99-CE
A MATHS
PAPER 1

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1999

ADDITIONAL MATHEMATICS PAPER 1

8.30 am – 10.30 am (2 hours)
This paper must be answered in English

1. Answer **ALL** questions in Section A and any **THREE** questions in Section B.

2. All working must be clearly shown.

3. Unless otherwise specified, numerical answers must be **exact**.

4. In this paper, vectors may be represented by bold-type letters such as **u**, but candidates are expected to use appropriate symbols such as $\vec{u}$ in their working.

5. The diagrams in the paper are not necessarily drawn to scale.

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)
Answer **ALL** questions in this section.

1. Find (a) $\dfrac{d}{dx}\sin(x^2+1)$,

(b) $\dfrac{d}{dx}[\dfrac{\sin(x^2+1)}{x}]$.

(4 marks)

2. Solve the inequality $\dfrac{x}{x-1}>2$.

(4 marks)

3. Solve $|x-3|=|x^2-4x+3|$.

(5 marks)

4. Let $f(x)=2x^2+2(k-4)x+k$, where k is real.

(a) Find the discriminant of the equation $f(x)=0$.

(b) If the graph of $y=f(x)$ lies above the x-axis for all values of x, find the range of possible values of k.

(5 marks)

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5. Express $1+i$ in polar form.

Hence find the three cube roots of $1+i$, giving your answers in polar form.

(5 marks)

6. The point $P(a, a)$ is on the curve $3x^2 - xy - y^2 - a^2 = 0$, where a is a non-zero constant.

(a) Find the value of $\frac{dy}{dx}$ at P.

(b) Find the equation of the tangent to the curve at P.

(6 marks)

7. Let **a**, **b** be two vectors such that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $|\mathbf{b}| = 4$. The angle between **a** and **b** is 60°.

(a) Find $|\mathbf{a}|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If the vector $(m\mathbf{a} + \mathbf{b})$ is perpendicular to **b**, find the value of m.

(6 marks)

8.

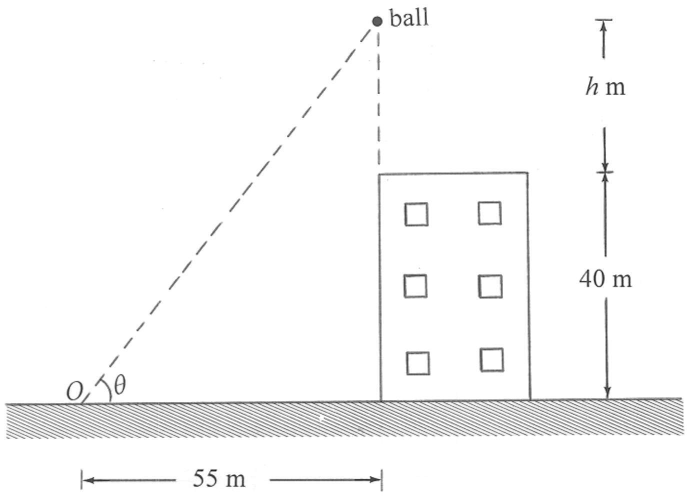

Figure 1

A ball is thrown vertically upwards from the roof of a building 40 metres in height. After t seconds, the height of the ball above the roof is h metres, where $h = 20t - 5t^2$. At this instant, the angle of elevation of the ball from a point O, which is at a horizontal distance of 55 metres from the building, is θ. (See Figure 1.)

(a) Find (i) $\tan\theta$ in terms of t,

(ii) the value of θ when $t = 3$.

(b) Find the rate of change of θ with respect to time when $t = 3$.

(7 marks)

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Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9. Let $f(x) = a\sin 2x + b\cos x$, where $0 \le x \le \pi$ and a, b are constants.
Figure 2 (a) shows the graph of $y = f'(x)$.

