FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

Section A (42 marks)

Answer **ALL** questions in this section.

Find the coefficient of x^2 in the expansion of $(x - \frac{2}{x})^6$.

(4 marks)

Given a line $L : x - 7y + 3 = 0$ and
a circle $C : (x-2)^2 + (y+5)^2 = a$, where a is a positive number.

(a) Find the distance from the centre of C to L.

(b) If L is a tangent to C, find a.

(4 marks)

Prove, by mathematical induction, that

$$1 \times 2 + 2 \times 3 + 2^2 \times 4 + \ldots + 2^{n-1}(n+1) = 2^n(n)$$

for all positive integers n.

(5 marks)

The slope at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = \cos^2 x .$$

If the curve passes through the point $(\frac{\pi}{2}, \pi)$, find its equation.

(5 marks)

5. Two lines $L_1 : 2x + y - 3 = 0$ and $L_2 : x - 3y + 1 = 0$ intersect at a point P.

(a) Write down an equation of the family of straight lines passing through P.

(b) Suppose L is a line passing through P and the origin, find

(i) the equation of L,

(ii) the acute angle between L and L_1 correct to the nearest degree.

(6 marks)

6. Using the substitution $u = \sin\theta$, evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5\theta \sin^2\theta \, d\theta .$$

(6 marks)

7. Show that $\sin(3x + \frac{\pi}{4})\cos(3x - \frac{\pi}{4}) = \frac{1 + \sin 6x}{2}$.

Hence, or otherwise, find the general solution of the equation

$$\sin(3x + \frac{\pi}{4})\cos(3x - \frac{\pi}{4}) = \frac{3}{4}.$$

(6 marks)

8.

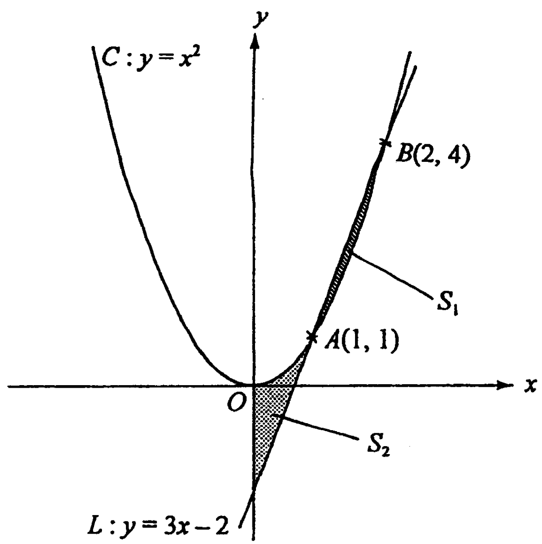

Figure 1

In Figure 1, the line $L : y = 3x - 2$ and the curve $C : y = x^2$ intersect at two points $A(1, 1)$ and $B(2, 4)$. Let S_1 denote the area of the region bounded by C and line segment AB, and S_2 denote the area bounded by C, L and the y-axis.

(a) Find S_1.

(b) Which of the following expressions represent(s) the total area $S_1 + S_2$? (Note : You are not required to give reasons.)

(I) $\int_0^2 (3x - 2 - x^2)\,dx$

(II) $\int_0^1 (x^2 - 3x + 2)\,dx + \int_1^2 (3x - 2 - x^2)\,dx$

(III) $\int_0^2 |3x - 2 - x^2|\,dx$

(IV) $\left| \int_0^2 (3x - 2 - x^2)\,dx \right|$

(6 marks)

Section B (48 marks)
Answer any **THREE** questions in this section.
Each question carries 16 marks.

9. (a) Let a be a positive number.

(i) Show that $\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(-x)dx$.

(ii) If $f(x) = f(-x)$ for $-a \le x \le a$, show that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

(5 marks)

(b) Using the substitution $t = \frac{\sqrt{3}}{3}\tan\theta$, show that

$$\int_{0}^{1} \frac{dt}{1+3t^2} = \frac{\sqrt{3}\pi}{9}.$$

(3 marks)

(c) Given $I_1 = \int_{0}^{1} \frac{1-t^2}{1+3t^2}dt$ and $I_2 = \int_{0}^{1} \frac{t^2}{1+3t^2}dt$.

(i) Without evaluating I_1 and I_2,

(1) show that $I_1 + 4I_2 = 1$, and

(2) using the result of (b), evaluate $I_1 + I_2$.

(ii) Using the result of (c) (i), or otherwise, evaluate I_2.

(4 marks)

(d) Evaluate $\int_{-1}^{1} \frac{1+t^2}{1+3t^2}dt$.

(4 marks)

10.

