94-AL
P MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG ADVANCED LEVEL EXAMINATION 1994

PURE MATHEMATICS A-LEVEL PAPER II

2.00 pm-5.00 pm (3 hours)
This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

SECTION A (40 marks)

Answer ALL questions in this section.
Write your answers in the AL(C1) answer book.

1. Evaluate

(a) $\lim_{x \to 1} \dfrac{1-\sqrt{x}}{1-\sqrt[5]{x}}$,

(b) $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$.

(4 marks)

2. Evaluate

(a) $\int \tan^3 x \, dx$,

(b) $\int \dfrac{x^2-x+2}{x(x-2)^2} \, dx$.

(6 marks)

3. Find the equations of the straight line which satisfies the following two conditions:

(i) passing through the point $(4, 2, -3)$,

(ii) parallel to the planes $x+y+z-10 = 0$ and $x+2y = 0$.

(4 marks)

4. The equation of a curve C in polar coordinates is

$$r = 1 + \sin\theta , \quad 0 \le \theta \le 2\pi .$$

(a) Sketch curve C .

(b) Find the area bounded by curve C .

(5 marks)

5. For $n = 1, 2, 3, \ldots$ and $\theta \in \mathbb{R}$, let $s_n = \sum_{k=1}^{n} 3^{k-1} \sin^3\left(\dfrac{\theta}{3^k}\right)$.

Using the identity $\sin^3\phi = \dfrac{3}{4}\sin\phi - \dfrac{1}{4}\sin 3\phi$, show that

$$s_n = \frac{3^n}{4}\sin\left(\frac{\theta}{3^n}\right) - \frac{1}{4}\sin\theta .$$

Hence, or otherwise, evaluate $\lim_{n \to \infty} s_n$.

(4 marks)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

Show that $\int_0^1 x f(xt) dt = \int_0^x f(t) dt$ for all $x \in \mathbb{R}$.

If $\int_0^1 f(xt) dt = 0$ for all $x \in \mathbb{R}$,
show that $f(x) = 0$ for all $x \in \mathbb{R}$.

(5 marks)

7. Let $f(x) = \int_1^x \sin(\cos t) dt$, where $x \in [0, \frac{\pi}{2})$.

(a) Show that f is injective.

(b) If g is the inverse function of f , find $g'(0)$.

(6 marks)

8. (a) Show that for any $a, y \in \mathbb{R}$, $e^y - e^a \ge e^a(y-a)$.

(b) By taking $y = x^2$ in the inequality in (a), prove that

$$\int_0^1 e^{x^2} dx \ge e^{\frac{1}{3}} .$$

(6 marks)

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SECTION B (60 marks)

Answer any **FOUR** questions from this section. Each question carries 15 marks.
Write your answers in the AL(C2) answer book.

9. Given an ellipse

$$(E): \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and a point $P(h, k)$ outside (E).

(a) If $y = mx + c$ is a tangent from P to (E), show that $(h^2 - a^2)m^2 - 2hkm + k^2 - b^2 = 0$.

(4 marks)

(b) Suppose the two tangents from P to (E) touch (E) at A and B.

(i) Find the equation of the line passing through A and B.

(ii) Find the coordinates of the mid-point of AB.

(6 marks)

(c) Show that the two tangents from P to (E) are perpendicular if and only if P lies on the circle $x^2 + y^2 = a^2 + b^2$.

(5 marks)

10. Let $f(x) = \dfrac{\sqrt[3]{x^2}}{x^2 + 1}$, $x \in \mathbb{R}$.

(a) (i) Evaluate $f'(x)$ for $x \neq 0$.
Prove that $f'(0)$ does not exist.

(ii) Determine those values of x for which $f'(x) > 0$ and those values of x for which $f'(x) < 0$.

(iii) Find the relative extreme points of $f(x)$.

(8 marks)

(b) (i) Evaluate $f''(x)$ for $x \neq 0$. Hence determine the points of inflexion of $f(x)$.

