Section A (42 marks)
Answer **ALL** questions in this section.

1. The slope at any point (x, y) of a curve C is given by

$$\frac{dy}{dx} = 2x\sqrt{x^2 + 1}$$

and C cuts the y-axis at the point (0, 1). Find the equation of C.

(Hint : Let $u = x^2 + 1$.)

(5 marks)

2. A family of straight lines is given by the equation

$$2x - 3y + 4 + k(4x + 2y - 1) = 0,$$

where k is any constant.

(a) Find the equation of the line in the family which passes through the point (1, 0).

(b) Find the equation of the line in the family with slope 2.

(6 marks)

3. $A(t, \frac{1}{2}t^2)$ is a point on the parabola $x^2 = 2y$. B is the point (2, 2).

(a) Find the equation of the locus of the mid-point of AB as A moves along the parabola.

(b) Sketch the locus in (a).

(6 marks)

4. Given $(x^2 + \frac{1}{x})^5 - (x^2 - \frac{1}{x})^5 = ax^7 + bx + \frac{c}{x^5}$, find the values of a, b and c.

Hence evaluate $(2 + \frac{1}{\sqrt{2}})^5 - (2 - \frac{1}{\sqrt{2}})^5$.

(6 marks)

5.

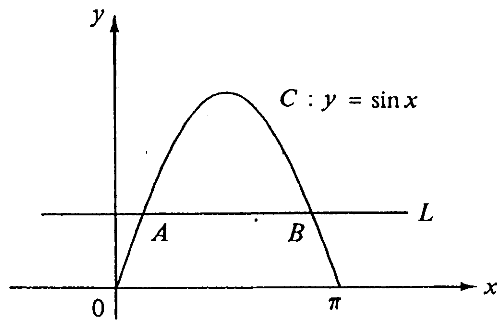

Figure 1

Figure 1 shows the curve $C : y = \sin x$ for $0 \le x \le \pi$.

(a) Find the area of the finite region bounded by the curve C and the x-axis.

(b) A horizontal line L cuts C at two points A and B. A is the point $(\frac{\pi}{6}, \frac{1}{2})$.

(i) Write down the coordinates of B.

(ii) Find the area of the finite region bounded by C and L.

(6 marks)

6. Prove, by mathematical induction, that $(8^n - 1)$ is divisible by 7 for all positive integers n.

(6 marks)

7.

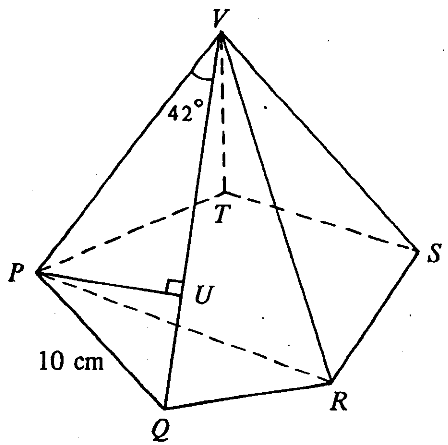

Figure 2

In Figure 2, *VPQRST* is a right pyramid whose base *PQRST* is a regular pentagon. $PQ = 10$ cm and $\angle PVQ = 42°$. U is a point on VQ such that PU is perpendicular to VQ. Find, correct to 3 significant figures,

(a) PU and PR,

(b) the angle between the faces VPQ and VQR.

(7 marks)

Section B (48 marks)

Answer any **THREE** questions in this section.
Each question carries 16 marks.

8. Let n be an integer greater than 1.

(a) Show that

$$\frac{d}{dx}[x^{n-1}(1-x^2)^{\frac{3}{2}}] = (n-1)x^{n-2}\sqrt{1-x^2} - (n+2)x^n\sqrt{1-x^2}.$$

(4 marks)

(b) Using (a), show that

$$\int_0^1 x^n\sqrt{1-x^2}\,dx = \frac{n-1}{n+2}\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx.$$

(3 marks)

(c) Using the substitution $x = \sin\theta$, evaluate

$$\int_0^1 \sqrt{1-x^2}\,dx.$$

(3 marks)

(d) Using (b) and (c), evaluate the following integrals :

(i) $$\int_0^1 x^4\sqrt{1-x^2}\,dx,$$

(ii) $$\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\,d\theta.$$

(6 marks)

9. (a) Show that $\cos^2 A - \cos^2 B = \sin(A + B)\sin(B - A)$.

