1989 PAPER I

Solution	Marks	Remarks
$\frac{dy}{dx} = \sin 5x + 5x\cos 5x$	2A	
$\frac{d^2y}{dx^2} = 5\cos 5x + 5\cos 5x - 25x\sin 5x$		
$= 10\cos 5x - 25x\sin 5x$	1A	
$\frac{d^2y}{dx^2} + 25y = 10\cos 5x$	1A	
	4	

Solution	Marks	Remarks
2. (a) $\overrightarrow{OC} = \frac{\underset{\sim}{i} + 3\underset{\sim}{j} + k(4\underset{\sim}{i} - \underset{\sim}{j})}{k + 1}$	1A	omit vector sign(pp-1)
$= \frac{1}{k+1}[(4k+1)\underset{\sim}{i} + (3-k)\underset{\sim}{j}]$	1A	
(b) $\overrightarrow{AB} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$	1A	Alt. Solution:
$OC \perp AB$		slope of AB $= -\frac{4}{3}$
$\overrightarrow{OC} \cdot \overrightarrow{AB} = 0$ (omit dot sign pp-1)		slope of OC $= \frac{3-k}{4k+1}$
$3 \cdot \frac{4k+1}{k+1} + (-4)(\frac{3-k}{k+1}) = 0$	1M	$(-\frac{4}{3})(\frac{3-k}{4k+1}) = -1$ 1M
$k = \frac{9}{16}$	1A	$k = \frac{9}{16}$ 1A
	5	

Solution	Marks	Remarks
3. $y = x^3$		
$y' = 3x^2$		
$3x^2 = \frac{3}{4}$	1M	
$x = \pm\frac{1}{2}$		
$x = \frac{1}{2}, y = \frac{1}{8}$	1A	
$x = -\frac{1}{2}, y = -\frac{1}{8}$	1A	
$\frac{y - \frac{1}{8}}{x - \frac{1}{2}} = \frac{3}{4}$		
$y = \frac{3}{4}x - \frac{1}{4}$ or $3x - 4y - 1 = 0$	1A	
$\frac{y + \frac{1}{8}}{x + \frac{1}{2}} = \frac{3}{4}$		
$y = \frac{3}{4}x + \frac{1}{4}$ or $3x - 4y + 1 = 0$	1A	
	5	

