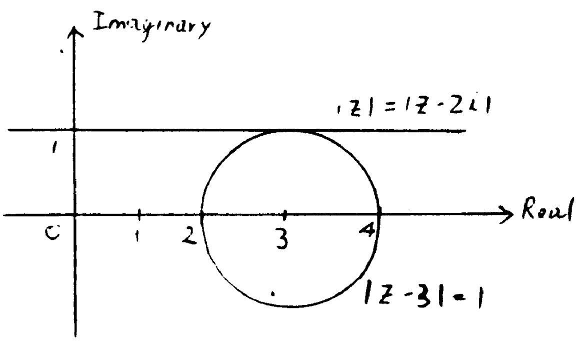
1990 HKCE Additional Mathematics I

Solutions	Marks	Remarks
1. $f'(x) = \sqrt{x^2+k}\dfrac{d}{dx}\sin 2x + \sin 2x\dfrac{d}{dx}\sqrt{x^2+k}$	1M	For product rule.
$= 2\sqrt{x^2+k}\cos 2x + \dfrac{x\sin 2x}{\sqrt{x^2+k}}$	1A+1A	
$f'(0) = 1$		
$2\sqrt{k} = 1$	1M	For substituting $x = 0$ in $f'(x)$
$k = \frac{1}{4}$	1A	
	5	
2. (a) $\vec{AB} = \vec{OB} - \vec{OA}$		Omit vector sign (pp − 1)
$= -\hat{i} + 2\hat{j}$	1A	
$\vec{OP} = \vec{OA} + \vec{AP}$	1M	$\vec{OB} + \vec{BP}$ acceptable
$= \vec{OA} + t\vec{AB}$		
$= -t\hat{i} + (2t+5)\hat{j}$	1A	
Alt. Solution		
$\lvert\vec{AP}\rvert : \lvert\vec{PB}\rvert = t : 1-t$	1A	$\vec{AP} : \vec{PB} = t : 1-t$ (pp − 1)
$\vec{OP} = \dfrac{t\vec{OB} + (1-t)\vec{OA}}{t + (1-t)}$	1M	
$= t(-\hat{i} + 7\hat{j}) + (1-t)(5\hat{j})$		
$= -t\hat{i} + (2t+5)\hat{j}$	1A	
(b) (i) $\vec{OP} \cdot \vec{AB} = 0$	1M	Omit dot sign (pp − 1)
$-t(-1) + (2t+5)(2) = 0$		
$t = -2$	1A	
(ii) $\vec{OP} = 2\hat{i} + \hat{j}$	1A	
	6	

Solutions	Marks	Remarks
3. (a) $\frac{1+i\tan\theta}{1-i\tan\theta} = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} \cdot \frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin\theta}$	1M	Multiplying $\frac{1+i\tan\theta}{1+i\tan\theta}$ acceptable
$= \frac{(\cos^2\theta - \sin^2\theta) + 2\sin\theta\cos\theta i}{\cos^2\theta + \sin^2\theta}$		
$= \cos 2\theta + i\sin 2\theta$	1A	
Alt. Solution		
$\frac{1+i\tan\theta}{1-i\tan\theta} = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$		
$= \frac{\cos\theta + i\sin\theta}{\cos(-\theta) + i\sin(-\theta)}$	1M	Can be omitted
$= \cos 2\theta + i\sin 2\theta$	1A	
(b) $\frac{1+\sqrt{3}i}{1-\sqrt{3}i} = \frac{1+i\tan\frac{\pi}{3}}{1-i\tan\frac{\pi}{3}}$		
$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$	1A	Accept $\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}$ or $\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}$ etc.
Alt. Solution		
$\frac{1+\sqrt{3}i}{1-\sqrt{3}i} = \frac{1+\sqrt{3}i}{1-\sqrt{3}i} \cdot \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$		
$= \frac{1}{2}(-1+\sqrt{3}i) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$	1A	
$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{1/3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{1/3}$		
$= \cos\frac{(3k+1)2\pi}{9} + i\sin\frac{(3k+1)2\pi}{9},$	1A	
$k = 0, 1, 2$	2A	2A for $k = 0, 1, 2$
OR $= \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$		
$\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9},$		
$\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}$ (or $\cos\frac{-4\pi}{9} + i\sin\frac{-4\pi}{9}$)	1A+1A +1A	Angles in degree acceptable
	6	

Solutions	Marks	Remarks
4. (a) $\alpha+\beta = k+2$, $\alpha\beta = k$ } (*)	1A	
(b) $(\alpha + 1)(\beta + 2) = 4$ ----- (1)		
$\alpha\beta + (\alpha+\beta) + \alpha + 2 = 4$		
$k + k + 2 + \alpha + 2 = 4$	1M	For eliminating β.
$\therefore \alpha = -2k$	1	
Subs. into the equation		
$(-2k)^2 - (k + 2)(-2k) + k = 0$	1M	
$6k^2 + 5k = 0$		
$k = 0$ or $-5/6$	1A+1A	
Alt. Solution 1		
Subs. $\alpha = -2k$ into (*)		
$\begin{cases} -2k + \beta = k + 2 \\ -2k\beta = k \end{cases}$	1M	
$k = 0$	1A	
or $\beta = -\frac{1}{2}$		
$-2k - \frac{1}{2} = k + 2$		
$k = \frac{-5}{6}$	1A	
Alt. Solution 2		
Subs. $\alpha = -2k$, $\beta = 3k + 2$ into (1)		
$(-2k + 1)(3k + 2 + 2) = 4$	1M	
$6k^2 + 5k = 0$		
$k = 0$ or $\frac{-5}{6}$	1A+1A	
	6	
5.  Unit circle	1A	Axes or curves not labelled (pp − 1)
Correct centre	1A	
Horizontal straight line	1A	Separate diagrams (pp − 1)
Position correct	1A	
Circle and line touch at correct point.	1A	Solve $\begin{cases}(x - 3)^2 + y^2 = 1 \\ y = 1\end{cases}$ 1A
The intersection is the complex no. $3 + i$	1A	Ans.: $3 + i$ 1A
	6	

