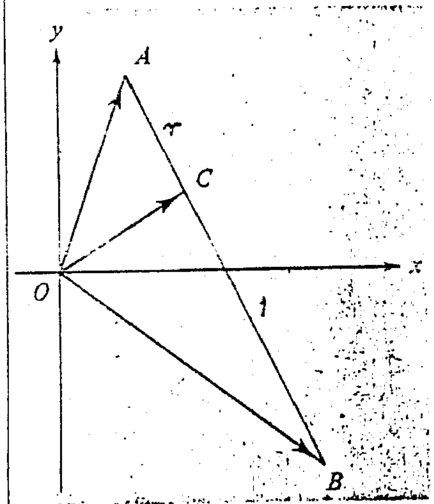
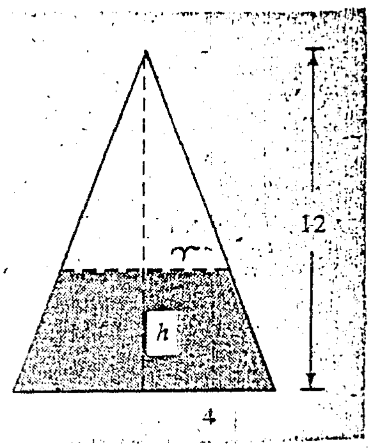
1985 PAPER I

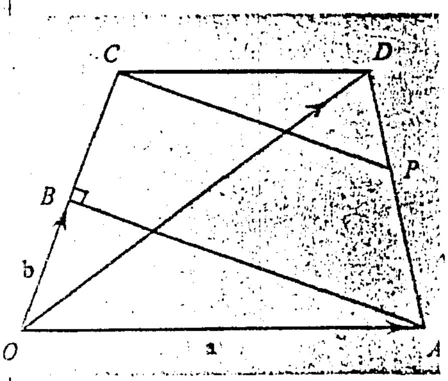
SOLUTIONS	MARKS	REMARKS
$f'(x) = \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}}(-2x)$	1+1+1A	1 for product rule, 1 for chain rule
$= \frac{(1-2x^2)}{\sqrt{1-x^2}}$		
$\therefore f'(\frac{1}{2}) = \frac{1-\frac{2}{4}}{\sqrt{1-\frac{1}{4}}}$	1M	
$= \frac{1}{\sqrt{3}}$ (0.577)	1A	
	5	
$1 - i = \sqrt{2}(\cos -\frac{\pi}{4} + i \sin -\frac{\pi}{4})$	1A+1A	1 for mod., 1 for argument
(or $\sqrt{2}\text{cis}\,\frac{7\pi}{4}$, $\sqrt{2}\text{cis}\,315°$, etc.)		
$(1-i)^{\frac{1}{3}} = \sqrt[6]{2}(\cos \frac{-\frac{\pi}{4}+2k\pi}{3} + i \sin \frac{-\frac{\pi}{4}+2k\pi}{3})$	1M+1M	1M for general form 1M for De Moivre's Theorem
$= \sqrt[6]{2}(\cos \frac{(8k-1)\pi}{12} + i \sin \frac{(8k-1)\pi}{12})$,		
$k = 0, 1, 2$	2A	
$= \sqrt[6]{2}(\cos -\frac{\pi}{12} + i \sin -\frac{\pi}{12})$ or		
$\sqrt[6]{2}(\cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12})$ or		
$\sqrt[6]{2}(\cos -\frac{3\pi}{4} + i \sin -\frac{3\pi}{4})$		
	6	
[Note other variants in arguments,		
e.g. $\theta = 105°, 225°, 345°$; $\theta = \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{23\pi}{12}$]		
$x^2 - ax - 4 \leqslant 0$		
$\Leftrightarrow (x - \frac{a+\sqrt{a^2+16}}{2})(x - \frac{a-\sqrt{a^2+16}}{2}) \leqslant 0$		
$\therefore \frac{a-\sqrt{a^2+16}}{2} \leqslant x \leqslant \frac{a+\sqrt{a^2+16}}{2}$	1M+1A	for $\alpha \leqslant x \leqslant \beta$
		Alt. Solution :
$\frac{a+\sqrt{a^2+16}}{2} = 4$	1M	Let $x^2-ax-4 = (x-\alpha)(x-\beta) \leqslant 0$, where $\alpha \leqslant \beta$
		$\alpha \leqslant x \leqslant \beta$ ——— 1M
$\Rightarrow \sqrt{a^2+16} = 8 - a$		Since $\alpha\beta = -4$ 1A
$\Rightarrow a^2 + 16 = 64 - 16a + a^2$		Sub. $\beta = 4$ ——— 1M
		$\alpha = -1$ 1A
$\Rightarrow a = 3$	1A	
$\therefore$ the least possible value of x is $\frac{3-\sqrt{9+16}}{2} = -1$	1M+1A	
	6	

