1990 HKCE Additional Mathematics II

Solution	Marks	Remarks
1. (a) $(1 + 2x - 3x^2)^n$		
$= [1 + x(2 - 3x)]^n$	1M	
$= 1 + nx(2 - 3x) + \frac{n(n-1)}{2}x^2(2 - 3x)^2 + \ldots\ldots$		Deduct 1 mark for missing .....
$= 1 + 2nx + [2n(n - 1) - 3n]x^2 + \ldots\ldots$		
$a = 2n$	1A	
$b = 2n^2 - 5n$	1A	
(b) $2n^2 - 5n = 63$	1M	
$(2n + 9)(n - 7) = 0$		
$n = 7$	1A	
	5	
2. For $n = 1$, L.H.S. $= 1^2 + 1 = 2$		
R.H.S. $= \frac{1}{3}(1)(2)(3) = 2$		
the statement is true for $n = 1$	1	
Assume the statement is true for some integer k.	1	
For $n = k + 1$		
L.H.S. $= T_1 + T_2 + \ldots + T_k + T_{k+1}$		
$= \frac{1}{3}k(k + 1)(k + 2) + (k + 1)^2 + (k + 1)$	1	
$= \frac{1}{3}(k + 1)[k(k + 2) + 3(k + 1) + 3]$		
$= \frac{1}{3}(k + 1)(k^2 + 5k + 6)$		
$= \frac{1}{3}(k + 1)(k + 2)(k + 3)$		
$\therefore$ the statement holds for $n = k + 1$.	1	
By the principle of mathematical induction, the statement holds for all +ve integer n.	1	Awarded if previous steps all correct.
	5	

Solution	Marks	Remarks
3. $du = 2\sin x\cos x dx$	1A	
$\int \frac{\sin x\cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}dx = \int \frac{1}{2\sqrt{5u+4}}du$	2A	Integrated must be in terms of u
$= \frac{1}{5}\sqrt{5u+4} + c$	1A	
$= \frac{1}{5}\sqrt{5\sin^2 x + 4} + c$	1A	Deduct 1 mark for omitting c
(or $\frac{1}{5}\sqrt{9\sin^2 x + 4\cos^2 x} + c$)	5	
4. $\int_0^{\pi/2} [\cos x - k(x - \frac{\pi}{2})^2]dx$	1A	
$= [\sin x - \frac{k}{3}(x - \frac{\pi}{2})^3]_0^{\pi/2}$	1A	
$= 1 - \frac{k\pi^3}{24} = 2$	1A+1M	
$k = \frac{-24}{\pi^3}$ (−0.774)	1A	
	5	

Alt. Solution

Solution	Marks
$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2}$	
$= 1$	1A
$\int_0^{\pi/2} k(x - \frac{\pi}{2})^2 ds = \frac{k}{3}(x - \frac{\pi}{2})^3]_0^{\pi/2}$	
$= \frac{k\pi^3}{24}$	1A
$1 - \frac{k\pi^3}{24} = 2$	1A+1M
$k = \frac{-24}{\pi^3}$ (−0.774)	1A

Solution	Marks	Remarks
5. $2\sin\frac{x}{2}\sin\frac{3x}{2} = 1$		
$\cos x - \cos 2x = 1$	1A	
$\cos x - (2\cos^2 x - 1) = 1$	1A	
$2\cos^2 x - \cos x = 0$		
$\cos x = 0$ or $\frac{1}{2}$	1A	
$x = 2n\pi \pm \frac{\pi}{2}$ $\left(\frac{2n+1}{2}\pi\right)$	1A	$360n° \pm 90°$, $(2n+1)\ 90°$
or $2n\pi \pm \frac{\pi}{3}$ where $n \in z$	1A	$360n° \pm 60°$. use different units (pp − 1)
	5	
Alt. Solution		
Let $\sin\frac{x}{2} = t$		
$t(3t - 4t^3) = \frac{1}{2}$	1A	
$8t^4 - 6t^2 + 1 = 0$	1A	
$(2t^2 - 1)(4t^2 - 1) = 0$		
$t = \pm\frac{\sqrt{2}}{2}$ or $\pm\frac{1}{2}$	1A	
$\frac{x}{2} = n\pi \pm \frac{\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$		
$x = 2n\pi \pm \frac{\pi}{2}$ or $2n\pi \pm \frac{\pi}{3}$	1A+1A	
6. (a) $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$	1A	
$\tan\ = \sqrt{3}$ $\therefore \alpha = 60°$	1A	no mark if in radian
(b) $x = \dfrac{1}{2\cos(\theta - 60°) + 5}$		
$-1 \leq \cos(\theta - 60°) \leq 1$	1M	
$\therefore \frac{1}{7} \leq x \leq \frac{1}{3}$	1A+1A	
	5	