Solutions	Marks	Remarks
6. $(x+2)^2 - 8\|x+2\| + 15 \geqslant 0$		
$\|x+2\|^2 - 8\|x+2\| + 15 \geqslant 0$	2M	
$(\|x+2\| - 3)(\|x+2\| - 5) \geqslant 0$	1A	
$\|x+2\| \geqslant 5$ or $\|x+2\| \leqslant 3$	1A	Omit 'or' (pp − 1) use 'and' (no mark)
$(x \geqslant 3$ or $x \leqslant -7)$ or $-5 \leqslant x \leqslant 1$	1A+1A 6	use ',' (pp − 1)
Alt. Solution		
Case (i) $x \geqslant -2$ (or $x > -2$)	1M	Notes : (1) $x \geqslant -2$, $x \leqslant -2$ (deduct no mark)
$(x+2)^2 - 8(x+2) + 15 \geqslant 0$	1A	
$(x-1)(x-3) \geqslant 0$		(2) Solve without stating range of x (no mark)
$x \geqslant 3$ or $x \leqslant 1$		
Since $x \geqslant -2$ $x \geqslant 3$ or $-2 \leqslant x \leqslant 1$	1A	
Case (ii) $x < -2$ (or $x \leqslant -2$)		
$(x+2)^2 + 8(x+2) + 15 \geqslant 0$	1A	
$(x+5)(x+7) \geqslant 0$		
$x \geqslant -5$ or $x \leqslant -7$		
Since $x < -2$, $x \leqslant -7$ or $-2 > x \geqslant -5$	1A	
Combining the 2 cases,		
$x \geqslant 3$ or $x \leqslant -7$ or $-5 \leqslant x \leqslant 1$	1A	
7. $2x + 4y + 4x\frac{dy}{dx} + 10y\frac{dy}{dx} = 0$	1M	For implicit differentiation
$\frac{dy}{dx} = \frac{-(x+2y)}{2x+5y}$	1A	
$\frac{-(x+2y)}{2x+5y} = \frac{-1}{2}$	1M	
$y = 0$	1A	
Subs. into the equation,		
$x = \pm 1$	1A	
The equations are		$x + 2y + 1 = 0$
$y = \frac{-1}{2}x + \frac{1}{2}$ and $y = \frac{-1}{2}x - \frac{1}{2}$	1A+1A 7	$x + 2y - 1 = 0$

Solutions	Marks	Remarks
8. (a) (i) $\vec{x}\cdot\vec{z} = \lvert\vec{x}\rvert\,\lvert\vec{z}\rvert\cos\theta$	1A	Omit vector sign (pp − 1)
$= \lvert\vec{z}\rvert\cos\theta$	1A	Omit dot sign (pp − 1)
$\vec{y}\cdot\vec{z} = \lvert\vec{y}\rvert\,\lvert\vec{z}\rvert\cos\theta$		
$= \lvert\vec{z}\rvert\cos\theta$		
$\vec{x}\cdot\vec{z} = \vec{y}\cdot\vec{z}$	1	
(ii) $\vec{x}\cdot\vec{z} = \vec{x}\cdot(m\vec{x}+n\vec{y})$		
$= m\vec{x}\cdot\vec{x} + n\vec{x}\cdot\vec{y}$		
$= m + n\cos 2\theta$	1A	
$\vec{y}\cdot\vec{z} = \vec{y}\cdot(m\vec{x}+n\vec{y})$		
$= m\cos 2\theta + n\vec{y}\cdot\vec{y}$		
$= m\cos 2\theta + n$	1A	
From (i), $m + n\cos 2\theta = m\cos 2\theta + n$	1M	
$(m-n)(1-\cos 2\theta) = 0$	1A	Accept $(m-n)(1-\vec{x}\cdot\vec{y}) = 0$
$\therefore m = n \quad (\because \cos 2\theta \neq 1)$	1	Accept omitting $\cos 2\theta \neq 1$
	8	
(b) (i) $\vec{OC} = \dfrac{\lambda(b\vec{v}) + (a\vec{u})}{1+\lambda}$	1A	
(ii) Using (a) (ii)		
$\dfrac{a}{\lambda+1} = \dfrac{b\lambda}{\lambda+1}$	1M	
$\lambda = \dfrac{a}{b}$	1	
(iii) $\lvert\vec{OA}\rvert = \sqrt{3^2+4^2} = 5$	1A	$\vec{OA} = 5$ (pp − 1)
$\dfrac{AC}{CB} = \dfrac{5}{25/3}$	1M	
$= \dfrac{3}{5}$	1A	
$\vec{OC} = \dfrac{\frac{3}{5}(\frac{25}{3}\hat{i}) + (3\hat{i}+4\hat{j})}{\frac{3}{5}+1}$	1M	
$= 5\hat{i} + \frac{5}{2}\hat{j}$	1A	
	8	