SOLUTIONS	MARKS	REMARKS
(a) $\overrightarrow{OC} = \frac{1}{1+r}(\overrightarrow{OA} + r\overrightarrow{OB})$	1A	
$= \frac{1}{1+r}[(\vec{i} + 3\vec{j}) + r(4\vec{i} - 3\vec{j})]$		
$= \frac{1}{1+r}[(1 + 4r)\vec{i} + (3 - 3r)\vec{j}]$	1A	
(b) $\overrightarrow{AB} = (4\vec{i} - 3\vec{j}) - (\vec{i} + 3\vec{j})$		
$= 3\vec{i} - 6\vec{j}$	1A	
$OC \perp AB \Rightarrow \overrightarrow{AB} \cdot \overrightarrow{OC} = 0$	1M	
$\Rightarrow \frac{1}{1+r}[(1 + 4r)3 - (3 - 3r)6] = 0$		
$\Rightarrow \frac{1}{1+r}(30r - 15) = 0$		
$\Rightarrow r = \frac{1}{2}$	1A	
$\therefore \overrightarrow{OC} = \frac{2}{3}[(1 + 2)\vec{i} + (3 - \frac{3}{2})\vec{j}] = 2\vec{i} + \vec{j}$		
i.e. C = (2 , 1)	1A 6	
Let the radius of the water surface be r centimetres.		
By similar triangles	1M	Attempt to use similar triangles
$\frac{r}{12 - h} = \frac{4}{12}$		
$r = \frac{1}{3}(12 - h)$	1A	
Volume of water $V = \frac{1}{3}(\pi)(4^2)(12) - \frac{1}{3}\pi r^2(12 - h)$	1M	
$= \frac{\pi}{3}(192 - \frac{(12 - h)^3}{9})$	1A	
$= \frac{\pi}{27}(432h - 36h^2 + h^3)$		
$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$	1	
$= \frac{\pi}{9}(12 - h)^2 \cdot \frac{dh}{dt}$ - - - - - - - - - - - -	1A	
$\frac{\pi}{9}(12 - h)^2 \cdot \frac{dh}{dt} = \pi$	1M	
$\therefore$ at h = 6 ,		
$\frac{dh}{dt} = \frac{9}{(12 - 6)^2}$		
$= \frac{1}{4}$		
$\therefore$ the water level is rising at $\frac{1}{4}$ cm/s	1A 8	Accept $\frac{dh}{dt} = \frac{1}{4}$ cm/s

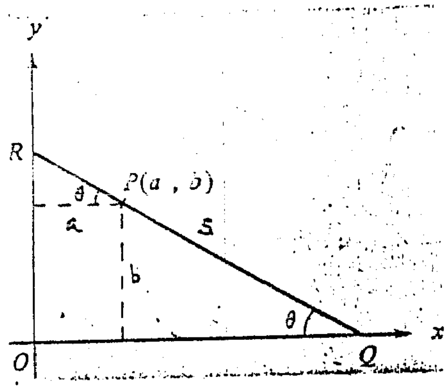
SOLUTIONS	MARKS	REMARKS
6. $\log_{10}\lvert x^2 + 2px\rvert = 0$ iff $\lvert x^2 + 2px\rvert = 1$	2A	'iff' optional
iff $x^2 + 2px = 1$ or $x^2 + 2px = -1$	1A+1A	-1A for 'and', accept ','
(i) Let $x^2 + 2px - 1 = 0$		
Discriminant $= 4p^2 + 4$		
> 0 for all real p	1A	
$\therefore$ the given equation has no double root.	1A	
(ii) Let $x^2 + 2px + 1 = 0$		
Discriminant $= 4p^2 - 4 = 0$	1A	
iff $p = \pm 1$	1A	
	8	
The given equation has a double root if $p = \pm 1$		