Solution	Marks	Remarks
7. Equation of CD : $y = mx+1$ ----- (1)	1A	
Equation of AB : $\frac{x}{3} + \frac{y}{5} = 1$ ----- (2)	1A	
Subs. (1) into (2) : $\frac{x}{3} + \frac{mx+1}{5} = 1$		
$x = \frac{12}{5+3m}$	1A	
Area of $\triangle BCD = \frac{1}{2}(5-1)\left(\frac{12}{5+3m}\right)$ $= \frac{24}{5+3m}$	1A	
$\frac{24}{5+3m} = \frac{1}{2} \cdot \frac{15}{2}$	1M	
$m = \frac{7}{15}$		
$\therefore$ Equation of CD is $y = \frac{7x}{15} + 1$	1A	$7x - 15y + 15 = 0$
	6	
Alt. Solution		
Let coordinates of D be (x, y)		
$\frac{4x}{2} = \frac{1}{2} \cdot \frac{15}{2}$	1M	
$x = \frac{15}{8}$	1A	
Equation of AB : $\frac{x}{3} + \frac{y}{5} = 1$	1A	$\frac{y}{\frac{15}{8} - 3} = \frac{5}{-3}$ 1A
Subs. $x = \frac{15}{8}$, $y = \frac{15}{8}$	1A	$y = \frac{15}{8}$ 1A
$\therefore$ Equation of CD		
$\frac{y-1}{x} = \frac{\frac{15}{8} - 1}{15/8}$	1M	
$y = \frac{7}{15}x + 1$	1A	

Solution	Marks	Remarks
8. Let coordinates of S and T be (a, 0), (b, b) respectively	1A	
coordinates of mid-point is $\left(\frac{a+b}{2}, \frac{b}{2}\right)$	1A	
Let $x = \frac{a+b}{2}$, $y = \frac{b}{2}$		
$b = 2y$, $a = 2(x - y)$	1M	For making a, b as subjects
$(a - b)^2 + (b - 0)^2 = 4$	1M	
$(2x - 4y)^2 + (2y)^2 = 4$	1A	
$(x - 2y)^2 + y^2 = 1$		
$x^2 - 4xy + 5y^2 - 1 = 0$	1A	
	6	
Alt. Solution		
Let coordinates of P be (x, y)		
then coordinates of T is (2y, 2y)	1A	
coordinates of S is (2x − 2y, 0)	2A	
$(2x - 4y)^2 + 4y^2 = 4$	1M+1A	
$x^2 - 4xy + 5y^2 - 1 = 0$	1A	

	Solution	Marks	Remarks
9. (a) (i)	$\int_0^{\pi} \cos^2 x\,dx = \int_0^{\pi} \frac{1}{2}(1 + \cos 2x)\,dx$	1A	
	$= [\frac{1}{2}(x + \frac{\sin 2x}{2})]_0^{\pi}$	1A	
	$= \pi/2$	1A	
(ii)	Put $x = \pi - y$		
	$\int_0^{\pi} x\cos^2 x\,dx = \int_{\pi}^{0} (\pi - y)\cos^2(\pi - y) - dy$	1A	
	$= \pi\int_0^{\pi} \cos^2 y\,dy - \int_0^{\pi} y\cos^2 y\,dy$	1M	For separating into 2 integrals
	$2\int_0^{\pi} x\cos^2 x\,dx = \pi\int_0^{\pi} \cos^2 x\,dx$	1M	
	$= \pi^2/2$		
	$\therefore \int_0^{\pi} x\cos^2 x\,dx = \pi^2/4$	1A	
		7	
(b) (i)	Put $x = \pi + y$	1A	
	$\int_{\pi}^{2\pi} x\cos^2 x\,dx = \int_0^{\pi} (\pi + y)\cos^2(\pi + y)\,dy$	1A	
	$= \pi\int_0^{\pi} \cos^2 y\,dy + \int_0^{\pi} y\cos^2 y\,dy$		
	$= \pi\int_0^{\pi} \cos^2 x\,dx + \int_0 x\cos^2 x\,dx$	1	
(ii)	$\int_0^{2\pi} x\cos^2 x\,dx = \int_0^{\pi} x\cos^2 x\,dx + \int_{\pi}^{2\pi} x\cos^2 x\,dx$	1A	
	$= \int_0^{\pi} x\cos^2 x\,dx + \pi\int_0^{\pi} \cos^2 x\,dx$ $+ \int_0^{\pi} x\cos^2 x\,dx$	1M	For subs. (b)(i)
	$= \frac{\pi^2}{4} + \pi(\frac{\pi}{2}) + \frac{\pi^2}{4}$		
	$= \pi^2$	1	
		6	
(c)	Put $x^2 = y$	1A	
	$2x\,dx = dy$		
	$\int_0^{\sqrt{2\pi}} x^3\cos^2 x^2\,dx = \int_0^{2\pi} y\cos^2 y \cdot \frac{1}{2}dy$	1A	
	$= \frac{1}{2}\int_0^{2\pi} y\cos^2 y\,dy$		
	$= \frac{\pi^2}{2}$	1A	
		3	