Solutions	Marks	Remarks
9. (a) (i) $f(x) = x^2 + 4x + 1$		
$= (x + 2)^2 - 3$	1A	
Vertex of C_1 is $(-2, -3)$	1A	
(ii) $x^2 + 4x + 1 = 0$		
$x = -2 \pm \sqrt{3}$	1A	Answer in decimal – no mark
$PQ = (-2 + \sqrt{3}) - (-2 - \sqrt{3})$	1M	For subtraction
$= 2\sqrt{3}$	1A	Accept $PQ = \sqrt{12}$
Alt. Solution		
$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$	1M	
$= (-4)^2 - 4 = 12$	1A	
$PQ = \lvert \alpha - \beta \rvert = 2\sqrt{3}$	1A	Accept $PQ = \alpha - \beta$
	5	
(b) (i) Vertex of C_2 is $(-2, -3 - m)$	1M	
$g(x) = (x + 2)^2 - (3 + m)$	1A	$x^2 + 4x + 1 - m$
(ii) $x^2 + 4x + 1 - m = 0$		
$x = -2 \pm \sqrt{m + 3}$	1A	
$P'Q' = 2\sqrt{m + 3}$	1A	Accept $P'Q' = \sqrt{4m + 12}$
Alt. Solution		
$(\alpha' - \beta')^2 = 4m + 12$	1A	
$P'Q' = \lvert \alpha' - \beta' \rvert = 2\sqrt{m + 3}$	1A	
(iii) $2\sqrt{m + 3} = 2(2\sqrt{3})$	1A	
$m = 9$	1A	
	6	
(c) (i) Vertex of C_3 is $(-2 + n, -3)$	1M	
$h(x) = (x + 2 - n)^2 - 3$	1A	
(ii) $h(0) = 0$		
$0 = (2 - n)^2 - 3$	1M	
$n = 2 \pm \sqrt{3}$	1A+1A	3.73, 0.268
	5	

Solutions	Marks	Remarks
10. (a) (i) $\frac{dy}{dx} = \frac{2(x^2+2) - 2x(2x+1)}{(x^2+2)^2}$	1A	
$= \frac{-2(x^2+x-2)}{(x^2+2)^2}$		
$\frac{-2(x^2+x-2)}{(x^2+2)^2} < 0$	1M	$\leqslant 0$, no mark.
$x^2 + x - 2 > 0$	1A	
$x > 1$ or $x < -2$	1A	
(ii) $\frac{-2(x^2+x-2)}{(x^2+2)^2} = 0$	1M	
$x = 1$ or -2	1A	
$x = 1$, $y = 1$ (1, 1) is a maximum point.	1A	
$x = -2$, $y = \frac{-1}{2}$ $(-2, \frac{-1}{2})$ is a minimum point.	1A 8	
(b) Curve C_1 : Shape	1A	Curve not labelled but position correct - deduct 1 mark only.
Intercepts	1A	
End points	1A	
Turning points	1A 4	Pure plotting without part (a) - no mark.
(c) Curve C_2 : Shape	1A	
Intercepts	1A	
End points	1A	
Turning points $(-2, 1\frac{1}{2})$, (1, 0)	1A 4	

Candidate Number	Centre Number	Seat Number	Total Marks on this page	

10. If you attempt Question 10, fill in the details in the first three boxes above and tie this sheet into your answer book.