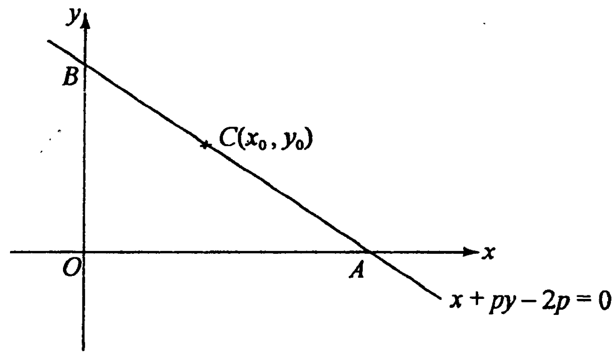

Figure 2

In Figure 2, the line $x + py - 2p = 0$, where $p > 0$, cuts the x-axis and y-axis at points A and B respectively. $C(x_0, y_0)$ is a point on AB such that $BC : CA = 1 : p^2$.

(a) Find x_0 and y_0 in terms of p.

(4 marks)

(b) Show that $\frac{y_0}{x_0} = p$.

Hence find the equation of the locus of C as p varies.

Sketch the locus of C.

(7 marks)

(c) Find the coordinates of A if the area of ΔOBC is greatest as p varies.

(5 marks)

11.

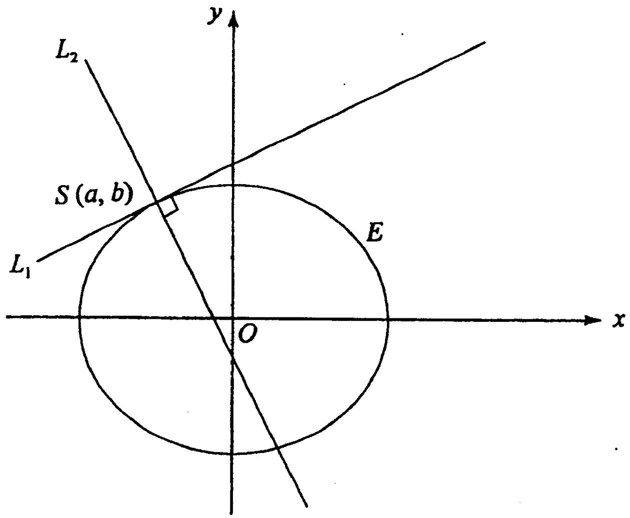

Figure 3(a)

$S(a,b)$ is a point on the second quadrant (i.e. $a < 0 < b$) and lies on the ellipse $E: \frac{x^2}{4}+\frac{y^2}{3}=1$ as shown in Figure 3(a). L_1 and L_2 are the tangent and normal to E at point S respectively. Let m_1 and m_2 be the slopes of L_1 and L_2 respectively.

(a) (i) Show that $3a^2+4b^2=12$.

(ii) Find m_1 and m_2 in terms of a and b.

(4 marks)

(b) P is the parabola $y^2=4cx$, where $c>0$.

(i) Show that the line $y=mx+\frac{c}{m}$, where $m \neq 0$, is a tangent to P.

(ii) If the line in (b) (i) passes through point $S(a,b)$, show that

$$am^2-bm+c=0 \ldots\ldots (*).$$

(b) **(continued)**

(iii)

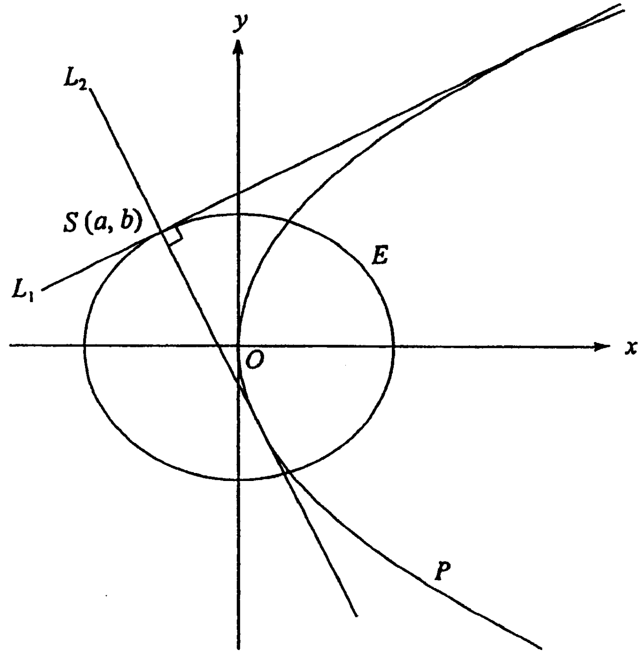

Figure 3(b)

Furthermore, the lines L_1 and L_2 are tangents to the parabola P as shown in Figure 3(b).

(1) By considering the quadratic equation (*) in (b) (ii), write down m_1+m_2 and m_1m_2 in terms of a, b and c.

Hence, and using (a) (ii), show that $9a^2=4b^2$ and $c=-a$.

(2) Find the value of a and the equation of P.