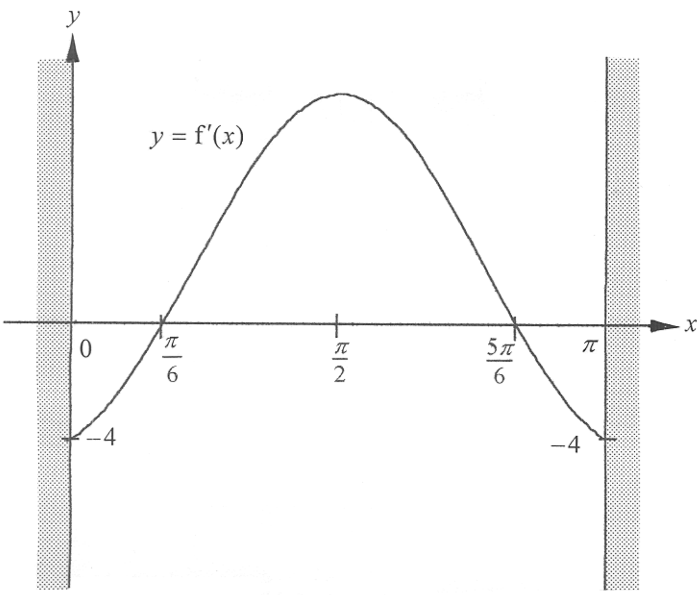

Figure 2(a)

(a) (i) Find $f'(x)$ in terms of a, b and x.

(ii) Using Figure 2(a), show that $a = -2$ and $b = -4$.
(4 marks)

(b) (i) Find the x- and y-intercepts of the curve $y = f(x)$.

(ii) Find the maximum and minimum points of the curve $y = f(x)$.
(7 marks)

(c) In Figure 2 (b) , sketch the curve $y = f(x)$. (3 marks)

(d) Let $g(x) = |\, a\sin 2x + b\cos x - 6\,|$, where $0 \le x \le \pi$. Using the result of (c), write down the range of possible values of $g(x)$.
(2 marks)

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If you attempt Question 9, fill in the details in the first three boxes above and tie this sheet into your answer book.

9. (c) **(continued)**

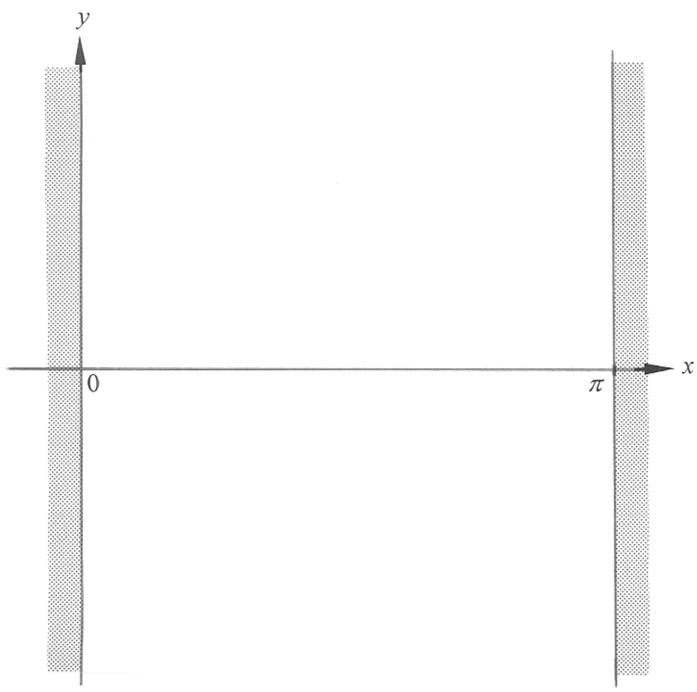

Figure 2(b)

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10.

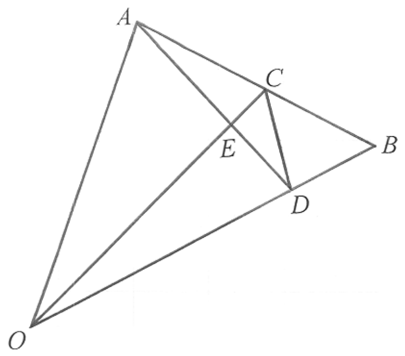


Figure 3

In Figure 3, OAB is a triangle. C and D are points on AB and OB respectively such that $AC : CB = 8 : 7$ and $OD : DB = 16 : 5$. OC and AD intersect at a point E. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{AD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(4 marks)

(b) Let $\overrightarrow{OE} = r\overrightarrow{OC}$ and $\overrightarrow{AE} = k\overrightarrow{AD}$.

(i) Express $\overrightarrow{OE}$ in terms of r, $\mathbf{a}$ and $\mathbf{b}$.