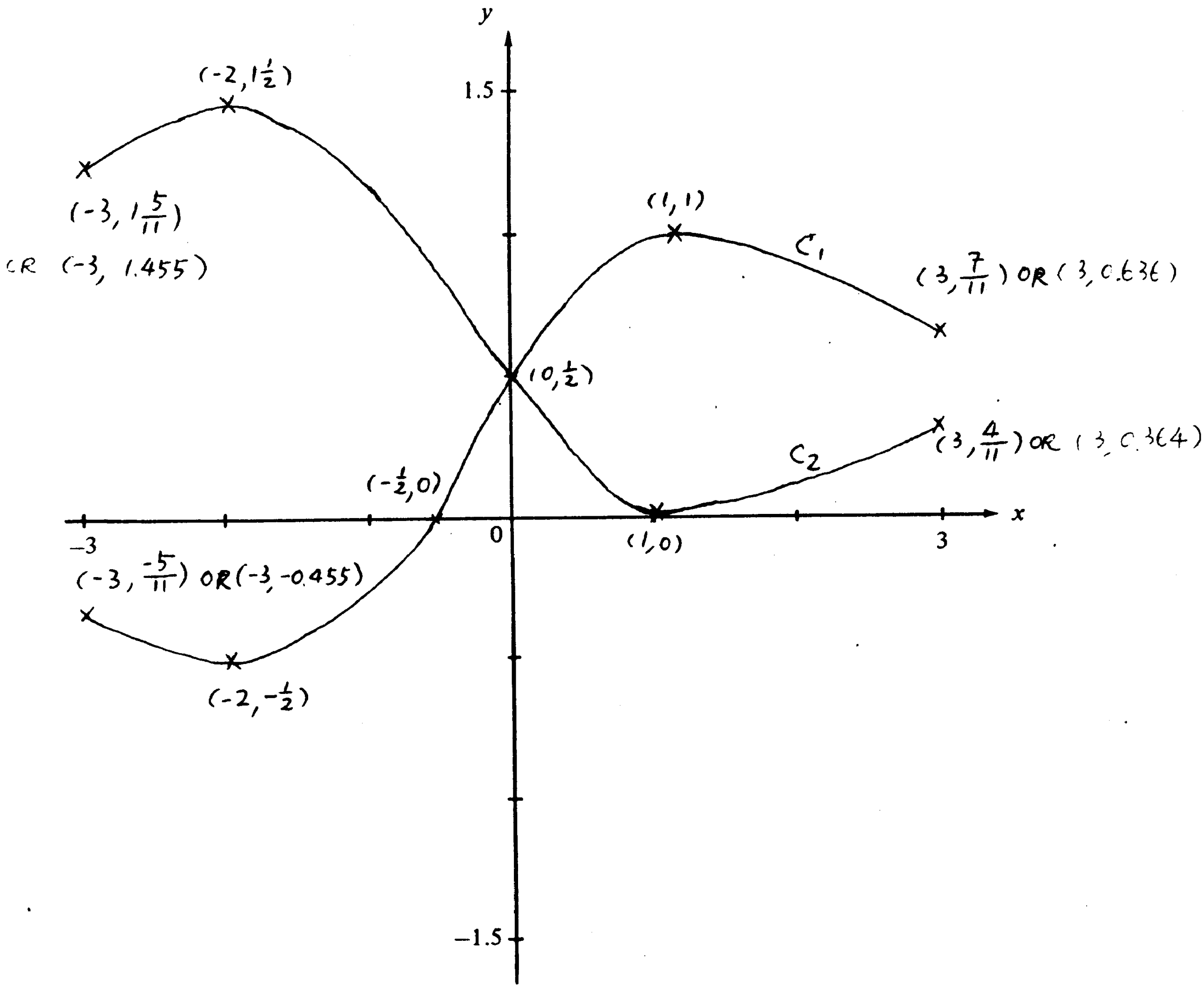


Figure 3

Solutions	Marks	Remarks
11. (a) By Sine Law,		
$\dfrac{x}{\sin(\pi - \frac{2\pi}{3} - \theta)} = \dfrac{3}{\sin\frac{2\pi}{3}}$	1M	
$x = 2\sqrt{3}\sin(\frac{\pi}{3} - \theta)$	1A 2	$x = 3\cos\theta - \sqrt{3}\sin\theta$
(b) $S = \frac{1}{2}3x\sin\theta$	1A	
$= 3\sqrt{3}\sin(\frac{\pi}{3} - \theta)\sin\theta$	1A	$S = \frac{9}{2}\sin\theta\cos\theta - \frac{3\sqrt{3}}{2}\sin^2\theta$ $= -\frac{3\sqrt{3}}{4} + \frac{3\sqrt{3}}{2}\cos(\frac{\pi}{3} - 2\theta)$
$\frac{dS}{d\theta} = 3\sqrt{3}[\cos\theta\sin(\frac{\pi}{3} - \theta) - \sin\theta\cos(\frac{\pi}{3} - \theta)]$		
$= 3\sqrt{3}\sin(\frac{\pi}{3} - 2\theta)$	1	
$\frac{dS}{d\theta} = 0$ when $\theta = \frac{\pi}{6}$ $\quad(\because 0 \le \theta \le \frac{\pi}{3})$	1M+1A	Accept omitting $0 \le \theta \le \frac{\pi}{3}$
$\frac{d^2S}{d\theta^2} = -6\sqrt{3}\cos(\frac{\pi}{3} - 2\theta)$	1A	
$\left.\frac{d^2S}{d\theta^2}\right\|_{\theta = \frac{\pi}{6}} = -6\sqrt{3}$ $\quad\therefore$ max.	1M	Awarded only when the 2nd derivative is correct.
Alt. Solution for checking maximum		
$\frac{dS}{d\theta} > 0$ for $0 < \theta < \frac{\pi}{6}$ $\frac{dS}{d\theta} < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{3}$	1A	for correct ranges of θ
$\theta = \frac{\pi}{6}$ is a maximum	1M	for slope change from +ve to -ve.
$S_{max} = 3\sqrt{3}\sin(\frac{\pi}{3} - \frac{\pi}{6})\sin\frac{\pi}{6}$		
$= \frac{3\sqrt{3}}{4}$.	1A 8	Only awarded if if max. is checked.

Solutions	Marks	Remarks
(c) (i) $x = 2\sqrt{3}\sin(\frac{\pi}{3} - \theta)$		
$\frac{dx}{dt} = -2\sqrt{3}\cos(\frac{\pi}{3} - \theta)\frac{d\theta}{dt}$	1A	
Since $\frac{dx}{dt} = -\frac{\sqrt{3}}{3}$	1A	Omit -ve sign (no mark)
$\frac{d\theta}{dt} = \frac{1}{6\cos(\frac{\pi}{3} - \theta)}$	1	
(ii) $\because$ $0 \leq \theta \leq \frac{\pi}{3}$	1A	
$\frac{1}{2} \leq \cos(\frac{\pi}{3} - \theta) \leq 1$	1A	
$\therefore$ greatest value of $\frac{d\theta}{dt} = \frac{1}{3}$	1A	
least value of $\frac{d\theta}{dt} = \frac{1}{6}$		
	6	

Solutions	Marks	Remarks
12. (a) Let $z = x + yi$		
(i) $z\bar{z} = (x + yi)(x - yi) = x^2 + y^2$ ∴ real	1	
(ii) $z + \bar{z} = (x + yi) + (x - yi) = 2x = 2\text{Re}(z)$	1 / 2	
(b) (i) (1) By (a) (ii)		
$\text{Re}(p\bar{r}) = \frac{1}{2}(p\bar{r} + \overline{p\bar{r}})$	1A	
$= \frac{1}{2}(p\bar{r} + \bar{p}r)$	1A	
$= 0$	1	
(2) $\text{Re}(\frac{p}{r}) = \frac{1}{2}(\frac{p}{r} + (\overline{\frac{p}{r}}))$	1A	$\text{Re}(\frac{p}{r}) = \text{Re}(\frac{p\bar{r}}{r\bar{r}})$ 1A
$= \frac{1}{2}(\frac{p}{r} + \frac{\bar{p}}{\bar{r}})$		$= \frac{\text{Re}(p\bar{r})}{r\bar{r}}$ 1A
$= \frac{1}{2}\,\frac{p\bar{r} + \bar{p}r}{r\bar{r}}$	1A	$= 0$ 1
$= 0$	1	

Alt. Solution

Let $p = a + bi$, $r = c + di$		
(1) $p\bar{r} + \bar{p}r = 0$		
$(a + bi)(c - di) + (a - bi)(c + di) = 0$		
$ac + bd = 0$	1A	
$\text{Re}(p\bar{r}) = ac + bd$	1A	
$= 0$	1	
(2) $\frac{p}{r} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}$	1M	
$= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$		
$\text{Re}(\frac{p}{r}) = \frac{ac + bd}{c^2 + d^2}$	1A	
$= 0$	1	