(12 marks)

12. (a)

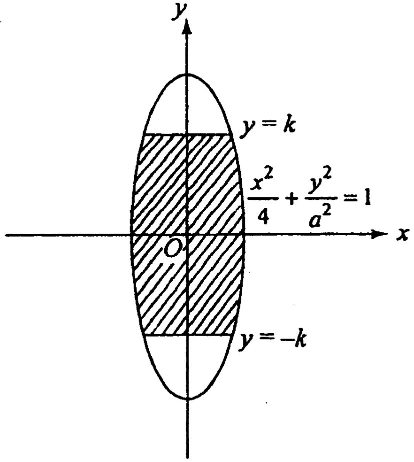

Figure 4(a)

In Figure 4(a), the shaded region enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{a^2} = 1$, the lines $y = k$ and $y = -k$, where $0 < k \le a$, is revolved about the y-axis. Show that the volume of the solid of revolution is $8k\left(1 - \frac{k^2}{3a^2}\right)\pi$.

(4 marks)

(b)

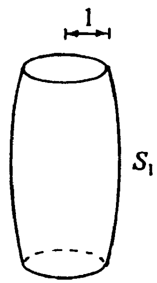

Figure 4(b)

A solid S_1 is in the shape of the solid of revolution described in (a). Furthermore, the radii of the plane circular faces of the solid are both equal to 1. (See Figure 4 (b).)

(i) Show that the height of S_1 is equal to $\sqrt{3}\,a$.

(ii) Find the volume of S_1 in terms of a and π.

(5 marks)

12. (c)

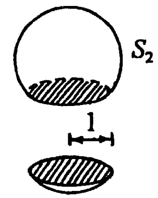

Figure 4(c)

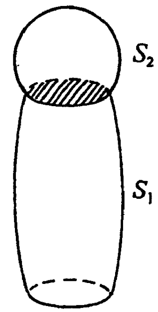

Figure 4(d)

A solid sphere of radius 2 is cut along a plane into two unequal portions as shown in Figure 4 (c). The radius of the plane circular face is equal to 1. The larger portion S_2 is joined to S_1 in (b) to form a toy as shown in Figure 4 (d).

(i) Show that the height of the toy is $2 + (a+1)\sqrt{3}$.

(ii) Using (b), or otherwise, find the volume of the toy in terms of a and π.

(7 marks)

13. (a)

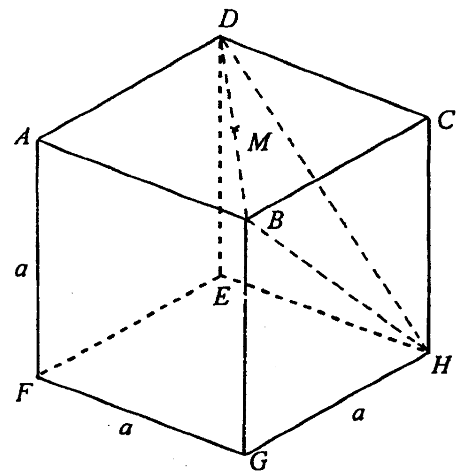

Figure 5(a)

Figure 5 (a) shows a solid cube $ABCDEFGH$ of side a. Let M be the mid-point of BD.

(i) Find CM.

(ii) Find the angle between the lines CM and HM to the nearest degree.

(4 marks)

(b)

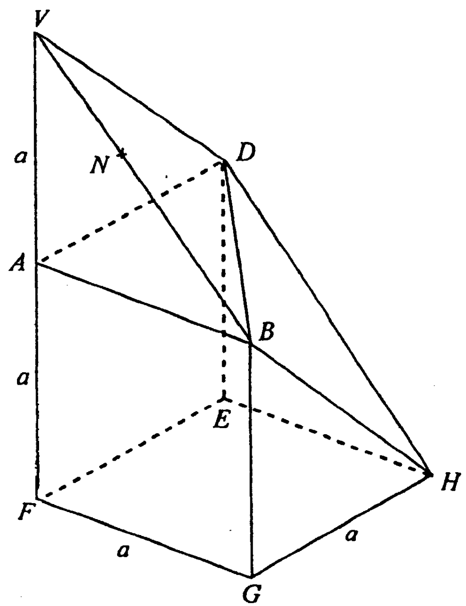

Figure 5(b)

13. (b) **(continued)**

The tetrahedron $BCDH$ is cut off from the cube in (a) and is then placed on top of the solid $ABDEFGH$ as shown in Figure 5 (b). The face BCD of the tetrahedron coincides with the face BAD of the solid $ABDEFGH$ such that vertex H of the tetrahedron moves to position V and vertex C coincides with A. The two faces BHD and BVD of the new solid lie on the same plane.

(i) Show that $\sin\angle FVH = \frac{\sqrt{3}}{3}$ and find the perpendicular distance from F to the face $BVDH$.

(ii) Let N be the point on VB such that DN and AN are both perpendicular to VB.

(1) Find DN.

(2) Find the angle between the faces BVD and BVA to the nearest degree.

(iii) A student says that the angle between the faces BHD and $ABGF$ is $\angle AND$. Explain briefly whether the student is correct.

(12 marks)

END OF PAPER