94-AL
P MATHS

PAPER I

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG ADVANCED LEVEL EXAMINATION 1994

# PURE MATHEMATICS A-LEVEL PAPER I

9.00 am-12.00 noon (3 hours)

This paper must be answered in English

1. This paper consists of Section A and Section B.
2. Answer ALL questions in Section A, using the AL(C1) answer book.
3. Answer any FOUR questions in Section B, using the AL(C2) answer book.

## SECTION A (40 marks)

Answer ALL questions in this section.

**Write your answers in the AL(C1) answer book.**

1. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

   (a) Find $P^{-1}AP$.

   (b) Find $A^n$, where $n$ is a positive integer.

   (6 marks)

2. Consider the following system of linear equations:

$$(*) \begin{cases} 4x + 3y + z = \lambda x \\ 3x - 4y + 7z = \lambda y \\ x + 7y - 6z = \lambda z \end{cases} .$$

   Suppose $\lambda$ is an integer and $(*)$ has nontrivial solutions. Find $\lambda$ and solve $(*)$.

   (6 marks)

3. (a) If $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 + px + q = 0$, find a cubic equation whose roots are $\alpha^2$, $\beta^2$ and $\gamma^2$.

   (b) Solve the equation $\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$.

   Hence, or otherwise, solve the equation $x^3 - 38x^2 + 361x - 900 = 0$.

   (6 marks)



4. $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are two sequences of real numbers. Define $s_k = a_1 + a_2 + \ldots + a_k$ for $k = 1, 2, \ldots, n$.

   (a) Prove that $\sum_{k=1}^{n} a_k b_k = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \ldots + s_{n-1}(b_{n-1} - b_n) + s_n b_n$.

   (b) If $b_1 \ge b_2 \ge \ldots \ge b_n \ge 0$ and there are constants $m$ and $M$ such that
   $m \le s_k \le M$ for $k = 1, 2, \ldots, n$,
   prove that $mb_1 \le \sum_{k=1}^{n} a_k b_k \le Mb_1$.

   (5 marks)

5. Let $\{a_n\}$ be a sequence of positive numbers such that

$$a_1 + a_2 + \ldots + a_n = \left(\frac{1 + a_n}{2}\right)^2$$

   for $n = 1, 2, 3, \ldots$.

   Prove by induction that $a_n = 2n - 1$ for $n = 1, 2, 3, \ldots$.

   (5 marks)

6. Let $\operatorname{Arg} z$ denote the principal value of the argument of the complex number $z$ ( $-\pi < \operatorname{Arg} z \le \pi$ ).

   (a) If $z \ne 0$ and $z + \bar{z} = 0$, show that $\operatorname{Arg} z = \pm\frac{\pi}{2}$.

   (b) If $z_1, z_2 \ne 0$ and $|z_1 + z_2| = |z_1 - z_2|$, show that $\frac{z_1}{z_2} + \overline{\left(\frac{z_1}{z_2}\right)} = 0$

   and hence find all possible values of $\operatorname{Arg}\frac{z_1}{z_2}$.

   (5 marks)



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7. (a) Let $m$ and $n$ be positive integers. Using the identity

$$(1+x)^n+(1+x)^{n+1}+\ldots+(1+x)^{n+m} = \frac{(1+x)^{n+m+1}-(1+x)^n}{x},$$

where $x \neq 0$, show that

$$C_n^n + C_n^{n+1} + \ldots + C_n^{n+m} = C_{n+1}^{n+m+1}.$$

(b) Using (a), or otherwise, show that

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24\left(C_5^{m+5}-1\right).$$

Hence evaluate $\sum_{r=0}^{k} r(r-1)(r-2)(r-3)$ for $k \geq 4$.

(7 marks)



## SECTION B (60 marks)

**Answer any FOUR questions from this section. Each question carries 15 marks. Write your answers in the AL(C2) answer book.**

8. Let $M = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$, where $a$, $b$ and $c$ are non-negative real numbers.

(a) Show that $\det(M) = \frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$ and $0 \leq \det(M) \leq (a+b+c)^3$.

(4 marks)

(b) Let $M^n = \begin{pmatrix} a_n & b_n & c_n \\ c_n & a_n & b_n \\ b_n & c_n & a_n \end{pmatrix}$ for any positive integer $n$, show that $a_n$, $b_n$ and $c_n$ are non-negative real numbers satisfying $a_n + b_n + c_n = (a+b+c)^n$.

(4 marks)

(c) If $a+b+c = 1$ and at least two of $a$, $b$ and $c$ are non-zero, show that

(i) $\lim_{n\to\infty} \det(M^n) = 0$,

(ii) $\lim_{n\to\infty} (a_n - b_n) = 0$ and $\lim_{n\to\infty} (a_n - c_n) = 0$,

(iii) $\lim_{n\to\infty} a_n = \frac{1}{3}$.

(7 marks)

9. (a) Consider

$$\text{(I)} \quad \begin{cases} a_{11}x + a_{12}y + a_{13}z = 0 \\ a_{21}x + a_{22}y + a_{23}z = 0 \\ a_{31}x + a_{32}y + a_{33}z = 0 \end{cases}$$

and

$$\text{(II)} \quad \begin{cases} a_{11}x + a_{12}y + a_{13} = 0 \\ a_{21}x + a_{22}y + a_{23} = 0 \\ a_{31}x + a_{32}y + a_{33} = 0 \end{cases} .$$

(i) Show that if (I) has a unique solution, then (II) has no solution.

(ii) Show that $(u, v)$ is a solution of (II) if and only if $(ut, vt, t)$ are solutions of (I) for all $t \in \mathbb{R}$ .

(iii) If (II) has no solution and (I) has nontrivial solutions, what can you say about the solutions of (I) ?