Solution	Marks	Remarks
4. (a) $\tan(\theta - x)$		
$= \frac{\tan\theta - \tan x}{1 + \tan\theta\tan x}$	1A	
$= \frac{\tan x}{1 + 2\tan^2 x}$	1A	
(b) $\frac{dy}{dx} = \frac{(1 + 2\tan^2 x)\sec^2 x - \tan x \cdot 4\tan x\sec^2 x}{(1 + 2\tan^2 x)^2}$	1M	For quotient rule.
$= \frac{\sec^2 x - 2\tan^2 x\sec^2 x}{(1 + 2\tan^2 x)^2}$		
$= 0$		
$\tan^2 x = \frac{1}{2}$	1A	
(or equivalent answers such as $\sec^2 x = \frac{3}{2}$, $\cos^2 x = \frac{2}{3}$, $\sin^2 x = \frac{1}{3}$)		
$x = 0.615$ (0.61548) 0.196π	1A	Do not accept answers given in degrees.
	5	
5. $\frac{x^2 + 5x + 1}{x^2 - x + 1} = r$	1A	
$(r - 1)x^2 - (r + 5)x + (r - 1) = 0$		
or $(1 - r)x^2 + (r + 5)x + (1 - r) = 0$	1A	?
$D = (r + 5)^2 - 4(1 - r)^2$		
For real values of x, $(r + 5)^2 - 4(1 - r)^2 \geq 0$	1M	(1M for using $D \geq 0$)
$r^2 + 10r + 25 - 4r^2 + 8r - 4 \geq 0$		
$3r^2 - 18r - 21 \leq 0$		
$r^2 - 6r - 7 \leq 0$		
$(r + 1)(r - 7) \leq 0$ or $(1 + r)(7 - r) \geq 0$	1A	
$7 \geq r \geq -1$	1A	
	5	
6. $(p + qi)^2 = 21 - 20i$		
$p^2 + 2pqi - q^2 = 21 - 20i$		
$p^2 - q^2 = 21$, $2pq = -20$	1M+1A	
Solving,		Alt. Solution:
$p^2 - \frac{100}{p^2} = 21$		$\frac{100}{q^2} - q^2 = 21$
$p^4 - 21p^2 - 100 = 0$	1A	$q^4 + 21q^2 - 100 = 0$
$(p^2 + 4)(p^2 - 25) = 0$		$(q^2 - 4)(q^2 + 25) = 0$
$p = \pm 5$	1A	$q = \pm 2$
$p = 5, q = -2$; $p = -5, q = 2$	1A	
The two square roots are $5 - 2i$ and $-5 + 2i$.	1A	
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Solutions	Marks	Remarks
7. a) $\dfrac{1-\sin\theta+i\cos\theta}{1-\sin\theta-i\cos\theta}$		Alt. Solution:
$=\dfrac{(1-\sin\theta+i\cos\theta)(1-\sin\theta+i\cos\theta)}{(1-\sin\theta-i\cos\theta)(1-\sin\theta+i\cos\theta)}$ for correct conjugate only	1M	$(1-s-ic)i(c+is)$ 1M
$=\dfrac{(1-\sin\theta)^2-\cos^2\theta+2\cos\theta(1-\sin\theta)i}{(1-\sin\theta)^2+\cos^2\theta}$		$=(-s+s^2+c^2)+i(c-cs+sc)$
$=\dfrac{(2\sin^2\theta-2\sin\theta)+2\cos\theta(1-\sin\theta)i}{2-2\sin\theta}$		$=1-s+ci$ 1
$=-\sin\theta+i\cos\theta$ $=i(\cos\theta+i\sin\theta)$	1	
b) $\left\{\dfrac{1-\sin\frac{7\pi}{36}+i\cos\frac{7\pi}{36}}{1-\sin\frac{7\pi}{36}-i\cos\frac{7\pi}{36}}\right\}^6=i^6\left(\cos\frac{7\pi}{36}+i\sin\frac{7\pi}{36}\right)^6$	1A	
$=-\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$	1M+1A	1M for DeMoivre's Thm. 1A for $(i)^6=-1$
$=\frac{\sqrt{3}}{2}+\frac{1}{2}i$	1 6	
8. $(x-2)^2-5\lvert x-2\rvert-6=0$		
Solution (1):		
$(x-2)^2=\lvert x-2\rvert^2$	1M	
$\lvert x-2\rvert^2-5\lvert x-2\rvert-6=0$		
$(\lvert x-2\rvert+1)(\lvert x-2\rvert-6)=0$	1A	
$\lvert x-2\rvert=-1$ or $\lvert x-2\rvert=6$		
No solution or ($x=-4$ or 8) $\therefore x=-4$ or 8	2A+1A +1A 6	
Solution (2):		
2 cases, (i) $x\geq 2$ (ii) $x<2$	1M	
Case (i) $x\geq 2$,		Notes: (1) $x\geq 2$, $x\leq 2$ (deduct no mark) (2) $x>2$, $x<2$ (pp-1) (3) missing 2 cases, ~~(pp-1)~~ (4) only 1 case without stating range of x (no mark)
$(x-2)^2-5(x-2)-6=0$	1A	
$[(x-2)-6][(x-2)+1]=0$ or $x^2-9x+8=0$		
$x=1$ or 8	1A	
Rejecting $x=1$, $x=8$	1A	
Case (ii) $x<2$,		
$(x-2)^2+5(x-2)-6=0$	1A	
$[(x-2)+6][(x-2)-1]=0$		
$x=3$ or -4		
Rejecting $x=3$, $x=-4$	1A	
$x=-4$ or 8		