Solution	Marks	Remarks
10. (a) $\left.\frac{dy}{dx}\right\|_{x=t} = 2t - 2$	1A	
y-coordinates of P $= t^2 - 2t + 3$	1A	
Equation of tangent : $y - (t^2 - 2t + 3)$		
$= (2t - 2)(x - t)$	1M	
$y = (2t - 2)x - t^2 + 3$ ...(*)	1A	
	4	
Alt. Solution		
Using the formula $\frac{y + y_1}{2} = xx_1 - (x + x_1) + 3$		
Equation of tangent : $\frac{y + (t^2 - 2t + 3)}{2}$		
$= tx - (t + x) + 3$	1M+1A +1A	1A for $y_1 = t^2 - 2t + 3$
$y = (2t - 2)x - t^2 + 3$	1A	
(b) (i) Put $t = \frac{1}{3}$ in (*)		
Equation of T_1 : $y = \frac{-4}{3}x + \frac{26}{9}$	1A	
(ii) Coordinates of C : (1, 2)	1A	
Coordinates of D : $(1, \frac{14}{9})$	1A	
(iii) Subs. $(1, \frac{14}{9})$ into (*)		
$\frac{14}{9} = 2t - 2 - t^2 + 3$	1M	
$9t^2 - 18t + 5 = 0$		
$t = \frac{1}{3}$ or $\frac{5}{3}$		
∴ x-coordinate of B $= \frac{5}{3}$	1A	
y-coordinate of B $= (\frac{5}{3})^2 - 2(\frac{5}{3}) + 3 = \frac{22}{9}$		
Coordinates of B $= (\frac{5}{3}, \frac{22}{9})$	1A	
	6	

Solution	Marks	Remarks
Alt. Solution		
(iii) Since S is symmetrical about $x = 1$ and x-coordinate of A $= \frac{1}{3}$, $\therefore$ by symmetry x coordinate of B $= 1 + (1 - \frac{1}{3}) = \frac{5}{3}$	1M+1A	
coordinates of B $= (\frac{5}{3}, \frac{22}{9})$	1A	
(c) Centre of circle lies on $x = 1$		
let its coordinates be $(1, a)$	1A	
Radius = Distance to T_1		
$= \left\| \dfrac{-\frac{4}{3} - a + \frac{26}{9}}{\sqrt{1 + (\frac{4}{3})^2}} \right\|$		
$= \left\| \dfrac{14 - 9a}{15} \right\|$	1A	
Since the circles pass through C (1, 2)		
Radius $= \|2 - a\|$	1M	
$\|2 - a\| = \left\| \dfrac{14 - 9a}{15} \right\|$	1M	
$a = \frac{8}{3}$ or $\frac{11}{6}$		
Coordinates of centres are $(1, \frac{8}{3})$ or $(1, \frac{11}{6})$	1A+1A 6	