(5 marks)

(b) Consider

$$\text{(III)} \quad \begin{cases} -(3+k)x + y - z = 0 \\ -7x + (5-k)y - z = 0 \\ -6x + 6y + (k-2)z = 0 \end{cases}$$

and

$$\text{(IV)} \quad \begin{cases} -(3+k)x + y - 1 = 0 \\ -7x + (5-k)y - 1 = 0 \\ -6x + 6y + (k-2) = 0 \end{cases} .$$

(i) Find the values of $k$ for which (III) has non-trivial solutions.

(ii) Find the values of $k$ for which (IV) is consistent. Solve (IV) for each of these values of $k$ .

(iii) Solve (III) for each $k$ such that (III) has non-trivial solutions.

(10 marks)



10. Let $\mathbf{a}$ , $\mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$ .

(a) Show that $\mathbf{a}$ , $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 ,$$

where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ .

(5 marks)

(b) Suppose $\mathbf{a}$ , $\mathbf{b}$ and $\mathbf{c}$ are linearly independent. Show that for any vector $\mathbf{x}$ in $\mathbb{R}^3$ , there are unique $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$\mathbf{x} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} .$$

(4 marks)

(c) Let $S = \{\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} : \alpha, \beta, \gamma \in \mathbb{R}\}$ .

Under what conditions on $\mathbf{a}$ , $\mathbf{b}$ and $\mathbf{c}$ will $S$ represent

(i) a point ?
(ii) a line ?
(iii) a plane ?
(iv) the whole space ?

[Note: You are not required to give reasons. ]

(6 marks)



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11. A function $f : \mathbf{C} \to \mathbf{C}$ is said to be *real linear* if

$$f(\alpha z_1 + \beta z_2) = \alpha f(z_1) + \beta f(z_2)$$

for all $\alpha, \beta \in \mathbf{R}$ and $z_1, z_2 \in \mathbf{C}$.

(a) Suppose f is a real linear function. Show that

(i) if $z = 0$ whenever $f(z) = 0$, then f is injective;

(ii) if $f(i) = i f(1)$ and $f(i) \neq 0$, then f is bijective.

(4 marks)

(b) Suppose $\lambda, \mu \in \mathbf{C}$ and

$$g(z) = \lambda z + \mu \bar{z} \quad \text{for all } z \in \mathbf{C}.$$

Show that

(i) g is real linear;

(ii) g is injective if and only if $|\lambda| \neq |\mu|$.

(8 marks)

(c) If f is a real linear function, find $a, b \in \mathbf{C}$ such that

$$f(z) = az + b\bar{z} \quad \text{for all } z \in \mathbf{C}.$$

(3 marks)



12. Let $p(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, where $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Suppose $z_1 = \cos\theta_1 + i\sin\theta_1$ and $z_2 = \cos\theta_2 + i\sin\theta_2$ are two roots of $p(x) = 0$, where $0 < \theta_1 < \theta_2 < \pi$.

(a) Show that

(i) $p(x) = (x^2 - 2x\cos\theta_1 + 1)(x^2 - 2x\cos\theta_2 + 1)$,

(ii) $p'(x) = 2p(x)\left(\dfrac{x - \cos\theta_1}{x^2 - 2x\cos\theta_1 + 1} + \dfrac{x - \cos\theta_2}{x^2 - 2x\cos\theta_2 + 1}\right)$.

(5 marks)

(b) Suppose $p(w) = 0$, by considering $p(x) - p(w)$, show that

$$\frac{p(x)}{x - w} = x^3 + (w + a_1)x^2 + (w^2 + a_1w + a_2)x + (w^3 + a_1w^2 + a_2w + a_3).$$

(3 marks)

(c) Let $s_n = z_1^n + \bar{z}_1^n + z_2^n + \bar{z}_2^n$, using (a)(ii) and (b), show that

$$p'(x) = 4x^3 + (s_1 + 4a_1)x^2 + (s_2 + a_1s_1 + 4a_2)x + (s_3 + s_2a_1 + s_1a_2 + 4a_3).$$

[Hint: $\dfrac{2(x - \cos\theta_r)}{x^2 - 2x\cos\theta_r + 1} = \dfrac{1}{x - z_r} + \dfrac{1}{x - \bar{z}_r}$, $r = 1, 2$.]

Hence show that

$$s_n + a_1s_{n-1} + \dots + a_{n-1}s_1 + na_n = 0 \quad \text{for } n = 1, 2, 3, 4.$$

(7 marks)

13. Let $\mathbb{Z}_+$ be the set of all positive integers and $m, n \in \mathbb{Z}_+$.

Let $A(m,n) = (1-x^m)(1-x^{m+1})\dots(1-x^{m+n-1})$,

$B(n) = (1-x)(1-x^2)\dots(1-x^n)$.

(a) Show that $A(m+1,n+1) - A(m,n+1)$ is divisible by $(1-x^{n+1})A(m+1,n)$.

(2 marks)

(b) Suppose $P(m,n)$ denote the statement

" $A(m,n)$ is divisible by $B(n)$. "

(i) Show that $P(1,n)$ and $P(m,1)$ are true.

(ii) Using (a), or otherwise, show that if $P(m,n+1)$ and $P(m+1,n)$ are true, then $P(m+1,n+1)$ is also true.

(iii) Let $k$ be a fixed positive integer such that $P(m,k)$ is true for all $m \in \mathbb{Z}_+$. Show by induction that $P(m,k+1)$ is true for all $m \in \mathbb{Z}_+$.

(10 marks)

(c) Using (b), or otherwise, show that $P(m,n)$ is true for all $m, n \in \mathbb{Z}_+$.

(3 marks)

END OF PAPER