Solutions	Marks	Remarks
(b) (ii) Method 1		
$\because \operatorname{Re}(\frac{p}{r}) = 0$		
$\arg(\frac{p}{r}) = \pm\frac{\pi}{2}$	1A	Accept omitting ± sign
$\arg p - \arg r = \pm\frac{\pi}{2}$ or $\pm\frac{3\pi}{2}$	1A	Accept omitting $\pm\frac{3\pi}{2}$
$\therefore OA \perp OC$		
$\therefore$ OABC is a rectangle	1	
Method 2		
$\lvert AC\rvert^2 = (p - r)(\overline{p - r})$		
$= p\bar{p} - p\bar{r} - \bar{p}r + r\bar{r}$		
$= p\bar{p} + r\bar{r}$	1A	
$\lvert OB\rvert^2 = q\bar{q} = (p + r)(\overline{p + r}) = p\bar{p} + r\bar{r}$	1A	
$AC = OB$		
$\therefore$ OABC is a rectangle	1	
Alt. Solution		
Method 1		
Slope of $OC = \frac{d}{c}$ $(p = a + bi,\ r = c + di)$	1A	
Slope of $OA = \frac{b}{a}$		
Product of slope $= \frac{d}{c} \cdot \frac{b}{a}$		
$= {}^{-ac}/ac$ (from (i))		
$= -1$	1A	Accept the negligence of considering $a = 0$ or $c = 0$
$OC \perp OA$ $\therefore$ OABC is a rectangle	1	
Method 2		
$OA^2 = a^2 + b^2,\quad OC^2 = c^2 + d^2$	1A	
$AC^2 = (a - c)^2 + (b - d)^2$		
$= a^2 + c^2 + b^2 + d^2 - 2(ac + bd)$		
$= (a^2 + b^2) + (c^2 + d^2)$		
$= OA^2 + OC^2$	1A	
$OA \perp OC$ (Converse of Pythagoras Theorem)		
$\therefore$ OABC is a rectangle.	1	

Solutions	Marks	Remarks
(b) (iii) $p = 2ri$		
$\frac{p-r}{p+r} = \frac{2ri-r}{2ri+r}$		
$= \frac{-1+2i}{1+2i}$	1A	
$= \frac{3}{5} + \frac{4}{5}i$	1A	Accept $\frac{3+4i}{5}$
$\arg(\frac{p-r}{p+r}) = \theta$	1A	$\arg(p-r) - \arg(p+r) = \theta$ (can be omitted)
$\tan\theta = \frac{4/5}{3/5}$	1M	
$= \frac{4}{3}$	1A	
	14	

1990 HKCE Additional Mathematics II

Solution	Marks	Remarks
1. (a) $(1 + 2x - 3x^2)^n$		
$= [1 + x(2 - 3x)]^n$	1M	
$= 1 + nx(2 - 3x) + \frac{n(n-1)}{2}x^2(2 - 3x)^2 + \ldots\ldots$		Deduct 1 mark for missing .....
$= 1 + 2nx + [2n(n-1) - 3n]x^2 + \ldots\ldots$		
$a = 2n$	1A	
$b = 2n^2 - 5n$	1A	
(b) $2n^2 - 5n = 63$	1M	
$(2n + 9)(n - 7) = 0$		
$n = 7$	1A	
	5	
2. For $n = 1$, L.H.S. $= 1^2 + 1 = 2$		
R.H.S. $= \frac{1}{3}(1)(2)(3) = 2$		
the statement is true for $n = 1$	1	
Assume the statement is true for some integer k.	1	
For $n = k + 1$		
L.H.S. $= T_1 + T_2 + \ldots + T_k + T_{k+1}$		
$= \frac{1}{3}k(k+1)(k+2) + (k+1)^2 + (k+1)$	1	
$= \frac{1}{3}(k+1)[k(k+2) + 3(k+1) + 3]$		
$= \frac{1}{3}(k+1)(k^2 + 5k + 6)$		
$= \frac{1}{3}(k+1)(k+2)(k+3)$		
$\therefore$ the statement holds for $n = k + 1$.	1	
By the principle of mathematical induction, the statement holds for all +ve integer n.	1	Awarded if previous steps all correct.
	5	

Solution	Marks	Remarks
3. $du = 2\sin x\cos x dx$	1A	
$\int \frac{\sin x\cos x}{\sqrt{9\sin^2 x + 4\cos^2 x}}dx = \int \frac{1}{2\sqrt{5u+4}}du$	2A	Integrated must be in terms of u
$= \frac{1}{5}\sqrt{5u+4} + c$	1A	
$= \frac{1}{5}\sqrt{5\sin^2 x + 4} + c$	1A	Deduct 1 mark for omitting c
(or $\frac{1}{5}\sqrt{9\sin^2 x + 4\cos^2 x} + c$)	5	
4. $\int_0^{\pi/2} [\cos x - k(x - \frac{\pi}{2})^2]dx$	1A	
$= [\sin x - \frac{k}{3}(x - \frac{\pi}{2})^3]_0^{\pi/2}$	1A	
$= 1 - \frac{k\pi^3}{24} = 2$	1A+1M	
$k = \frac{-24}{\pi^3}$ (-0.774)	1A	
	5	

Alt. Solution

$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2}$		
$= 1$	1A	
$\int_0^{\pi/2} k(x - \frac{\pi}{2})^2 ds = \frac{k}{3}(x - \frac{\pi}{2})^3]_0^{\pi/2}$		
$= \frac{k\pi^3}{24}$	1A	
$1 - \frac{k\pi^3}{24} = 2$	1A+1M	
$k = \frac{-24}{\pi^3}$ (-0.774)	1A	

Solution	Marks	Remarks
5. $2\sin\frac{x}{2}\sin\frac{3x}{2} = 1$		
$\cos x - \cos 2x = 1$	1A	
$\cos x - (2\cos^2 x - 1) = 1$	1A	
$2\cos^2 x - \cos x = 0$		
$\cos x = 0$ or $\frac{1}{2}$	1A	
$x = 2n\pi \pm \frac{\pi}{2}$ $\quad (\frac{2n+1}{2}\pi)$	1A	$360n° \pm 90°$, $(2n+1)\ 90°$
or $2n\pi \pm \frac{\pi}{3}$ where $n \in Z$	1A	$360n° \pm 60°$
	5	use different units (pp − 1)