Solution	Marks	Remarks
11. (a) Equation of family of circles		
$2x^2 + 2y^2 - 4x + 8y - 13 + k(x - y) = 0$	1A	$x^2 + y^2 - 2x + 4y - \frac{13}{2} + k(x - y) = 0$
$2x^2 + 2y^2 + (k - 4)x + (8 - k)y - 13 = 0$		$(x - y) + k(2x^2 + 2y^2 - 4x + 8y - 13) = 0$
$(\text{Radius})^2 = (\frac{k-4}{4})^2 + (\frac{8-k}{4})^2 + \frac{13}{2}$	1M	Area $A = \pi r^2$
$= \frac{1}{8}(k - 6)^2 + 7$	1M+1A	$= \frac{\pi}{8}(k^2 - 12k + 92)$ 1M
For minimum area, $k = 6$	1A	$\frac{dA}{dk} = \frac{\pi}{4}(k - 6)$ 1M $\frac{dA}{dk} = 0$ at $k = 6$ 1A
Equation of C_1 is $2x^2 + 2y^2 + 2x + 2y - 13 = 0$	1A	$\frac{d^2A}{dk^2} = \frac{\pi}{4}$
	6	$k = 6$ is a min. 1A

Alt. Solution	Marks
The centre of C_1 lies on $y = x$	
Centre of C_1 is $(\frac{4-k}{2}, \frac{k-8}{2})$	1A
The circle is smallest if C_1 lies on $y = x$	
$\frac{4-k}{2} = \frac{k-8}{2}$	2M
$k = 6$	1A
Equation of C_1 is $2x^2 + 2y^2 + 2x + 2y - 13 = 0$	1A

Solution	Marks	Remarks
(b) (i) Let equation of L_1 be $y = mx + 2$	1A	
centre of C_1 is $(-\frac{1}{2}, -\frac{1}{2})$, radius $r = \sqrt{7}$	1A	
Distance from centre to L_1		
$d = \left\lvert \frac{m(\frac{-1}{2}) - (\frac{-1}{2}) + 2}{\sqrt{1 + m^2}} \right\rvert$	1M	
$= \left\lvert \frac{5 - m}{2\sqrt{1 + m^2}} \right\rvert$		
Since $d^2 = r^2 - (\frac{\sqrt{2}}{2})^2$	1M	
$(\frac{5 - m}{2\sqrt{1 + m^2}})^2 = (\sqrt{7})^2 - (\frac{\sqrt{2}}{2})^2$		
$25m^2 + 10m + 1 = 0$		
$(5m + 1)^2 = 0$		
$m = \frac{-1}{5}$		
Equation of L_1 is $y = \frac{-1}{5}x + 2$	1A	$x + 5y - 10 = 0$
Alt. Solution		
Let equation of L_1 be $y = mx + 2$	1A	
Subs. into C_1		
$2x^2 + 2(mx + 2)^2 + 2x + 2(mx + 2) - 13 = 0$	1M	
$(2m^2 + 2)x^2 + (10m + 2)x - 1 = 0$		
Let coordinates of intersecting points be (x_1, y_1), (x_2, y_2)		
$x_1 + x_2 = \frac{-(5m + 1)}{1 + m^2}$, $x_1 x_2 = \frac{-1}{2(1 + m^2)}$	1M	
$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$		
$= (1 + m^2)(x_1 - x_2)^2$	1A	
$= (1 + m^2)[(x_1 + x_2)^2 - 4x_1x_2]$		
$= \frac{(5m + 1)^2}{1 + m^2} + 2 = 2$		
$m = \frac{-1}{5}$		
Equation of L_1 is $y = \frac{-1}{5}x + 2$	1A	

Solution	Marks	Remarks
(ii) The locus is the perpendicular bisector of AB.	2M	
Since AB is a chord of C_1, the perpendicular bisector of AB passes through centre of $C_1(-\frac{1}{2}, -\frac{1}{2})$	2M	
Equation of locus is $y + \frac{1}{2} = 5(x + \frac{1}{2})$		
$y = 5x + 2$	1A 10	
Alt. Solution		
$x^2 + y^2 + x + y - \frac{13}{2} + k(\frac{1}{5}x + y - 2) = 0$	1M	
$x^2 + y^2 + (1 + \frac{k}{5})x + (1 + k)y - (2k + \frac{13}{2}) = 0$		
coordinates of centre is $(\frac{-(1 + k/5)}{2}, \frac{-(k + 1)}{2})$	1M+1A	
Let coordinates of centre be (x, y)		
$\begin{cases} x = -\frac{1}{2}(1 + \frac{k}{5}) \\ y = \frac{-1}{2}(k + 1) \end{cases}$	1M	
Eliminating k,		
$y = 5x + 2$	1A	

