12. (a) Let $\bar{z}$ and $\text{Re}(z)$ denote the conjugate and the real part of a complex number z respectively.

Show that

(i) $z\bar{z}$ is real,

(ii) $z + \bar{z} = 2\text{Re}(z)$.

(2 marks)

(b)

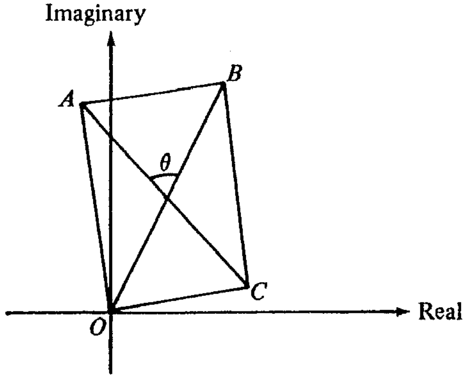


Figure 5

A, B and C are three points in the Argand diagram representing three distinct non-zero complex numbers p, q and r respectively, as shown in Figure 5. Let $p\bar{r} + \bar{p}r = 0$ and $OABC$ be a parallelogram.

(i) Show that $\text{Re}(p\bar{r}) = 0$ and $\text{Re}(\frac{p}{r}) = 0$.

(ii) Show that $OABC$ is a rectangle.

(iii) Let $\frac{p}{r} = 2i$.

Find $\frac{p-r}{p+r}$ in standard form.

Hence find the value of $\tan\theta$, where θ is the angle between the diagonals of $OABC$, as shown in Figure 5.

(14 marks)

END OF PAPER

90-CE
A MATHS
PAPER II

HONG KONG EXAMINATIONS AUTHORITY
HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1990

ADDITIONAL MATHEMATICS PAPER II

11.15 am-1.15 pm (2 hours)
This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.

SECTION A (42 marks)
Answer ALL questions in this section.

1. Given $(1 + 2x - 3x^2)^n = 1 + ax + bx^2 +$ terms involving higher powers of x, where n is a positive integer.

   (a) Express a and b in terms of n.

   (b) If $b = 63$, find the value of n. (5 marks)

2. Let $T_n = n^2 + n$ for any positive integer n.

   Prove, by mathematical induction, that

   $$T_1 + T_2 + \ldots + T_n = \frac{1}{3}n(n+1)(n+2)$$

   for any positive integer n. (5 marks)

3. Using the substitution $u = \sin^2 x$, find

   $$\int \frac{\sin x \cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}\,dx.$$

   (5 marks)

4.

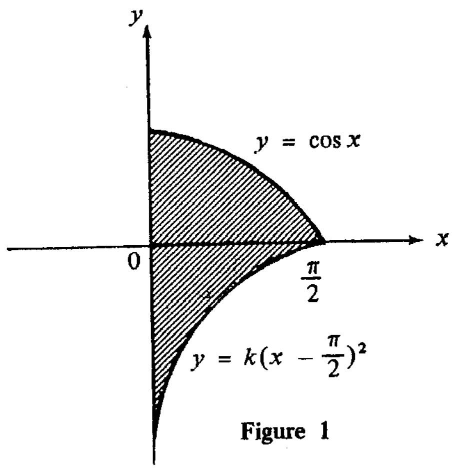


Figure 1

In Figure 1, the shaded area enclosed by the curves $y = \cos x$, $y = k(x - \frac{\pi}{2})^2$ and the y-axis is 2 square units. Find the value of k.

(5 marks)

5. Find the general solution of the equation

   $$2\sin\frac{x}{2}\sin\frac{3x}{2} = 1.$$

   (5 marks)

6. (a) If $\cos\theta + \sqrt{3}\sin\theta = r\cos(\theta - \alpha)$, where $r > 0$ and $0° \leqslant \alpha \leqslant 90°$, find r and α.

   (b) Let $x = \dfrac{1}{\cos\theta + \sqrt{3}\sin\theta + 5}$, find the range of values of x.

   (5 marks)

7.