(ii) Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

Hence show that $r = \frac{6}{7}$ and $k = \frac{3}{5}$.

(6 marks)

(c) It is given that $EC : ED = 1 : 2$.

(i) Using (b), or otherwise, find $EA : EO$.

(ii) Explain why $OACD$ is a cyclic quadrilateral.

(6 marks)

11. Figure 4 shows a parallelogram $OABC$ in an Argand diagram. $OA = 2$ and OA makes an angle $60°$ with the positive real axis. Let z_1 , z_2 and z_3 be the complex numbers represented by vertices A, B and C respectively. It is given that $z_3 = (\sqrt{3}i)\, z_1$.

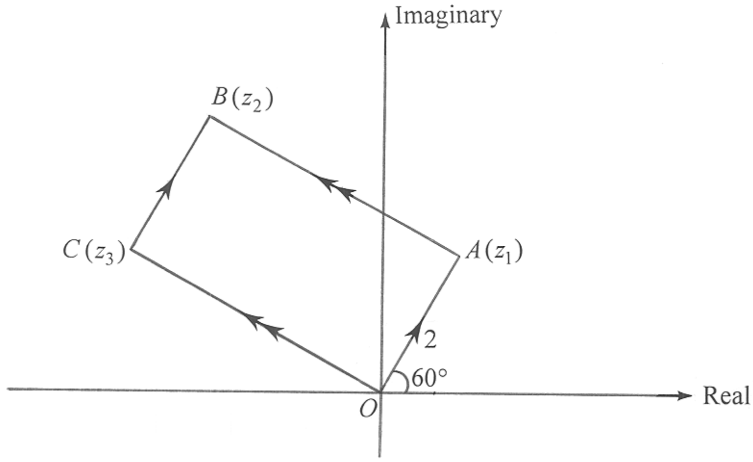


Figure 4

(a) Find z_1 and z_3 in standard form.

(3 marks)

(b) Show that $\dfrac{z_2}{z_1} = 1 + \sqrt{3}i$.

Hence, or otherwise, find $\angle AOB$.

(5 marks)

(c) Let $w = \cos\theta + i\sin\theta$, where $0° \le \theta < 360°$. Point E is a point in the Argand diagram representing the complex number wz_3 . Find the value(s) of θ in each of the following cases :

(i) E represents the complex number $\bar{z}_3$.

(ii) Points E, O and A lie on the same straight line.

(8 marks)

12.

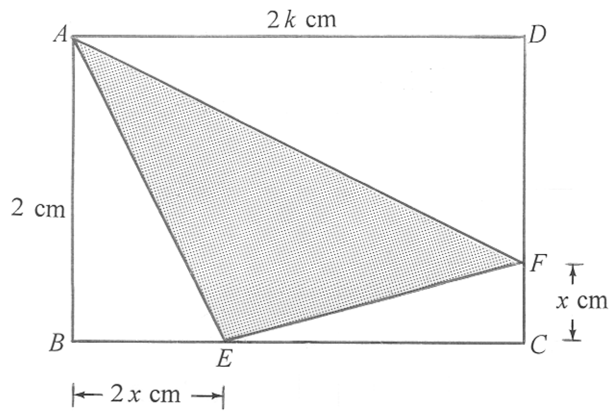


Figure 5

Figure 5 shows a rectangle $ABCD$ with $AB = 2$ cm and $AD = 2k$ cm, where k is a positive number. E and F are two variable points on the sides BC and CD respectively such that $CF = x$ cm and $BE = 2x$ cm, where x is a non-negative number. Let S cm^2 denote the area of ΔAEF.

(a) Show that $S = x^2 - 2x + 2k$. (3 marks)

(b) Suppose $k = \frac{3}{2}$.

(i) By considering that points E and F lie on the sides BC and CD respectively, show that $0 \le x \le \frac{3}{2}$.

(ii) Find the least value of S and the corresponding value of x.

(iii) Find the greatest value of S. (9 marks)

(c) Suppose $k = \frac{3}{8}$. A student says that S is least when $x = 1$.

(i) Explain whether the student is correct.

(ii) Find the least value of S. (4 marks)

13.