Solution	Marks	Remarks
12. (a) Volume $= \pi\int_{-b}^{-(b-h)} x^2 dy$	1A+1A	1A for $\pi\int x^2dy$ 1A for limit
$= \pi\int_{-b}^{-(b-h)} a^2(1 - \frac{y^2}{b^2})dy$	1M	
$= \pi a^2[y - \frac{y^3}{3b^2}]_{-b}^{-(b-h)}$	1A	
$= \pi a^2[-b + h + (\frac{b-h}{3b^2})^3 + b - \frac{b^3}{3b^2}]$		
$= \frac{\pi a^2}{3b^2}h^2(3b - h)$	1	
	5	
(b) (i) Put $a = b = 2$	1M	
$h = 2k$	1M	
Vol. of water $= \frac{\pi}{3}(2k)^2[3(2) - 2k]$		
$= \frac{8\pi}{3}k^2(3 - k)$	1A	
(ii) Depth of object immersed $= \frac{3}{4}k + \frac{1}{4}k$		
$= k$	1A	
Put $a = 1$, $b = h = k$	1M	
Vol. of object immersed $= \frac{\pi}{3k^2}k^2(3k - k)$		
$= \frac{2}{3}\pi k$	1	
$\frac{8\pi}{3}k^2(3 - k) + \frac{2}{3}\pi k = \frac{\pi}{3}(2k + \frac{k}{4})^2[3(2) - (2k + \frac{k}{4})]$	1M+1A	1A for RHS
$8k^2(3 - k) + 2k = k^2(\frac{9}{4})^2(6 - \frac{9k}{4})$		
$128 + 1536k - 512k^2 = 81k(24 - 9k)$		
$217k^2 - 408k + 128 = 0$	2A	
$k = 0.40$ or 1.48 (rejected)	1A	
	11	

Solution	Marks	Remarks
13. (a) By Sine Law		
$\frac{AB}{\sin\theta} = \frac{AQ}{\sin\angle ABQ}$		
$\sin\angle ABQ = \frac{AQ}{AB}\sin\theta$	1A	
$\sin\angle APQ = \frac{AQ}{PQ}$	1A	
$\angle APQ = \angle ABQ$	1A	
$\therefore \frac{AQ}{AB}\sin\theta = \frac{AQ}{PQ}$		
$PQ = \frac{AB}{\sin\theta}$	1A	
	4	
(b) By Cosine Law,		
$AB^2 = AP^2 + BP^2 - 2AP \cdot BP\cos(\pi - \theta)$	1M	
$= AP^2 + BP^2 + 2AP \cdot BP\cos\theta$	1A	
$\therefore PQ = \frac{\sqrt{AP^2 + BP^2 + 2AP \cdot BP\cos\theta}}{\sin\theta}$	1	
	3	
(c) (i) $\cot^2\phi = \frac{PQ^2}{VP^2}$	1A	
$= \frac{AP^2 + BP^2 + 2AP \cdot BP\cos\theta}{VP^2\sin^2\theta}$	1M	
$= \frac{1}{\sin^2\theta}[(\frac{AP}{VP})^2 + (\frac{BP}{VP})^2 + 2(\frac{AP}{VP})(\frac{BP}{VP})\cos\theta]$	1M	
$= \frac{\cot^2\alpha + \cot^2\beta + 2\cot\alpha\cot\beta\cos\theta}{\sin^2\theta}$	1	
(ii) $\cot^2\frac{\pi}{6} = \frac{1}{\sin^2\theta}(\cot^2\frac{\pi}{4} + \cot^2\frac{\pi}{3} + 2\cot\frac{\pi}{4}\cot\frac{\pi}{3}\cos\theta)$		
$3\sin^2\theta = \frac{4}{3} + \frac{2}{\sqrt{3}}\cos\theta$	1A	
$9\cos^2\theta + 2\sqrt{3}\cos\theta - 5 = 0$	1A	
$\cos\theta = \frac{\sqrt{3}}{3}$ or $\frac{-5\sqrt{3}}{9}$	1A	
$\theta = 0.955$ or 2.87 (rejected)	1A	
$\therefore \theta = 0.955$	1A	
	9	