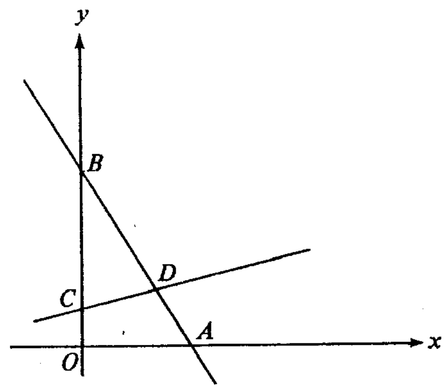


Figure 2

In Figure 2, $A(3, 0)$, $B(0, 5)$ and $C(0, 1)$ are three points and O is the origin. D is a point on AB such that the area of ΔBCD equals half of the area of ΔOAB. Find the equation of the line CD.

(6 marks)

8. S and T are variable points on the lines $y = 0$ and $x - y = 0$ respectively, such that the length of ST is always equal to 2 units. Find the equation of the locus of the mid-point of ST.

   (6 marks)

SECTION B (48 marks)
Answer any THREE questions from this section.
Each question carries 16 marks.

9. (a) (i) Evaluate $\int_0^{\pi} \cos^2 x \, dx$.

(ii) Using the substitution $x = \pi - y$,

evaluate $\int_0^{\pi} x \cos^2 x \, dx$.

(7 marks)

(b) Show that

(i) $\int_{\pi}^{2\pi} x \cos^2 x \, dx = \pi \int_0^{\pi} \cos^2 x \, dx + \int_0^{\pi} x \cos^2 x \, dx$.

(ii) $\int_0^{2\pi} x \cos^2 x \, dx = \pi^2$.

(6 marks)

(c) Using the result of (b) (ii),

evaluate $\int_0^{\sqrt{2\pi}} x^3 \cos^2 x^2 \, dx$.

(3 marks)

10. The equation of a parabola S is $y = x^2 - 2x + 3$.

(a) Let t be the x-coordinate of any point P on S. Find the equation of the tangent to S at P.

(4 marks)

(b)

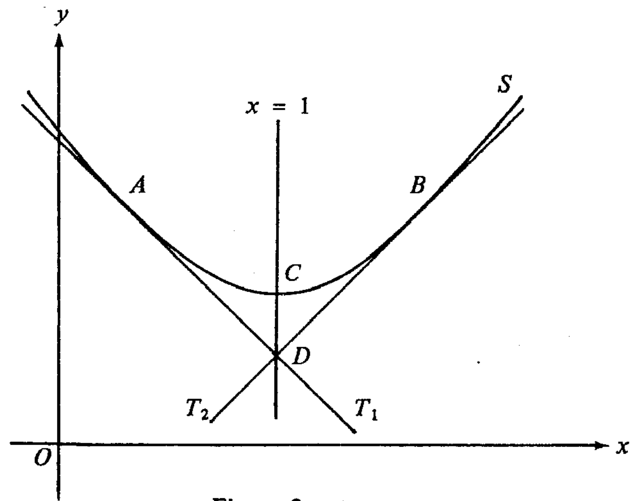


Figure 3

In Figure 3, the x-coordinate of a point A on S is $\frac{1}{3}$.

(i) Find the equation of the tangent T_1 to S at A.

(ii) The line $x = 1$ cuts S and T_1 at points C and D respectively. Find the coordinates of C and D.

(iii) Find the coordinates of another point B on S such that the tangent T_2 to S at B passes through D.

(6 marks)

(c) It is known that the line $x = 1$ bisects $\angle ADB$ and there are two circles each of which passes through C and touches both T_1 and T_2. Find the coordinates of the centres of these circles.

(6 marks)

11. Given the circle $C : 2x^2 + 2y^2 - 4x + 8y - 13 = 0$

and the line $L : x - y = 0$.

(a) Write down the equation of the family of circles passing through the points of intersection of C and L.

If C_1 is the smallest circle of the family, find the equation of C_1.

(6 marks)

(b) (i) A straight line L_1, with slope m and passing through the point $P(0, 2)$, cuts C_1 at points A and B such that $AB = \sqrt{2}$ units. Find the equation of L_1.