Alt. Solution

Solution	Marks
Let $\sin\frac{x}{2} = t$	
$t(3t - 4t^3) = \frac{1}{2}$	1A
$8t^4 - 6t^2 + 1 = 0$	1A
$(2t^2 - 1)(4t^2 - 1) = 0$	
$t = \pm\frac{\sqrt{2}}{2}$ or $\pm\frac{1}{2}$	1A
$\frac{x}{2} = n\pi \pm \frac{\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$	
$x = 2n\pi \pm \frac{\pi}{2}$ or $2n\pi \pm \frac{\pi}{3}$	1A+1A

Solution	Marks	Remarks
6. (a) $r = \sqrt{1^2 + (\sqrt{3})^2} = 2$	1A	
$\tan = \sqrt{3}$ $\quad \therefore \alpha = 60°$	1A	no mark if in radian
(b) $x = \frac{1}{2\cos(\theta - 60°) + 5}$		
$-1 \leq \cos(\theta - 60°) \leq 1$	1M	
$\therefore \frac{1}{7} \leq x \leq \frac{1}{3}$	1A+1A	
	5	

Solution	Marks	Remarks
7. Equation of CD : $y = mx+1$ ----- (1)	1A	
Equation of AB : $\frac{x}{3} + \frac{y}{5} = 1$ ----- (2)	1A	
Subs. (1) into (2) : $\frac{x}{3} + \frac{mx+1}{5} = 1$		
$x = \frac{12}{5+3m}$	1A	
Area of $\triangle BCD = \frac{1}{2}(5-1)\left(\frac{12}{5+3m}\right) = \frac{24}{5+3m}$	1A	
$\frac{24}{5+3m} = \frac{1}{2} \cdot \frac{15}{2}$	1M	
$m = \frac{7}{15}$		
$\therefore$ Equation of CD is $y = \frac{7x}{15} + 1$	1A	$7x - 15y + 15 = 0$
	6	
Alt. Solution		
Let coordinates of D be (x, y)		
$\frac{4x}{2} = \frac{1}{2} \cdot \frac{15}{2}$	1M	
$x = \frac{15}{8}$	1A	
Equation of AB : $\frac{x}{3} + \frac{y}{5} = 1$	1A	$\frac{y}{\frac{15}{8} - 3} = \frac{5}{-3}$ 1A
Subs. $x = \frac{15}{8}$, $y = \frac{15}{8}$	1A	$y = \frac{15}{8}$ 1A
$\therefore$ Equation of CD		
$\frac{y-1}{x} = \frac{\frac{15}{8} - 1}{15/8}$	1M	
$y = \frac{7}{15}x + 1$	1A	

Solution	Marks	Remarks
8. Let coordinates of S and T be (a, 0), (b, b) respectively	1A	
coordinates of mid-point is $(\frac{a+b}{2}, \frac{b}{2})$	1A	
Let $x = \frac{a+b}{2}$, $y = \frac{b}{2}$		
$b = 2y$, $a = 2(x - y)$	1M	For making a, b as subjects
$(a - b)^2 + (b - 0)^2 = 4$	1M	
$(2x - 4y)^2 + (2y)^2 = 4$	1A	
$(x - 2y)^2 + y^2 = 1$		
$x^2 - 4xy + 5y^2 - 1 = 0$	1A	
	6	
Alt. Solution		
Let coordinates of P be (x, y)		
then coordinates of T is (2y, 2y)	1A	
coordinates of S is (2x − 2y, 0)	2A	
$(2x - 4y)^2 + 4y^2 = 4$	1M+1A	
$x^2 - 4xy + 5y^2 - 1 = 0$	1A	

Solution	Marks	Remarks
9. (a) (i) $\int_0^\pi \cos^2 x dx = \int_0^\pi \frac{1}{2}(1+\cos 2x)dx$	1A	
$= [\frac{1}{2}(x+\frac{\sin 2x}{2})]_0^\pi$	1A	
$= \pi/2$	1A	
(ii) Put $x = \pi - y$		
$\int_0^\pi x\cos^2 x dx = \int_\pi^0 (\pi - y)\cos^2(\pi - y) - dy$	1A	
$= \pi\int_0^\pi \cos^2 y dy - \int_0^\pi y\cos^2 y dy$	1M	For separating into 2 integrals
$2\int_0^\pi x\cos^2 x dx = \pi\int_0^\pi \cos^2 x dx$	1M	
$= \pi^2/2$		
$\therefore \int_0^\pi x\cos^2 x dx = \pi^2/4$	1A	
	7	
(b) (i) Put $x = \pi + y$	1A	
$\int_\pi^{2\pi} x\cos^2 x dx = \int_0^\pi (\pi + y)\cos^2(\pi + y)dy$	1A	
$= \pi\int_0^\pi \cos^2 y dy + \int_0^\pi y\cos^2 y dy$		
$= \pi\int_0^\pi \cos^2 x dx + \int_0 x\cos^2 x dx$	1	
(ii) $\int_0^{2\pi} x\cos^2 x dx = \int_0^\pi x\cos^2 x dx + \int_\pi^{2\pi} x\cos^2 x dx$	1A	
$= \int_0^\pi x\cos^2 x dx + \pi\int_0^\pi \cos^2 x dx + \int_0^\pi x\cos^2 x dx$	1M	For subs. (b)(i)
$= \frac{\pi^2}{4} + \pi(\frac{\pi}{2}) + \frac{\pi^2}{4}$		
$= \pi^2$	1	
	6	
(c) Put $x^2 = y$	1A	
$2xdx = dy$		
$\int_0^{\sqrt{2\pi}} x^3\cos^2 x^2 dx = \int_0^{2\pi} y\cos^2 y.\frac{1}{2}dy$	1A	
$= \frac{1}{2}\int_0^{2\pi} y\cos^2 y dy$		
$= \frac{\pi^2}{2}$	1A	
	3	

