HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1985

附加數學 試卷二

ADDITIONAL MATHEMATICS PAPER II

11.15 am–1.15 pm (2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

Unless otherwise specified in a question, it is sufficient for numerical answers to be given correct to three significant figures.

SECTION A (39 marks)

Answer ALL questions in this section.

1. $(ax + \frac{1}{x^2})^n$ is expanded in descending powers of x, where n is a positive integer and $a > 0$. If the fourth term of the expansion is independent of x and is equal to $\frac{21}{2}$, find the values of n and a. (5 marks)

2. Let $T_n = \frac{n(n+2)}{(n+1)^2}$, where n is a positive integer. Prove by mathematical induction that

$$T_1 \times T_2 \times \ldots \times T_n = \frac{n+2}{2(n+1)}$$

for all n.

(5 marks)

3. Using the substitution $u = 25 - x^2$, evaluate $\int_3^4 \frac{x}{\sqrt{25 - x^2}}\, dx$. (5 marks)

4. In Figure 1, the straight line L cuts the x-axis at the point $(-1, 0)$ and the curve $y = 4 - x^2$ at the point $(1, 3)$. Find the area of the shaded part.

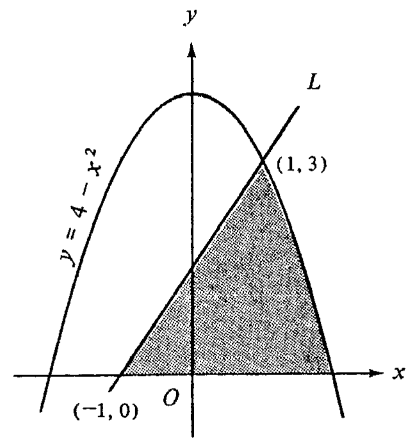

Figure 1

(5 marks)

5. The line $y = x$ and the circle $x^2 + y^2 - 2y = 0$ intersect at the points A and B. Write down the equation of the family of circles passing through A and B.

Hence find the equations of the two circles passing through these two points and with radius $\sqrt{5}$ (6 marks)

6. Find the equations of the two tangents drawn from the point $(-1, 0)$ to the parabola $y^2 = 4x$. (6 marks)

7. In triangle ABC, $\angle A$ and $\angle B$ are acute, $\sin A = \frac{5}{13}$ and $\sin B = \frac{3}{5}$.

(a) Show that $\sin C = \sin(A + B)$ and hence find the value of $\sin C$ without using calculators.

(b) If the perimeter of the triangle is 12 cm, find the length of the longest side. (7 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

8. (a) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$. (4 marks)

(b) By using the substitution $t = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt .$$

(4 marks)

(c) Show that $\int_{-\frac{\pi}{2}}^{0} \cos^3 t \sin^4 t \, dt = \int_0^{\frac{\pi}{2}} \cos^3 t \sin^4 t \, dt$

and $\int_{-\frac{\pi}{2}}^{0} \sin^3 t \cos^4 t \, dt = -\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t \, dt$.

(6 marks)

(d) Using the above results, or otherwise, evaluate

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{8} \sin^3 2t \, (\sin t + \cos t) \, dt .$$

(6 marks)

9. In Figure 2, P and Q are two points on the semi-circle $y = \sqrt{1 - x^2}$. OP and OQ make angles θ and β respectively with the positive x-axis, where $\theta \leqslant \beta$.

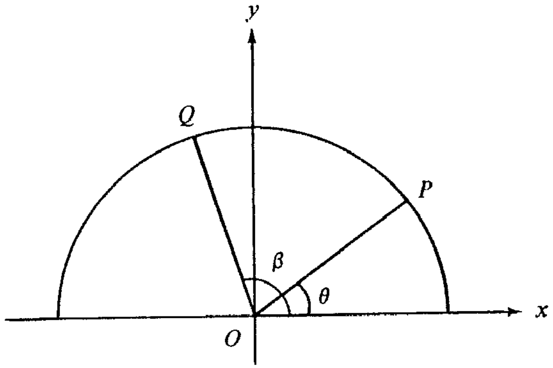

Figure 2

(a) The region bounded by OP, OQ and arc PQ is revolved about the x-axis. Show that the volume of the solid generated is $\frac{2\pi}{3}(\cos\theta - \cos\beta)$.
(10 marks)

(b) If P and Q move along the semi-circle such that $\beta = 2\theta$, find the maximum volume of the solid.
(5 marks)

(c) If P and Q move along the semi-circle such that $\beta - \theta = \frac{\pi}{3}$, find the maximum volume of the solid.
(5 marks)

10. $A(0, 2)$, $B(-3, 0)$ and $C(1, 0)$ are the vertices of a triangle. $PQRS$ is a variable rectangle inscribed in the triangle with PQ on the x-axis, R on AC and S on AB, as shown in Figure 3. Let the length of PS be h.