Solutions	Marks	Remarks
8. Solution (3):		
$(x-2)^2-5\lvert x-2\rvert-6=0$		
$x-2=u$		
$u^2-5\lvert u\rvert-6=0$		
$u^2-6=5\lvert u\rvert$		
$u^4-12u^2+36=25u^2$	1M	
$u^4-37u^2+36=0$		
$(u^2-1)(u^2-36)=0$	1A	
$u=\pm 1$, $u=\pm 6$		
$x=2+u$		
$x=3$ or 1 $x=8$ or -4	2A	
Rejecting, $x=8$ or -4 $x\neq 1, -4$	2A	

$\alpha\beta = q$) 1A

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= p^2 - 2q$ 1A

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$ 1A

$= -p(p^2 - 2q - q)$

$= -p(p^2 - 3q)$ $-p^3+3pq$ 1A

Alternative: $(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= -p^3 + 3pq$

$= -p(p^2 - 3q)$

(iii) $(\alpha^2 - \beta - 1)(\beta^2 - \alpha - 1)$

$= \alpha^2\beta^2 - \alpha^3 - \alpha^2 - \beta^3 + \alpha\beta + \beta - \beta^2 + \alpha + 1$

$= (\alpha\beta)^2 - (\alpha^3 + \beta^3) - (\alpha^2 + \beta^2) + \alpha\beta + (\alpha + \beta) + 1$ 1M

$= q^2 + p(p^2 - 3q) - (p^2 - 2q) + q - p + 1$ 1A

$= p^3 - 3pq + q^2 - p^2 + 3q - p + 1$

$= q^2 - 3(p - 1)q + p^3 - p^2 - p + 1$

$= q^2 - 3(p - 1)q + (p - 1)^2(p + 1)$

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(b) If the square of one root of (*) minus the other root equals 1,

i.e. $\alpha^2 - \beta = 1$ or $\beta^2 - \alpha = 1$ 1A Accept omitting either one

$(\alpha^2 - \beta - 1)(\beta^2 - \alpha - 1) = 0$ 1A

From (a)(iii), $q^2 - 3(p - 1)q + (p - 1)^2(p + 1) = 0$ 1

3

(c) $D = 9(p - 1)^2 - 4(p - 1)^2(p + 1)$ 1A

≥ 0 1M no mark if omitting equality sign

$(p - 1)^2 \geq 0$

$\therefore\ 9 - 4(p + 1) \geq 0$ 1A

$p \leq \frac{5}{4}$ 1A

4

(d) $4x^2 + 5x + k = 0$

$x^2 + \frac{5}{4}x + \frac{k}{4} = 0$ or $p = \frac{5}{4}$, $q = \frac{k}{4}$ 1A

(**) becomes $(\frac{k}{4})^2 - (\frac{3}{4})(\frac{k}{4}) + (\frac{1}{4})^2(\frac{9}{4}) = 0$

or $k^2 - 3k + \frac{9}{4} = 0$ 1A

$k = \frac{3}{2}$ 1A

12. (a) (i) $T = k\sqrt{a^2 + x^2} + (3a - x)$ 2A

(ii) $\frac{dT}{dx} = \frac{kx}{\sqrt{a^2 + x^2}} - 1$ 1A

$= 0$ 1M

$k^2x^2 = a^2 + x^2$

$x^2 = \frac{a^2}{k^2 - 1}$

$x = \frac{a}{\sqrt{k^2 - 1}}$ $x = \frac{a}{\sqrt{3}}$ 1A

$k = 2$, $\frac{d^2T}{dx^2} = \frac{2\sqrt{x^2 + a^2} - \frac{2x^2}{\sqrt{a^2 + x^2}}}{a^2 + x^2}$ 1M