Solution	Marks	Remarks
10. (a) $\left.\dfrac{dy}{dx}\right\rvert_{x=t} = 2t - 2$	1A	
y-coordinates of $P = t^2 - 2t + 3$	1A	
Equation of tangent : $y - (t^2 - 2t + 3)$		
$= (2t - 2)(x - t)$	1M	
$y = (2t - 2)x - t^2 + 3 \quad \ldots(*)$	1A	
	4	
Alt. Solution		
Using the formula $\dfrac{y + y_1}{2} = xx_1 - (x + x_1) + 3$		
Equation of tangent : $\dfrac{y + (t^2 - 2t + 3)}{2}$		
$= tx - (t + x) + 3$	1M+1A +1A	1A for $y_1 = t^2 - 2t + 3$
$y = (2t - 2)x - t^2 + 3$	1A	
(b) (i) Put $t = \dfrac{1}{3}$ in (*)		
Equation of T_1 : $y = \dfrac{-4}{3}x + \dfrac{26}{9}$	1A	
(ii) Coordinates of C : (1, 2)	1A	
Coordinates of D : $(1, \dfrac{14}{9})$	1A	
(iii) Subs. $(1, \dfrac{14}{9})$ into (*)		
$\dfrac{14}{9} = 2t - 2 - t^2 + 3$	1M	
$9t^2 - 18t + 5 = 0$		
$t = \dfrac{1}{3}$ or $\dfrac{5}{3}$		
$\therefore$ x-coordinate of B $= \dfrac{5}{3}$	1A	
y-coordinate of B $= (\dfrac{5}{3})^2 - 2(\dfrac{5}{3}) + 3 = \dfrac{22}{9}$		
Coordinates of B $= (\dfrac{5}{3}, \dfrac{22}{9})$	1A	
	6	

Solution	Marks	Remarks
Alt. Solution		
(iii) Since S is symmetrical about x = 1 and x-coordinate of A = $\frac{1}{3}$, ∴ by symmetry x coordinate of B = $1 + (1 - \frac{1}{3}) = \frac{5}{3}$	1M+1A	
coordinates of B = $(\frac{5}{3}, \frac{22}{9})$	1A	
(c) Centre of circle lies on x = 1		
let its coordinates be (1, a)	1A	
Radius = Distance to T_1		
$= \left\lvert \dfrac{-\frac{4}{3} - a + \frac{26}{9}}{\sqrt{1 + (\frac{4}{3})^2}} \right\rvert$		
$= \left\lvert \dfrac{14 - 9a}{15} \right\rvert$	1A	
Since the circles pass through C (1, 2)		
Radius = $\lvert 2 - a \rvert$	1M	
$\lvert 2 - a \rvert = \left\lvert \dfrac{14 - 9a}{15} \right\rvert$	1M	
$a = \frac{8}{3}$ or $\frac{11}{6}$		
Coordinates of centres are $(1, \frac{8}{3})$ or $(1, \frac{11}{6})$	1A+1A	
	6	

Solution	Marks	Remarks
11. (a) Equation of family of circles		
$2x^2 + 2y^2 - 4x + 8y - 13 + k(x - y) = 0$	1A	$x^2 + y^2 - 2x + 4y - \frac{13}{2} + k(x - y) = 0$
$2x^2 + 2y^2 + (k - 4)x + (8 - k)y - 13 = 0$		$(x - y) + k(2x^2 + 2y^2 - 4x + 8y - 13) = 0$
$(\text{Radius})^2 = (\frac{k-4}{4})^2 + (\frac{8-k}{4})^2 + \frac{13}{2}$	1M	Area $A = \pi r^2$
$= \frac{1}{8}(k - 6)^2 + 7$	1M+1A	$= \frac{\pi}{8}(k^2 - 12k + 92)$ 1M
For minimum area, $k = 6$	1A	$\frac{dA}{dk} = \frac{\pi}{4}(k - 6)$ 1M $\frac{dA}{dk} = 0$ at $k = 6$ 1A
Equation of C_1 is $2x^2 + 2y^2 + 2x + 2y - 13 = 0$	1A 6	$\frac{d^2A}{dk^2} = \frac{\pi}{4}$ $k = 6$ is a min. 1A
Alt. Solution		
The centre of C_1 lies on $y = x$		
Centre of C_1 is $(\frac{4-k}{2}, \frac{k-8}{2})$	1A	
The circle is smallest if C_1 lies on $y = x$		
$\frac{4-k}{2} = \frac{k-8}{2}$	2M	
$k = 6$	1A	
Equation of C_1 is $2x^2 + 2y^2 + 2x + 2y - 13 = 0$	1A	

Solution	Marks	Remarks
(b) (i) Let equation of L_1 be $y = mx + 2$	1A	
centre of C_1 is $(-\frac{1}{2}, -\frac{1}{2})$, radius $r = \sqrt{7}$	1A	
Distance from centre to L_1		
$d = \left\lvert \frac{m(\frac{-1}{2}) - (\frac{-1}{2}) + 2}{\sqrt{1 + m^2}} \right\rvert$	1M	
$= \left\lvert \frac{5 - m}{2\sqrt{1 + m^2}} \right\rvert$		
Since $d^2 = r^2 - (\frac{\sqrt{2}}{2})^2$	1M	
$(\frac{5 - m}{2\sqrt{1 + m^2}})^2 = (\sqrt{7})^2 - (\frac{\sqrt{2}}{2})^2$		
$25m^2 + 10m + 1 = 0$		
$(5m + 1)^2 = 0$		
$m = \frac{-1}{5}$		
Equation of L_1 is $y = \frac{-1}{5}x + 2$	1A	$x + 5y - 10 = 0$
Alt. Solution		
Let equation of L_1 be $y = mx + 2$	1A	
Subs. into C_1		
$2x^2 + 2(mx + 2)^2 + 2x + 2(mx + 2) - 13 = 0$	1M	
$(2m^2 + 2)x^2 + (10m + 2)x - 1 = 0$		
Let coordinates of intersecting points be (x_1, y_1), (x_2, y_2)		
$x_1 + x_2 = \frac{-(5m + 1)}{1 + m^2}$, $x_1x_2 = \frac{-1}{2(1 + m^2)}$	1M	
$AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$		
$= (1 + m^2)(x_1 - x_2)^2$	1A	
$= (1 + m^2)[(x_1 + x_2)^2 - 4x_1x_2]$		
$= \frac{(5m + 1)^2}{1 + m^2} + 2 = 2$		
$m = \frac{-1}{5}$		
Equation of L_1 is $y = \frac{-1}{5}x + 2$	1A	