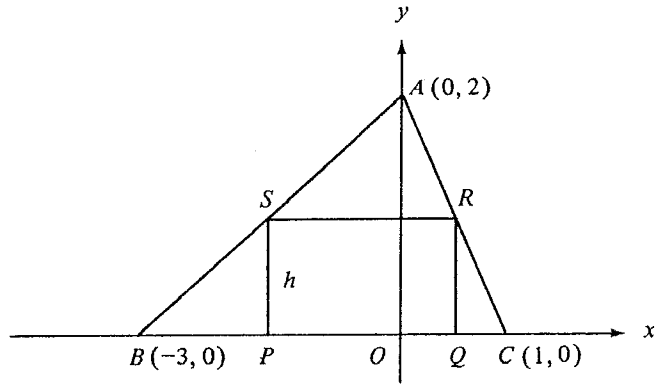

Figure 3

(a) Find the coordinates of S and R in terms of h.
(5 marks)

(b) Let A_1 be the area of $PQRS$ when it is a square, A_2 be the maximum possible area of rectangle $PQRS$, and A_3 be the area of ΔABC. Find the ratios $A_1 : A_2 : A_3$.
(8 marks)

(c) The centre of $PQRS$ is the point $M(x, y)$. Express x and y in terms of h.

Hence find the equation of the locus of M.

Show the locus on a diagram.
(7 marks)

11. The line $y = 2x + c$ cuts the ellipse $x^2 + \frac{y^2}{16} = 1$ at the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

(a) (i) Show that $PQ = \sqrt{5}\,|x_1 - x_2|$.

(ii) Show that x_1 and x_2 are the roots of the equation

$$20x^2 + 4cx + (c^2 - 16) = 0 .$$

(iii) Determine the two values of c such that the length of the chord PQ is $2\sqrt{2}$.

(11 marks)

(b) Let the two chords determined in (a)(iii) be $P'Q'$ and $P''Q''$. $P'Q'Q''P''$ is a parallelogram as shown in Figure 4.

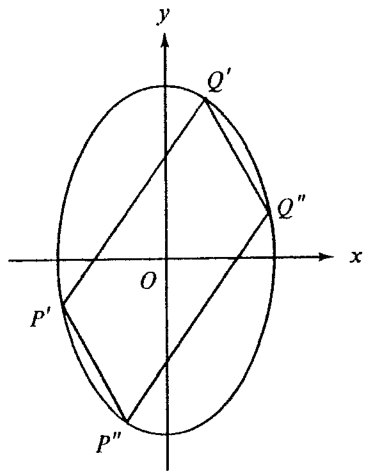

Figure 4

(i) By finding the distance between the chords $P'Q'$ and $P''Q''$, or otherwise, calculate the area of the parallelogram.

(ii) By finding the relation between the coordinates of P'' and the coordinates of Q', or otherwise, calculate the length of the side $P'P''$.

(9 marks)

12. In Figure 5, $ABCDE$ is a regular pentagon of side 1 cm inscribed in a circle with centre O. H is the foot of the perpendicular drawn from C to the diagonal BE.

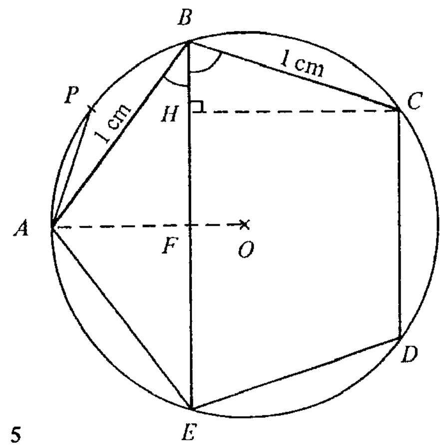

Figure 5

(a) Find $\angle ABE$ and $\angle CBE$.