(ii) Find the asymptote of the graph of $f(x)$.

(4 marks)

(c) Using the above results, sketch the graph of $f(x)$.

(3 marks)

11. For any non-negative integer n, let

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx .$$

(a) (i) Show that $\frac{1}{n+1}\left(\frac{\pi}{4}\right)^{n+1} \le I_n \le \frac{1}{n+1}\left(\frac{\pi}{4}\right)$.

[Note: You may assume without proof that
$x \le \tan x \le \frac{4x}{\pi}$ for $x \in [0, \frac{\pi}{4}]$.]

(ii) Using (i), or otherwise, evaluate $\lim_{n\to\infty} I_n$.

(iii) Show that $I_n + I_{n-2} = \frac{1}{n-1}$ for $n = 2, 3, 4, \dots$.

(8 marks)

(b) For $n = 1, 2, 3, \dots$, let $a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}$.

(i) Using (a)(iii), or otherwise, express a_n in terms of I_{2n} .

(ii) Evaluate $\lim_{n\to\infty} a_n$.

(7 marks)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying the following conditions for all $x \in \mathbb{R}$:

A. $f(x) > 0$;
B. $f(x+1) = f(x)$;
C. $f(\frac{x}{4})f(\frac{x+1}{4}) = f(x)$.

Define $g(x) = \frac{d}{dx}\ln f(x)$ for $x \in \mathbb{R}$.

(a) Show that for all $x \in \mathbb{R}$,

(i) $f'(x+1) = f'(x)$;

(ii) $g(x+1) = g(x)$;

(iii) $\frac{1}{4}[g(\frac{x}{4}) + g(\frac{x+1}{4})] = g(x)$.

(8 marks)

(b) Let M be a constant such that $|g(x)| \le M$ for all $x \in [0, 1]$.

(i) Using (a), or otherwise, show that
$|g(x)| \le \frac{M}{2}$ for all $x \in \mathbb{R}$.
Hence deduce that
$g(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Show that $f(x) = 1$ for all $x \in \mathbb{R}$.

(7 marks)

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13. Let $L_n = \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx \right\}^{\frac{1}{n}}$ for any positive integer n.

(a) Show that $L_n \le \pi^{\frac{1}{n}}$.

(3 marks)

(b) For $n = 1, 2, 3, \ldots$, let $r_n = \cos \frac{1}{2n}$.

Find the values of x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\cos x \ge r_n$.

Hence show that $L_n \ge r_n \left(\frac{1}{n} \right)^{\frac{1}{n}}$.

(5 marks)

(c) Show that

(i) $\lim_{n \to \infty} n^{\frac{1}{n}} = 1$,

(ii) $\lim_{n \to \infty} L_n = 1$.

(7 marks)

14. (a) $f(x)$ is a continuously differentiable and strictly increasing function on $[0, c]$ such that $f(0) = 0$.
Let $b \in [0, f(c)]$.
Define $g(t) = tb - \int_0^t f(x) dx$, $t \in [0, c]$.

(i) Determine the interval on which $g(t)$ is strictly increasing and the interval on which $g(t)$ is strictly decreasing. Hence show that

$$g(t) \le g(f^{-1}(b)) \quad \text{for all } t \in [0, c].$$

(ii) Using the substitution $y = f(x)$ and integration by parts, show that

$$\int_0^b f^{-1}(y) dy = g(f^{-1}(b)).$$

(iii) If $a \in [0, c]$, prove the inequality

$$\int_0^a f(x) dx + \int_0^b f^{-1}(x) dx \ge ab.$$

(10 marks)

(b) If a, b, p, q are positive numbers and $\frac{1}{p} + \frac{1}{q} = 1$, prove that

$$\frac{1}{p} a^p + \frac{1}{q} b^q \ge ab.$$

(5 marks)

END OF PAPER