Solution	Marks	Remarks
(ii) The locus is the perpendicular bisector of AB.	2M	
Since AB is a chord of C_1, the perpendicular bisector of AB passes through centre of $C_1(-\frac{1}{2}, -\frac{1}{2})$	2M	
Equation of locus is $y + \frac{1}{2} = 5(x + \frac{1}{2})$		
$y = 5x + 2$	1A 10	

Alt. Solution

Alt. Solution	Marks
$x^2 + y^2 + x + y - \frac{13}{2} + k(\frac{1}{5}x + y - 2) = 0$	1M
$x^2 + y^2 + (1 + \frac{k}{5})x + (1 + k)y - (2k + \frac{13}{2}) = 0$	
coordinates of centre is $(\frac{-(1 + k/5)}{2}, \frac{-(k + 1)}{2})$	1M+1A
Let coordinates of centre be (x, y)	
$\begin{cases} x = -\frac{1}{2}(1 + \frac{k}{5}) \\ y = \frac{-1}{2}(k + 1) \end{cases}$	1M
Eliminating k,	
$y = 5x + 2$	1A

Solution	Marks	Remarks
12. (a) Volume $= \pi\int_{-b}^{-(b-h)} x^2dy$	1A+1A	1A for $\pi\int x^2dy$ 1A for limit
$= \pi\int_{-b}^{-(b-h)} a^2(1 - \frac{y^2}{b^2})dy$	1M	
$= \pi a^2[y - \frac{y^3}{3b^2}]_{-b}^{-(b-h)}$	1A	
$= \pi a^2[-b + h + \frac{(b-h)^3}{3b^2} + b - \frac{b^3}{3b^2}]$		
$= \frac{\pi a^2}{3b^2}h^2(3b - h)$	1	
	5	
(b) (i) Put $a = b = 2$	1M	
$h = 2k$	1M	
Vol. of water $= \frac{\pi}{3}(2k)^2[3(2) - 2k]$		
$= \frac{8\pi}{3}k^2(3 - k)$	1A	
(ii) Depth of object immersed $= \frac{3}{4}k + \frac{1}{4}k$		
$= k$	1A	
Put $a = 1$, $b = h = k$	1M	
Vol. of object immersed $= \frac{\pi}{3k^2}k^2(3k - k)$		
$= \frac{2}{3}\pi k$	1	
$\frac{8\pi}{3}k^2(3 - k) + \frac{2}{3}\pi k = \frac{\pi}{3}(2k + \frac{k}{4})^2[3(2) - (2k + \frac{k}{4})]$	1M+1A	1A for RHS
$8k^2(3 - k) + 2k = k^2(\frac{9}{4})^2(6 - \frac{9k}{4})$		
$128 + 1536k - 512k^2 = 81k(24 - 9k)$		
$217k^2 - 408k + 128 = 0$	2A	
$k = 0.40$ or 1.48 (rejected)	1A	
	11	

Solution	Marks	Remarks
13. (a) By Sine Law		
$\dfrac{AB}{\sin\theta} = \dfrac{AQ}{\sin\angle ABQ}$		
$\sin\angle ABQ = \dfrac{AQ}{AB}\sin\theta$	1A	
$\sin\angle APQ = \dfrac{AQ}{PQ}$	1A	
$\angle APQ = \angle ABQ$	1A	
$\therefore \dfrac{AQ}{AB}\sin\theta = \dfrac{AQ}{PQ}$		
$PQ = \dfrac{AB}{\sin\theta}$	1A 4	
(b) By Cosine Law,		
$AB^2 = AP^2 + BP^2 - 2AP \cdot BP\cos(\pi - \theta)$	1M	
$= AP^2 + BP^2 + 2AP \cdot BP\cos\theta$	1A	
$\therefore PQ = \dfrac{\sqrt{AP^2 + BP^2 + 2AP \cdot BP\cos\theta}}{\sin\theta}$	1 3	
(c) (i) $\cot^2\phi = \dfrac{PQ^2}{VP^2}$	1A	
$= \dfrac{AP^2 + BP^2 + 2AP \cdot BP\cos\theta}{VP^2\sin^2\theta}$	1M	
$= \dfrac{1}{\sin^2\theta}[(\dfrac{AP}{VP})^2 + (\dfrac{BP}{VP})^2 + 2(\dfrac{AP}{VP})(\dfrac{BP}{VP})\cos\theta]$	1M	
$= \dfrac{\cot^2\alpha + \cot^2\beta + 2\cot\alpha\cot\beta\cos\theta}{\sin^2\theta}$	1	
(ii) $\cot^2\dfrac{\pi}{6} = \dfrac{1}{\sin^2\theta}(\cot^2\dfrac{\pi}{4} + \cot^2\dfrac{\pi}{3} + 2\cot\dfrac{\pi}{4}\cot\dfrac{\pi}{3}\cos\theta)$		
$3\sin^2\theta = \dfrac{4}{3} + \dfrac{2}{\sqrt{3}}\cos\theta$	1A	
$9\cos^2\theta + 2\sqrt{3}\cos\theta - 5 = 0$	1A	
$\cos\theta = \dfrac{\sqrt{3}}{3}$ or $\dfrac{-5\sqrt{3}}{9}$	1A	
$\theta = 0.955$ or 2.87 (rejected)	1A	
$\therefore \theta = 0.955$	1A 9	