By expressing the length of BE in two different forms, prove that $\cos 36° - \cos 72° = \frac{1}{2}$.

Hence find the value of $\cos 36°$ in surd form.

(9 marks)

(b) Show that the radius of the circle is $\frac{2}{\sqrt{10 - 2\sqrt{5}}}$ cm.

(5 marks)

(c) Let AP be one side of a regular decagon (10-sided polygon) inscribed in the same circle. Find $\angle PAO$, and hence show that

$$AP = \frac{2}{\sqrt{10 + 2\sqrt{5}}} \text{ cm}.$$

(6 marks)

END OF PAPER

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Additional Mathematics I

1. $\frac{1}{\sqrt{3}}$

2. $\sqrt{2}\left(\cos -\frac{\pi}{4} + i\sin -\frac{\pi}{4}\right)$

$\sqrt[6]{2}\left(\cos\frac{(8k-1)\pi}{12} + i\sin\frac{(8k-1)\pi}{12}\right)$,

$k = 0, 1, 2$

3. $\frac{a - \sqrt{a^2+16}}{2} \leqslant x \leqslant \frac{a + \sqrt{a^2+16}}{2}$

-1

4. (a) $\frac{1}{1+r}\,[(1+4r)\mathbf{i} + (3-3r)\mathbf{j}]$

(b) $r = \frac{1}{2}$

$C = (2, 1)$

5. $\frac{\pi}{27}(432h - 36h^2 + h^3)$ cm³

$\frac{1}{4}$ cm/s

6. ± 1

7. (a) (i) $a\left(x + \frac{b - \sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b + \sqrt{b^2-4ac}}{2a}\right)$

(iii) The roots are $\frac{5}{3i}$ and $\frac{-1}{i}$

(b) $\lambda = \mu$

8. (a) (i) $\overrightarrow{OD} = 2\mathbf{b} + k\mathbf{a}$

$\overrightarrow{DA} = (1-k)\mathbf{a} - 2\mathbf{b}$

(ii) $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

$\overrightarrow{CP} = [k + \lambda(1-k)]\mathbf{a} - 2\lambda\mathbf{b}$

$\lambda = \frac{k}{1+k}$

(b) (ii) $k = \frac{1}{4}$ or $\frac{1}{2}$

$\lambda = \frac{1}{5}$ or $\frac{1}{3}$

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Additional Mathematics I

9. (a) (i) $s = a\sec\theta + b\,\mathrm{cosec}\,\theta \quad (0 < \theta < \frac{\pi}{2})$

(b) (i) 4.69 m

(ii) 5.57 m

10. (b) (i) $z - \bar{z} = 0$

The locus of z is the real axis, excluding $z = -1$.

(ii) $z\bar{z} - 1 = 0$

The locus of z is the circle, centre O, radius 1, excluding the points $z = \pm 1$.

(iii) $z + \bar{z} = 0$

The locus of z is the imaginary axis.

11. (b) $\sqrt{300}$

(c) -0.0050

3.5

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Additional Mathematics II

1. $n = 9$

$a = \frac{1}{\sqrt{2}}$

3. 1

4. $4\frac{2}{3}$

5. $x^2 + y^2 - 2y + k(x - y) = 0$

$x^2 + y^2 + 2x - 4y = 0$

$x^2 + y^2 - 4x + 2y = 0$

6. $x - y + 1 = 0$

$x + y + 1 = 0$

7. (a) $\frac{56}{65}$

(b) 5.6 cm

8. (a) $\frac{2}{35}$

(d) $\frac{4}{35}$

9. (b) $\frac{3}{4}\pi$

(c) $\frac{2\pi}{3}$

10. (a) $S = (\frac{3h}{2} - 3, h)$

$R = (1 - \frac{h}{2}, h)$

(b) 8 : 9 : 18

(c) $x = \frac{1}{2}(h - 2)$

$y = \frac{h}{2}$

$x - y = -1$

11. (a) (iii) $\pm\sqrt{10}$

(b) (i) 8 (sq. units)

(ii) $\sqrt{26}$

12. (a) $\angle ABE = 36°$

$\angle CBE = 72°$

$\frac{1+\sqrt{5}}{4}$

(c) $\angle PAO = 72°$