$= \frac{2a^2}{(a^2 + x^2)^{3/2}} > 0$ if correct give marks. 1A

$T_{min} = 2\sqrt{a^2 + \frac{a^2}{3}} + 3a - \frac{a}{\sqrt{3}}$

$= \frac{4}{\sqrt{3}}a + 3a - \frac{a}{\sqrt{3}}$

$= \sqrt{3}a + 3a$ 1A

Alt. Solution:

$\frac{dT}{dx} = \frac{2x}{\sqrt{a^2 + x^2}} - 1 = 0$ ...1M+1A

$4x^2 = a^2 + x^2$

$x = \frac{a}{\sqrt{3}}$ 1A

$\frac{d^2T}{dx^2} = \frac{2a^2}{(a^2 + x^2)^{3/2}}$ 1M+1A

> 0

(iii) $\frac{a}{\sqrt{k^2 - 1}} > 3a$

$\frac{1}{k^2 - 1} > 9$

$k^2 - 1 < \frac{1}{9}$ 1A

$k^2 < \frac{10}{9}$

$(1 <)\ k < \frac{\sqrt{10}}{3}$ 1A Accept $k < \frac{\sqrt{10}}{3}$; accept $\frac{-\sqrt{10}}{3} < k < \frac{\sqrt{10}}{3}$

It would cost more to transport goods via P if the minimum point is beyond B, 1M

i.e. $\frac{a}{\sqrt{k^2 - 1}} > 3a$

$(1 <)\ k < \frac{\sqrt{10}}{3}$ 1A

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(b) $k = 2$

(i) $b = 2a$, $T = 2\sqrt{a^2 + x^2} + (2a - x)$

$\frac{dT}{dx} = \frac{2x}{\sqrt{a^2 + x^2}} - 1$ omit steps. (pp-1)

$= 0$

$x = \frac{a}{\sqrt{3}}$ 1A

$T_{min} = 2a + \sqrt{3}a$ 1A

Alt. Solution:
This exp. is the same as the exp. for T in (a)(i) except for the constant 2a and this will not affect the value of x for $\frac{dT}{dx} = 0$

$\therefore\ x = \frac{a}{\sqrt{3}}$

(ii) $b = \frac{1}{2}a < \frac{a}{\sqrt{3}}$

For minimum value of T, go directly from A to B.