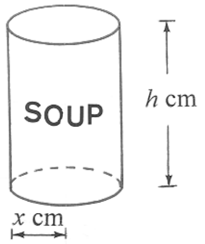


Figure 6

A food company produces cans of instant soup. Each can is in the form of a right cylinder with a base radius of x cm and a height of h cm (see Figure 6) and its capacity is V cm^3, where V is constant. The cans are made of thin metal sheets. The cost of the curved surface of the can is 1 cent per cm^2 and the cost of the plane surfaces is k cents per cm^2. Let C cents be the production cost of one can. For economic reasons, the value of C is minimised.

(a) Express h in terms of π, x and V.

Hence show that $C = \frac{2V}{x} + 2\pi k x^2$. (3 marks)

(b) If $\frac{dC}{dx} = 0$, express x^3 in terms of π, k and V.

Hence show that C is a minimum when $\frac{x}{h} = \frac{1}{2k}$. (6 marks)

(c) Suppose $k = 2$ and $V = 256\pi$.

(i) Find the values of x and h.

(ii) If the value of k increases, how would the dimensions of the can be affected ? Explain your answer. (5 marks)

(d) The company intends to produce a bigger can of capacity $2V$ cm^3, which is also in the form of a right cylinder. Suppose the costs of the curved surface and plane surfaces of the bigger can are maintained at 1 cent and k cents per cm^2 respectively. A worker suggests that the ratio of base radius to height of the bigger can should be twice that of the smaller can in order to minimize the production cost. Explain whether the worker is correct. (2 marks)

END OF PAPER

Outlines of Solutions

1999 Additional Mathematics

Paper 1

Section A

1. (a) $2x\cos(x^2+1)$

   (b) $2\cos(x^2+1)-\dfrac{1}{x^2}\sin(x^2+1)$

2. $1<x<2$

3. 0, 2 or 3

4. (a) $4k^2-40k+64$

   (b) $2<k<8$

5. $\sqrt{2}(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4})$

   $2^{\frac{1}{6}}\left[\cos\left(\dfrac{2k\pi}{3}+\dfrac{\pi}{12}\right)+i\sin\left(\dfrac{2k\pi}{3}+\dfrac{\pi}{12}\right)\right]$, $k=-1, 0, 1$

6. (a) $\dfrac{5}{3}$

   (b) $5x-3y-2a=0$

7. (a) 5

   (b) 10

   (c) –1.6

8. (a) (i) $\tan\theta=\dfrac{4t-t^2+8}{11}$

   (ii) $\dfrac{\pi}{4}$

   (b) $-\dfrac{1}{11}\text{s}^{-1}$

Section B

Q.9 (a) (i) $f'(x) = 2a\cos 2x - b\sin x$

(ii) From figure 2 (a), $f'(0) = -4$ and $f'(\frac{\pi}{6}) = 0$

$2a\cos 0 - b\sin 0 = -4$
$a = -2$

$2(-2)\cos\frac{\pi}{3} - b\sin\frac{\pi}{6} = 0$

$b = -4$

(b) (i) $f(0) = -4$ $\therefore$ the y-intercept is -4.

Put $f(x) = 0$: $-2\sin 2x - 4\cos x = 0$
$-4\sin x\cos x - 4\cos x = 0$
$-4\cos x(1 + \sin x) = 0$
$\cos x = 0$ or $\sin x = -1$ (rejected)
$x = \frac{\pi}{2}$

$\therefore$ the x-intercept is $\frac{\pi}{2}$.

(ii) From Figure 2 (a), $f'(x) = 0$ when $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.