(ii) Find the equation of the locus of the centres of the circles passing through A and B. (10 marks)

12. (a)

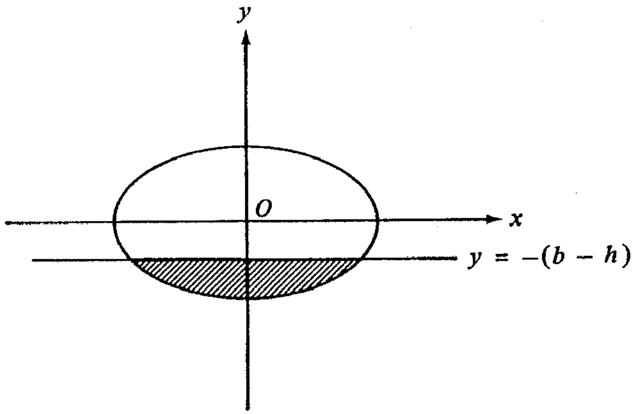


Figure 4(a)

In Figure 4(a), the shaded region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the line $y = -(b - h)$ where $0 < h \leqslant b$ is revolved about the y-axis. Show that the volume of the solid of revolution is $\frac{\pi a^2}{3b^2} h^2 (3b - h)$.

(5 marks)

(b)

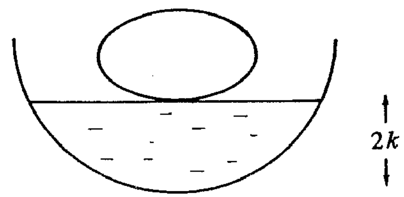


Figure 4(b)

A hemispherical bowl of inner radius 2 units contains water to a depth of $2k$ units.

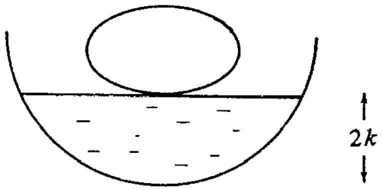
(i) Using (a) or otherwise, find the volume of water in the bowl in terms of k.

(ii) An object in the shape of the solid of revolution of the ellipse $x^2 + \frac{y^2}{k^2} = 1$ about the y-axis is placed with its axis of revolution vertical and its lowest end touching the surface of the water in the bowl as shown in Figure 4(b). The object is now lowered vertically by $\frac{3}{4}k$ units and as a result the water level rises by $\frac{1}{4}k$ units.

By showing that the volume of the immersed part of the object is $\frac{2}{3}\pi k$ cubic units, find the value of k correct to 2 decimal places.

(11 marks)

(b)



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By showing that the volume of the immersed part of the object is $\frac{2}{3}\pi k$ cubic units, find the value of k correct to 2 decimal places.

(11 marks)

13. In Figure 5(a), A, P, B, Q are four points on a circle in a horizontal plane. $\angle AQB = \theta$, $\angle PAQ = \frac{\pi}{2}$.

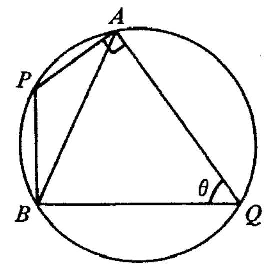

Figure 5(a)

(a) Express $\sin \angle ABQ$ in terms of AB, AQ and θ. Hence find PQ in terms of AB and θ.

(4 marks)

(b) Using the result of (a), show that

$$PQ = \frac{\sqrt{AP^2 + BP^2 + 2AP \cdot BP \cos\theta}}{\sin\theta}.$$

(3 marks)

(c)

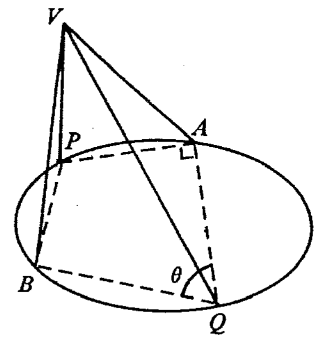

Figure 5(b)

Furthermore, V is a point vertically above P (See Figure 5(b)). Let $\angle VAP = \alpha$, $\angle VBP = \beta$ and $\angle VQP = \phi$.

(i) Using the result of (b), show that

$$\cot^2\phi = \frac{(\cot^2\alpha + \cot^2\beta + 2\cot\alpha\cot\beta\cos\theta)}{\sin^2\theta}.$$

(ii) If $\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$ and $\phi = \frac{\pi}{6}$, find θ, where $\theta < \frac{\pi}{2}$.