i.e. $x = \frac{1}{2}a$ 1A

$T_{min} = 2\sqrt{a^2 + (\frac{1}{2}a)^2}$

$= \sqrt{5}a$ 1A

4

Solution	Marks	Remarks
13. (a) $\arg(z-k) = \angle OKP$	1A	can be omitted
$\arg\left(\frac{z-h}{z-k}\right) = \arg(z-h) - \arg(z-k)$	1M	
$= \angle HPK$	1A	can be omitted
$= \pm 90°$	1A	($\pm$ can be omitted)
$\therefore$ Real part of $\frac{z-h}{z-k} = 0$	1	
	5	
Alt. Solution:		
$\frac{z-h}{z-k} = \frac{x+iy-h}{x+iy-k}$		
$= \frac{[(x-h)+iy][(x-k)-iy]}{[(x-k)+iy][(x-k)-iy]}$	1M	
$= \frac{[(x-h)(x-k)+y^2]+i[(x-k)y-(x-h)y]}{(x-k)^2+y^2}$		
Real part of $\frac{z-h}{z-k} = \frac{(x-h)(x-k)+y^2}{(x-k)^2+y^2}$	1A	
P lies on the circle with HK as diameter.		
$\therefore \frac{y-0}{x-h} \cdot \frac{y-0}{x-k} = -1$	1A	
$y^2 + (x-h)(x-k) = 0$	1A	
$\therefore$ Real part of $\frac{z-h}{z-k} = 0$	1	
(b) (i) $x^2 - 2x + 2 = 0$		
$x = 1 \pm i$	1A	
Since $-\pi < \arg z_2 < \arg z_1 < \pi$		
$z_1 = 1 + i$)	1A	
$z_2 = 1 - i$)		
(ii) $x^2 + 2tx - 4 = 0$		
$D = (2t)^2 + 16$	1A	
$\because D > 0 \quad \therefore \alpha$ and β are real and distinct	1	
$\because \alpha\beta = -4 < 0 \quad \therefore$ opp. sign	1	
(iii) $\frac{z_1-\alpha}{z_1-\beta} = \frac{1+i-\alpha}{1+i-\beta}$		
$= \frac{[(1-\alpha)+i][(1-\beta)-i]}{[(1-\beta)+i][(1-\beta)-i]}$		
$= \frac{(1-\alpha)(1-\beta)+1+(\alpha-\beta)i}{(1-\beta)^2+1}$	1	
$\frac{z_2-\alpha}{z_2-\beta} = \frac{[(1-\alpha)(1-\beta)+1]+(\beta-\alpha)i}{(1-\beta)^2+1}$	1A	
(iv) C lies on the circle with AB as diameter.		
From (a), real part of $\frac{z_1-\alpha}{z_1-\beta} = 0$.	1M	
$\frac{(1-\alpha)(1-\beta)+1}{(1-\beta)^2+1} = 0$		
$(1-\alpha)(1-\beta)+1 = 0$	1	
$1 - (\alpha+\beta) + \alpha\beta + 1 = 0$		
$1 + 2t - 4 + 1 = 0$	1A	For $\alpha+\beta = -2t$, $\alpha\beta = -4$
$t = 1$	1A	
	11	
Alt. Solution:		
$\alpha = -t + \sqrt{t^2+4}$, $\beta = -t - \sqrt{t^2+4}$		
$(1+t-\sqrt{t^2+4})(1+t+\sqrt{t^2+4}) + 1 = 0$	1A	
$(1+t)^2 - (t^2+4) + 1 = 0$		
$t = 1$	1A	

(a) (i) $\overrightarrow{OM} = \frac{1}{2}(\underline{a} + \underline{b})$ 1A

$\overrightarrow{OD} = \frac{1}{3}\underline{a}$ 1A

(ii) $\overrightarrow{OK} = \lambda\overrightarrow{OD} + (1-\lambda)\overrightarrow{OB}$ 1M

$= \lambda\cdot\frac{1}{3}\underline{a} + (1-\lambda)\underline{b}$ 1A

$\overrightarrow{OK} = \mu\overrightarrow{OM}$

$= \frac{\mu}{2}\underline{a} + \frac{\mu}{2}\underline{b}$

$\therefore \frac{\lambda}{3} = \frac{\mu}{2}$ and $(1-\lambda) = \frac{\mu}{2}$ 1M+1A

$\lambda = \frac{3}{4}$)

) 1A

$\mu = \frac{1}{2}$) 7

(b) (i) $\overrightarrow{OM} = 7\underline{i} + 4\underline{j}$ 1A

$\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD}$

$= (2\underline{i} + 8\underline{j}) - \frac{1}{3}(12\underline{i})$

$= -2\underline{i} + 8\underline{j}$ 1A

$\underline{i} \to i$, $\underline{j}$ – pp −1

(ii) $\overrightarrow{OM}\cdot\overrightarrow{DB} = (-2)(7) + (8)(4)$ 1M

$= 18$ 1A

$\cos\angle BKM = \frac{\overrightarrow{OM}\cdot\overrightarrow{DB}}{|\overrightarrow{OM}||\overrightarrow{DB}|}$ 1M

$= \frac{18}{\sqrt{7^2+4^2}\sqrt{8^2+(-2)^2}}$

$= 0.2707$

$\angle BKM = 74.3°$ (or 1.30 rad.) 1A

74.294°, 74°18'
(1.2967 rad.)