SOLUTIONS	MARKS	REMARKS
7. (a) (i) $ax^2 + bx + c$		
$= a(x^2 + \frac{b}{a}x + \frac{c}{a})$	1A	
$= a[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}]$	1M+1A	1M completing square
$= a(x + \frac{b - \sqrt{b^2-4ac}}{2a})(x + \frac{b + \sqrt{b^2-4ac}}{2a})$	1A	
(ii) The roots of the given equation are		
$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$	1A	
Since a, b are real, if $b^2 - 4ac < 0$, the roots are imaginary.	1A	Must mention a, b real.
(iii) If $a = 3i$, $b = -2$, $c = 5i$,		
$b^2 - 4ac = 4 - 4 \times 3 \times 5i^2$		
$= 64$	1A	
> 0		
But the roots $= \frac{2 \pm \sqrt{64}}{6i}$	1A+1A	
$= \frac{5}{3i}$ or $\frac{-1}{i}$ (or $\frac{-5i}{3}$, i),		
which are imaginary.	9	
(b) The discriminant $= 4\lambda^2 - 4(2\lambda^2 - 2\lambda\mu + \mu^2)$	1A	
$= -4(\lambda^2 - 2\lambda\mu + \mu^2)$		
$= -4(\lambda - \mu)^2$	1A	
Since the roots are real,		
$-4(\lambda - \mu)^2 \geqslant 0$	1M	
$\therefore \lambda = \mu$ (Since λ and μ are real)	1A 4	
(c) Since (1) and (2) have imaginary roots		
$a^2 < 4b$)	1A	
and $c^2 < 4d$)		
The discriminant of (3) $= (a + c)^2 - 3(b + d)$	1A	
$< (a + c)^2 - 2(a^2 + c^2)$	1M+1A	1M using $a^2 < 4b$ or $c^2 < 4d$
$= -(a - c)^2$	1A	
$\leqslant 0$	1A	
$\therefore$ the discriminant < 0		
As the coefficients of (3) are real, it has imaginary roots.	1M 7	Must mention coeff. real

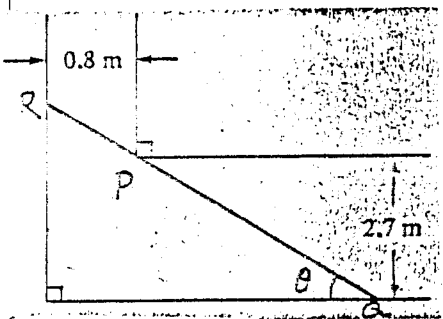
SOLUTIONS	MARKS	REMARKS
3. (a) (i) $\vec{OD} = \vec{OC} + \vec{CD}$		
$= 2\vec{b} + k\vec{a}$	1A	
$\vec{DA} = \vec{OA} - \vec{OD}$		
$= \vec{a} - (2\vec{b} + k\vec{a})$	1M	Sub. in correct expression
$= (1 - k)\vec{a} - 2\vec{b}$	1A	
(ii) $\vec{BA} = \vec{a} - \vec{b}$	1A	
$\vec{CP} = \vec{CD} + \vec{DP}$		
$= k\vec{a} + \lambda[(1 - k)\vec{a} - 2\vec{b}]$	1M	Same as above
$= (k + \lambda(1 - k))\vec{a} - 2\lambda\vec{b}$	1A	
Since CP // BA, $\dfrac{k + \lambda(1 - k)}{1} = \dfrac{-2\lambda}{-1}$	2M }	**Alt. Solution :** $t\,\vec{BA} = \vec{CP}$ 1M
$k = \lambda(1+k)$		$t(\vec{a}-\vec{b}) = (k+\lambda(1-k))\vec{a} - 2\lambda\vec{b}$
$\lambda = \dfrac{k}{1 + k}$	1A	$k + \lambda(1-k) = t$
	9	$-2\lambda = -t$ 1M
(b) (i) $\vec{a}\cdot\vec{b} = \lvert\vec{a}\rvert\lvert\vec{b}\rvert\cos AOB$	1A	
$= OB \times OA\cos AOB$)		Should not be omitted
)	1A	
$= OB^2$)		
(ii) $\vec{OD}\cdot\vec{DA} = (2\vec{b} + k\vec{a})\cdot((1 - k)\vec{a} - 2\vec{b})$		
$= k(1-k)\vec{a}\cdot\vec{a} - 4\vec{b}\cdot\vec{b} + [2(1-k) - 2k]\vec{a}\cdot\vec{b}$	1A	
$= k(1-k)\vec{a}\cdot\vec{a} - 4\vec{b}\cdot\vec{b} + (2-4k)OB^2$	1M	
$= 16k(1-k)OB^2 - 4OB^2 + (2-4k)OB^2$	1M+1M	
$= (-16k^2 + 12k - 2)OB^2$	1A	
If $OD \perp AD$, $-16k^2 + 12k - 2 = 0$	1M	
$(4k - 1)(2k - 1) = 0$		
$k = \frac{1}{4}$ or $\frac{1}{2}$	1A	
$\lambda = \dfrac{k}{1 + k}$		
$= \frac{1}{5}$ or $\frac{1}{3}$	1A+1A	
	11	