(3 marks)

(b) ABC is a triangle.

(i) Using (a), show that

$$\cos^2 A - \cos^2 B + \sin^2 C = 2\cos A \sin B \sin C.$$

(ii) If $\cos^2 A - \cos^2 B - \cos^2 C = -1$, show that ABC is a right-angled triangle.

(6 marks)

(c) Using (a), or otherwise, show that

$$\cos^2 x - \sin^2 y = \cos(x + y)\cos(x - y).$$

Hence find the general solution of

$$\cos^2 2\theta - \sin^2 3\theta + \cos\theta \sin 5\theta = 0.$$

(7 marks)

10.

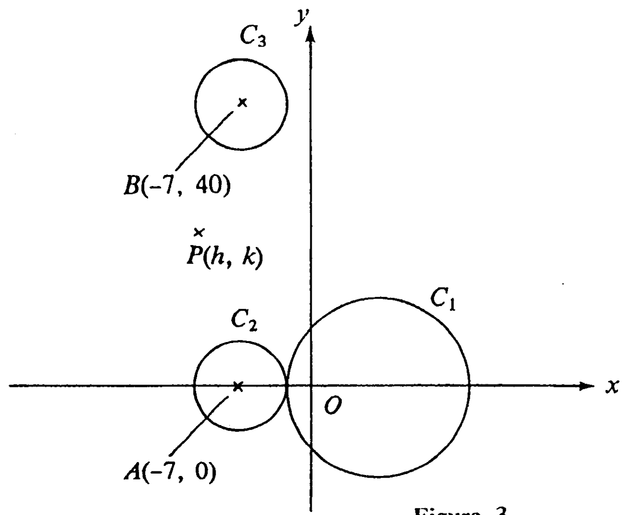

Figure 3

C_1 is the circle $x^2 + y^2 - 16x - 36 = 0$ and C_2 is a circle centred at the point $A(-7, 0)$. C_1 and C_2 touch externally as shown in Figure 3. $P(h, k)$ is a point in the second quadrant.

(a) Find the centre and radius of C_1.

Hence find the radius of C_2. (4 marks)

(b) If P is the centre of a circle which touches both C_1 and C_2 externally, show that

$$8h^2 - k^2 - 8h - 48 = 0.$$ (5 marks)

(c) C_3 is a circle centred at the point $B(-7, 40)$ and of the same radius as C_2.

(i) If P is the centre of a circle which touches both C_2 and C_3 externally, write down the equation of the locus of P.

(ii) Find the equation of the circle, with centre P, which touches all the three circles C_1, C_2 and C_3 externally.

(7 marks)

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11. (a)

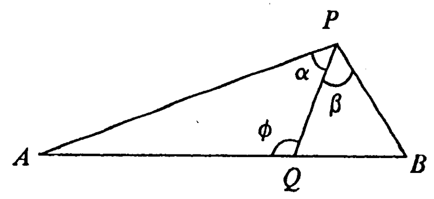

Figure 4(a)

Figure 4(a) shows a triangle PAB and Q is a point on AB. Let $\angle AQP = \phi$, $\angle APQ = \alpha$ and $\angle QPB = \beta$.

(i) Express $\frac{QA}{PA}$ in terms of α and ϕ.

(ii) If $\frac{QA}{PA} = \frac{QB}{PB}$, show that $\alpha = \beta$.