(iii) $\overrightarrow{AP} = \frac{\overrightarrow{AB} + 2\overrightarrow{AO}}{3}$

$= \frac{(-10\underline{i} + 8\underline{j}) + 2(-12\underline{i})}{3}$

$= \frac{-34\underline{i} + 8\underline{j}}{3}$ 1A

Alt. Solution:

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \frac{1}{3}(2\underline{i} + 8\underline{j}) - 12\underline{i}$

$= \frac{1}{3}(-34\underline{i} + 8\underline{j})$ 1A

$\overrightarrow{AK} = \overrightarrow{OK} - \overrightarrow{OA}$

$= \frac{1}{2}\overrightarrow{OM} - \overrightarrow{OA}$

$= \frac{1}{2}(7\underline{i} + 4\underline{j}) - 12\underline{i}$

$= -\frac{17}{2}\underline{i} + 2\underline{j}$ 1A

Alt. Solution:

1. calculate $\angle PAK = 0$

2. show slope of AP = slope of AK $= \frac{-4}{17}$

must

$\overrightarrow{AK} = \frac{3}{4}\overrightarrow{AP}$)

) 1A

$\therefore$ A, P, K are collinear.) 9

10. (a) $\ell + 2\pi r = 2$ 1A

use ℓ

$V = \pi r^2\ell$

$= \pi r^2(2 - 2\pi r)$ 1A

reject irrelevant answers

$\frac{dV}{dr} = 2\pi(2r - 3\pi r^2)$ 1A

$= 0$ 1M

$r \neq 0$, $r = \frac{2}{3\pi}$ 1A

$\frac{d^2V}{dr^2} = 2\pi(2 - 6\pi r)$)

) 1M

When $r = \frac{2}{3\pi}$, $\frac{d^2V}{dr^2} = 2\pi(-2) < 0$)

V is a max. when $r = \frac{2}{3\pi}$

$V_{max} = \frac{8}{27\pi}$ (m³) 1A no unit (pp-1)

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(b) (i) $S = 2\pi r^2 + 2\pi r\ell$
$= 2\pi r^2 + 2\pi r(2 - 2\pi r)$
$= (2\pi - 4\pi^2)r^2 + 4\pi r$ 1A

(ii) $\frac{dS}{dr} = (4\pi - 8\pi^2)r + 4\pi$ 1A

$= 0$

$r = \frac{1}{2\pi - 1}$ (or 0.189) 1A

$\frac{d^2S}{dr^2} = 4\pi - 8\pi^2 < 0$ 1M

$\therefore$ S is a max. when $r = \frac{1}{2\pi - 1}$

Alt. Solution:

S is quadratic }
$2\pi - 4\pi^2 < 0$ } 1M

x-coord. of centre
$= \frac{-4\pi}{2(2\pi - 4\pi^2)}$ 1A
$= \frac{1}{2\pi - 1}$

This corresponds to a ma[x]
$\therefore$ S is a max. when
$r = \frac{1}{2\pi - 1}$ 1A

(iii) $0.15 \le r \le 0.25$

(1) S is increasing, $\frac{dS}{dr} \ge 0$

$(4\pi - 8\pi^2)r + 4\pi \ge 0$ 1M

$0.15 \le r \le \frac{1}{2\pi - 1}$ 1A

Accept no equality sign

Accept $r \le \frac{1}{2\pi - 1}$

(2) S is decreasing, $(4\pi - 8\pi^2)r + 4\pi \le 0$

$\frac{1}{2\pi - 1} \le r \le 0.25$ 1A

Accept no equality sign

Accept $r \ge \frac{1}{2\pi - 1}$

$S_{r=0.25} = 1.07$

$S_{r=0.15} = 1.14$

From above, S increasing for $0.15 \le r \le 0.189$
and decreasing for $0.189 \le r \le 0.25$ 1M

[Alternatively, consider the shape of the graph of $S = (2\pi - 4\pi^2)r^2 + 4\pi r$ which is a quadratic in r.]

Smallest value of S occurs when $r = 0.25$

Smallest value of $S = 1.07$ or $(\frac{9\pi}{8} - \frac{\pi^2}{4})$ 1A

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