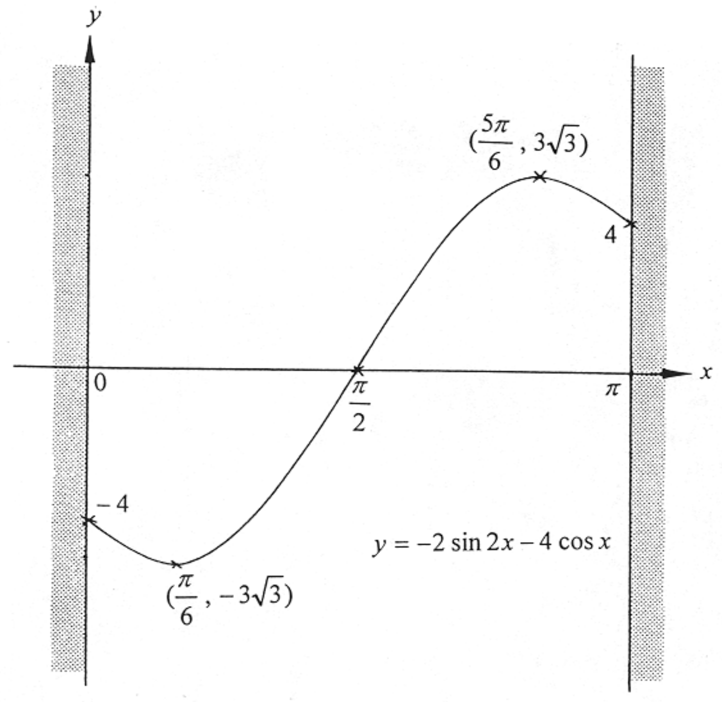
As $f'(x)$ changes from negative to positive as x increases through $\frac{\pi}{6}$,

so $(\frac{\pi}{6}, -3\sqrt{3})$ is a minimum point.

As $f'(x)$ changes from positive to negative as x increases through $\frac{5\pi}{6}$,

so $(\frac{5\pi}{6}, 3\sqrt{3})$ is a maximum point.

(c)



(d) $6 - 3\sqrt{3} \le g(x) \le 6 + 3\sqrt{3}$

Q.10 (a) $\overrightarrow{OC} = \dfrac{7\mathbf{a}+8\mathbf{b}}{15}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \dfrac{16}{21}\mathbf{b} - \mathbf{a}$

(b) (i) $\overrightarrow{OE} = r\overrightarrow{OC} = \dfrac{7r}{15}\mathbf{a} + \dfrac{8r}{15}\mathbf{b}$

(ii) $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$

$= \mathbf{a} + k\overrightarrow{AD}$

$= \mathbf{a} + k(\dfrac{16\mathbf{b}}{21} - \mathbf{a}) = (1-k)\mathbf{a} + \dfrac{16k}{21}\mathbf{b}$

Comparing the two expressions :

$$\begin{cases} \dfrac{7r}{15} = 1-k & \text{------(1)} \\ \dfrac{8r}{15} = \dfrac{16}{21}k & \text{------(2)} \end{cases}$$

(1) ÷ (2) : $\dfrac{7}{8} = \dfrac{21(1-k)}{16k}$

$14k = 21 - 21k$

$k = \dfrac{3}{5}$

Substitute $k = \dfrac{3}{5}$ into (1) : $\dfrac{7}{15}r = 1 - \dfrac{3}{5}$

$r = \dfrac{6}{7}$

$\therefore k = \dfrac{3}{5}$ and $r = \dfrac{6}{7}$.

(c) (i) Let $EC = x$.

Since $EC : ED = 1 : 2$, $ED = 2x$.

From (b), $\overrightarrow{OE} = \frac{6}{7}\overrightarrow{OC}$.

$\therefore$ $EO : EC = 6 : 1$, i.e. $EO = 6x$.

From (b), $\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AD}$.

$\therefore$ $EA : ED = 3 : 2$, i.e. $EA = 3x$.

$\therefore$ $EA : EO = 3x : 6x$

$= 1 : 2$.

(ii) In ΔEAC and ΔEOD,

$\angle AEC = \angle OED$

From (b), $\frac{EA}{EO} = \frac{1}{2} = \frac{EC}{ED}$

$\therefore \Delta EAC \sim \Delta EOD$.

$\angle EAC = \angle EOD$ (Corr $\angle$s of similar Δs)

$\therefore OACD$ is a cyclic quadrilateral.

(Converse of $\angle$s in the same segment)

Q.11 (a) $z_1 = 2(\cos 60° + i \sin 60°)$

$= 1 + \sqrt{3}i$

$z_3 = (\sqrt{3}i)\, z_1$

$= -3 + \sqrt{3}i$

(b) $\frac{z_2}{z_1} = \frac{z_1 + z_3}{z_1} = 1 + (\frac{z_3}{z_1})$

$= 1 + \sqrt{3}i \quad (\because\ z_3 = (\sqrt{3}i)z_1)$

$\angle AOB = \arg(\frac{z_2}{z_1}) = \arg(1 + \sqrt{3}i)$

$= 60°$

(c) (i) $\arg(z_3) = 150°$

$\arg(wz_3) = 150° + \theta$

$\arg(\bar{z}_3) = -150°$

If E represents the complex number $\bar{z}_3$,

$150° + \theta = -150° + 360°$

$\theta = 60°$

(ii) If E, O and A lie on a straight line,

$150° + \theta = 60° + 360\,k°$ or $150° + \theta = -120° + 360\,k°$
(k is an integer)

$\theta = 270°$ or $90°$.