SOLUTIONS	MARKS	REMARKS
9. (a) (i) $RP = a\sec\theta \quad (= \frac{a}{\cos\theta})$	1A	
$PQ = b\operatorname{cosec}\theta \quad (= \frac{b}{\sin\theta})$	1A	
$\therefore\ s = RP + PQ$		
$= a\sec\theta + b\operatorname{cosec}\theta \quad (0 < \theta < \frac{\pi}{2})$	1A	
$(= \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$ or $\sqrt{(\frac{a\tan\theta + b}{\tan\theta})^2 + (a\tan\theta + b)^2}$		
(ii) $\frac{ds}{d\theta} = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta$	1A+1A	
$\frac{ds}{d\theta} = 0 \Rightarrow a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta = 0$	1M	
$\Rightarrow \frac{a\tan\theta}{\cos\theta} = \frac{b}{\sin\theta\tan\theta}$		
$\Rightarrow \tan^3\theta = \frac{b}{a}$		
$\Rightarrow \tan\theta = \sqrt[3]{\frac{b}{a}}$	1A	
$\frac{d^2s}{d\theta^2} = a(\sec\theta\tan^2\theta+\sec^3\theta)-b(-\operatorname{cosec}\theta\cot^2\theta-\operatorname{cosec}^3\theta)$	2A	Alt. Solution : $\frac{ds}{d\theta} = \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta}$
$= a\sec\theta(\tan^2\theta+\sec^2\theta)+b\operatorname{cosec}\theta(\cot^2\theta+\operatorname{cosec}^2\theta)$		$= \frac{a\sin^3\theta - b\cos^3\theta}{\sin^2\theta\cos^2\theta}$
If $\tan\theta = \sqrt[3]{\frac{b}{a}}$, $0° < \theta < 90°$, $\sec\theta$, $\operatorname{cosec}\theta > 0$,		$= \frac{\cos^3\theta(a\tan^3\theta - b)}{\sin^2\theta\cos^2\theta}$ 2A
$\therefore \frac{d^2s}{d\theta^2} > 0$	1M (Knowledge of Test)	
$\therefore$ s will be least when $\tan\theta = \sqrt[3]{\frac{b}{a}}$.	1A	If $\theta < \tan^{-1}\sqrt[3]{\frac{b}{a}}$ slightly, $\frac{ds}{d\theta} < 0$.
	11	If $\theta > \tan^{-1}\sqrt[3]{\frac{b}{a}}$ slightly, $\frac{ds}{d\theta} > 0$ .... 1M (Knowledge of test)
		$\therefore$ s is least when $\tan\theta = \sqrt[3]{\frac{b}{a}}$ 1A



SOLUTIONS	MARKS	REMARKS
9. (b) (i) When being moved horizontally, the longest pipe will just touch the outside walls of both corridors while it is negotiating the corner P. The length of the pipe must not be longer than the shortest distance between Q and R. From (a), this occurs when		
$\tan\theta = \sqrt[3]{\frac{2.7}{0.8}}$	1M+1A	1M for attempting to use (a)
$= \frac{3}{2}$ $(\theta = 56.3°)$	1A	
∴ the length of the longest pipe that can be carried round the corner horizontally is		
$0.8\sec\theta + 2.7\operatorname{cosec}\theta$ $(\theta = 56.3°)$	1M+1M	1M for sub. a, b, 1M for sub. θ .
$= 0.8 \times \frac{\sqrt{13}}{2} + 2.7 \times \frac{\sqrt{13}}{3}$		
$= 4.69$ m (4.687)	1A	
(ii) If the height of the ceiling is 3 m, the length of the longest pipe that can be carried round the corner is		
$\sqrt{3^2 + 4.687^2}$	2M	
$= 5.57$ m	1A	
	9	

