h^3+4h+16)\pi$ cubic units

(c) (i) Let h units be the depth of water at time t.

$\frac{\mathrm{d}V}{\mathrm{d}t}=\pi(3h^2+4)\frac{\mathrm{d}h}{\mathrm{d}t}$

Put $\frac{\mathrm{d}V}{\mathrm{d}t}=-2\pi$ and $h=3$:

$-2\pi=\pi[3(3)^2+4]\frac{\mathrm{d}h}{\mathrm{d}t}$

$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-2}{31}$ units per sec.

$\therefore$ the depth decreases at a rate $\frac{2}{31}$ units per sec.

(ii) When $h = 1$, the water remained is in the cylindrical part only.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\text{base area of cylilnder}}$$

$$= \frac{-2\pi}{\pi(4)^2}$$

$$= -\frac{1}{8} \text{ units per sec.}$$

$\therefore$ the depth decreases at a rate $\frac{1}{8}$ units per sec.