Q.12 (a) S = Area of $ABCD$ – Area of ΔABE – Area of ΔCEF – Area of ΔADF

$$= 2(2k) - \frac{1}{2}(2)(2x) - \frac{1}{2}(x)(2k-2x) - \frac{1}{2}(2k)(2-x)$$

$$= x^2 - 2x + 2k$$

(b) (i) As E lies on BC, so $0 \le 2x \le 2k$, i.e. $0 \le x \le \frac{3}{2}$.

As F lies on CD, so $0 \le x \le 2$.

Combining the two inequalities, $0 \le x \le \frac{3}{2}$.

(ii) $S = x^2 - 2x + 2k$

$= x^2 - 2x + 3$

$= (x-1)^2 + 2$

As $x = 1$ lies in the range of possible value of x ($0 \le x \le \frac{3}{2}$),

$\therefore$ the least value of $S = 2$, which occurs when $x = 1$.

(iii) Since $S = x^2 - 2x + 3$ is a parabola and there is only a minimum in the range $0 \le x \le \frac{3}{2}$, so greatest value of S occurs at the end points.

At $x = 0, S = 3$.

At $x = \frac{3}{2}, S = (\frac{3}{2})^2 - 2(\frac{3}{2}) + 3 = \frac{9}{4}$.

$\therefore$ the greatest value of S is 3.

(c) (i) Put $k = \frac{3}{8}, S = x^2 - 2x + \frac{3}{4}$.

The range of possible values of x is $0 \le x \le \frac{3}{8}$.

As $x = 1$ does not lie in the above interval, the least value of S will not happen when $x = 1$.

$\therefore$ the student is incorrect.

(ii) As S is monotonic decreasing on $0 \le x \le \frac{3}{8}$,

least value of S occurs when $x = \frac{3}{8}$.

$\therefore$ least value of $S = (\frac{3}{8})^2 - 2(\frac{3}{8}) + \frac{3}{4} = \frac{9}{64}$

Q.13 (a) $h = \frac{V}{\pi x^2}$

$C = (2\pi xh) + k(\pi x^2)2$

$= 2\pi x(\frac{V}{\pi x^2}) + 2\pi x^2 k$

$= \frac{2V}{x} + 2\pi k x^2$

(b) $\frac{dC}{dx} = -\frac{2V}{x^2} + 4\pi k x$

$\frac{dC}{dx} = 0 \quad -\frac{2V}{x^2} + 4\pi k x = 0$

$x^3 = \frac{V}{2\pi k}$

$\frac{d^2C}{dx^2} = \frac{4V}{x^3} + 4\pi k$

Put $x^3 = (\frac{V}{2\pi k})$: $\frac{d^2C}{dx^2} = 12\pi k > 0.$

$\therefore$ C is a minimum.

$\frac{x}{h} = \frac{x}{V/\pi x^2} = \frac{\pi x^3}{V}$

$= \frac{\pi}{V}(\frac{V}{2\pi k}) = \frac{1}{2k}$

(c) (i) From (b), $x^3 = (\frac{V}{2\pi k}) = (\frac{256\pi}{2\pi(2)}) = 64$

$x = 4$

Since $\frac{x}{h} = \frac{1}{2k}$, $\frac{4}{h} = \frac{1}{2(2)}$

$h = 16$

(ii) Since $x^3 = \frac{V}{2\pi k}$, so x decreases when k increases.

As $h = \frac{V}{\pi x^2}$, so h increases when x decreases.

$\therefore$ the base radius of the can decreases and the height of the can increases.

(d) As the costs of the curved and plane surfaces remain unchanged, the ratio $\frac{x}{h} = \frac{1}{2k}$ is independent of the volume of the can.

$\therefore$ the ratio $\frac{\text{base radius}}{\text{height}}$ of the bigger can should remain identical to that of the smaller can in order to minimise the cost. So the worker is incorrect.