(3 marks)

(b)

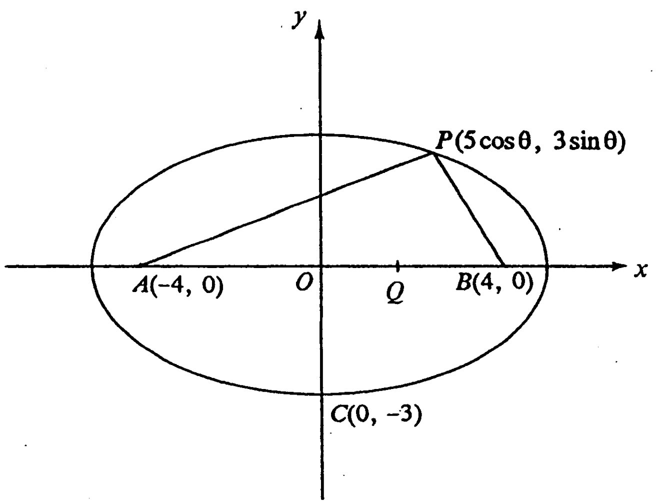

Figure 4(b)

In Figure 4(b), $P(5\cos\theta, 3\sin\theta)$ is a point on the ellipse $9x^2 + 25y^2 = 225$, where $\sin\theta \neq 0$. A and B are the points $(-4, 0)$ and $(4, 0)$ respectively. Q is a point on AB such that the normal to the ellipse at P passes through Q.

(i) Show that the equation of PQ is

$$5x\sin\theta - 3y\cos\theta - 16\sin\theta\cos\theta = 0.$$

Hence find the coordinates of Q in terms of θ.

(ii) Show that $PA = 5 + 4\cos\theta$.

Hence, using (a) (ii), show that PQ bisects $\angle APB$.

(13 marks)

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12. (a) Using the substitution $y - k = \cos\theta$, show that

$$\int_{k-1}^{k+1} \sqrt{1-(y-k)^2}\,dy = \frac{\pi}{2}.$$

Hence show that

(i) $$\int_0^2 \left[2+\sqrt{1-(y-1)^2}\right]^2 dy = \frac{28}{3} + 2\pi,$$

(ii) $$\int_2^4 \left[2-\sqrt{1-(y-3)^2}\right]^2 dy = \frac{28}{3} - 2\pi.$$

(8 marks)

(b)

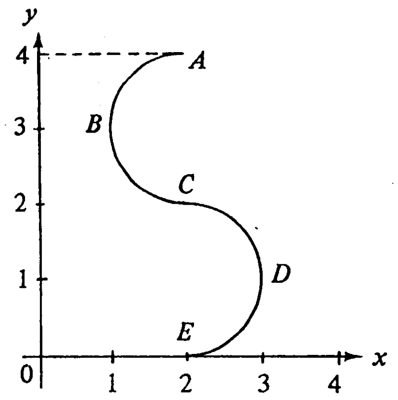

Figure 5(a)

Figure 5(a) shows two semicircles ABC and CDE centred at (2, 3) and (2, 1) respectively. Their radii are both equal to 1.

(i) Show that the equation of the semicircle ABC is

$$x = 2 - \sqrt{1-(y-3)^2},$$

and that of the semicircle CDE is

$$x = 2 + \sqrt{1-(y-1)^2}.$$

(ii) A pot is formed by revolving the curve $ABCDE$ and the line segment OE about the y-axis, where O is the origin. Using (a), find the capacity of the pot.

(5 marks)

(c)

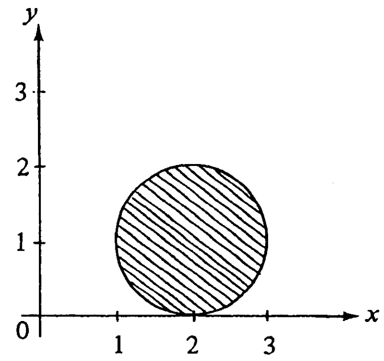

Figure 5(b)

The shaded region enclosed by the circle $(x-2)^2 + (y-1)^2 = 1$, as shown in Figure 5(b), is revolved about the y-axis to form a solid. Using (a) and (b), or otherwise, find the volume of the solid.

(3 marks)

END OF PAPER