(9 marks)

END OF PAPER

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Additional Mathematics I

1. $\frac{1}{4}$

2. (a) $-t\mathbf{i} + (2t + 5)\mathbf{j}$

   (b) (i) -2

   (ii) $2\mathbf{i} + \mathbf{j}$

3. (a) $\cos 2\theta + i\sin 2\theta$

   (b) $\cos\frac{(3k+1)2\pi}{9} + i\sin\frac{(3k+1)2\pi}{9}$, $k = 0, 1, 2$

4. (a) $k + 2, k$

   (b) $0, \frac{-5}{6}$

5. $3 + i$

6. $x > 3$ or $x < -7$ or $-5 < x < 1$

7. $\frac{dy}{dx} = \frac{-(x + 2y)}{2x + 5y}$

   $y = -\frac{1}{2}x + \frac{1}{2}$ and $y = -\frac{1}{2}x - \frac{1}{2}$

8. (b) (i) $\frac{1}{1 + \lambda}(a\mathbf{u} + \lambda b\mathbf{v})$

   (iii) $3 : 5$, $5\mathbf{i} + \frac{5}{2}\mathbf{j}$

9. (a) (i) $f(x) = (x + 2)^2 - 3$

   $(-2, -3)$

   (ii) $2\sqrt{3}$

   (b) (i) $(-2, -3 - m)$

   $g(x) = (x + 2)^2 - (3 + m)$

   (ii) $2\sqrt{m + 3}$

   (iii) 9

   (c) (i) $(-2 + n, -3)$

   $h(x) = (x + 2 - n)^2 - 3$

   (ii) $2 \pm \sqrt{3}$

10. (a) (i) $x > 1$ or $x < -2$

   (ii) $(1, 1)$ is a maximum point.

   $(-2, -\frac{1}{2})$ is a minimum point.

11. (a) $x = 2\sqrt{3}\sin(\frac{\pi}{3} - \theta)$

   (b) $\frac{3\sqrt{3}}{4}$

   (c) (ii) $\frac{1}{2} < \cos(\frac{\pi}{3} - \theta) < 1$

   $\frac{1}{6} < \frac{d\theta}{dt} < \frac{1}{3}$

12. (b) (iii) $\frac{3}{5} + \frac{4}{5}i$, $\frac{4}{3}$

1990

Additional Mathematics II

1. (a) $2n$, $2n^2 - 5n$

   (b) 7

3. $\frac{1}{5}\sqrt{5\sin^2 x + 4} + c$

4. $\frac{-24}{\pi^3}$

5. $2n\pi \pm \frac{\pi}{2}$, $2n\pi \pm \frac{\pi}{3}$

6. (a) $2, 60°$

   (b) $\frac{1}{7} < x < \frac{1}{3}$

7. $y = \frac{7}{15}x + 1$

8. $x^2 - 4xy + 5y^2 - 1 = 0$

9. (a) (i) $\frac{\pi}{2}$

   (ii) $\frac{\pi^2}{4}$

   (c) $\frac{\pi^2}{2}$

10. (a) $y = (2t - 2)x - t^2 + 3$

   (b) (i) $y = -\frac{4}{3}x + \frac{26}{9}$

   (ii) $C(1, 2)$, $D(1, \frac{14}{9})$

   (iii) $B(\frac{5}{3}, \frac{22}{9})$

   (c) $(1, \frac{8}{3})$, $(1, \frac{11}{6})$

11. (a) $2x^2 + 2y^2 - 4x + 8y - 13 + k(x - y) = 0$

   $2x^2 + 2y^2 + 2x + 2y - 13 = 0$

   (b) (i) $y = -\frac{1}{5}x + 2$

   (ii) $y = 5x + 2$

12. (b) (i) $\frac{8}{3}\pi k^2(3 - k)$

   (ii) 0.40

13. (a) $\sin\angle ABQ = \frac{AQ}{AB}\sin\theta$

   $PQ = \frac{AB}{\sin\theta}$

   (c) (ii) 0.955