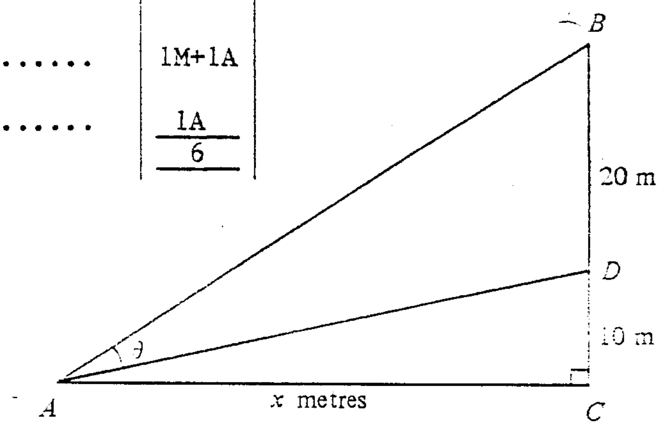


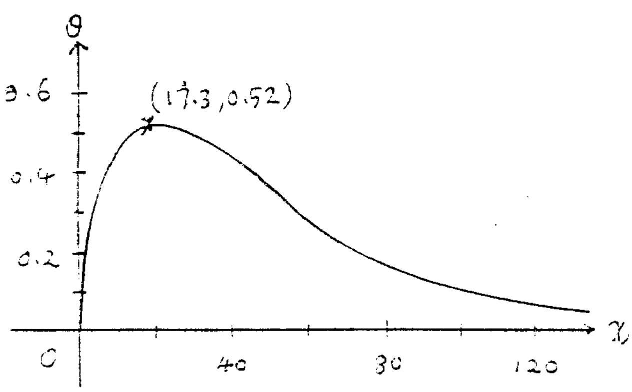
SOLUTIONS	MARKS	REMARKS
10. (a) $\frac{1}{2}(w+\overline{w}) = \frac{1}{2}[(p+qi)+(p-qi)]$		
$= p$	1A	
$\frac{1}{2i}(w-\overline{w}) = \frac{1}{2i}[(p+qi)-(p-qi)]$		
$= q$	1A	
$p = \frac{1}{2}(w+\overline{w})$		
$= \frac{1}{2}\left[\frac{z-1}{z+1} + \left(\overline{\frac{z-1}{z+1}}\right)\right]$	1M	
$= \frac{(z-1)(\overline{z}+1)+(\overline{z}-1)(z+1)}{2(z+1)(\overline{z}+1)}$	1A	
$= \frac{z\overline{z}-\overline{z}+z-1+\overline{z}z-z+\overline{z}-1}{2(z\overline{z}+z+\overline{z}+1)}$		Show working
$= \frac{z\overline{z}-1}{z\overline{z}+z+\overline{z}+1}$	1A	
$q = \frac{1}{2i}(w-\overline{w})$		
$= \frac{1}{2i}\left[\frac{z-1}{z+1} - \left(\overline{\frac{z-1}{z+1}}\right)\right]$	1M	
$= \frac{1}{2i}\,\frac{(z-1)(\overline{z}+1)-(\overline{z}-1)(z+1)}{(z+1)(\overline{z}+1)}$		
$= \frac{1}{2i}\,\frac{z\overline{z}+z-\overline{z}-1-\overline{z}z+z-\overline{z}+1}{z\overline{z}+z+\overline{z}+1}$		Show working
$= \frac{i(\overline{z}-z)}{z\overline{z}+z+\overline{z}+1}$	1A 7	3+2 marks for p, q
(b) (i) w is real $\Leftrightarrow$ $q = 0$.	1	Optional
$\therefore z - \overline{z} = 0$	1A	
The locus of z is the real axis, excluding $z = -1$.	1A+1A	
(ii) w is purely imaginary $\Leftrightarrow$ $p = 0$, $q \neq 0$	1	Optional
$\therefore$ $z\overline{z} - 1 = 0$ i.e. $x^2 + y^2 = 1$	1A	
The locus of z is the circle, centre 0, radius 1, excluding the points $z = \pm 1$.	1A+1A	
(iii) $\lvert w\rvert^2 = w\overline{w}$		
$= \frac{z-1}{z+1} \times \left(\overline{\frac{z-1}{z+1}}\right)$		
$= \frac{(z-1)(\overline{z}-1)}{(z+1)(\overline{z}+1)}$	1A	
$\lvert w\rvert = 1$		
$\Leftrightarrow$ $1 = \frac{z\overline{z}-z-\overline{z}+1}{z\overline{z}+z+\overline{z}+1}$	1M	
$z\overline{z}+z+\overline{z}+1 = z\overline{z}-z-\overline{z}+1$		
$\therefore$ $z + \overline{z} = 0$	1A	
The locus of z is the imaginary axis.	2A 13	

SOLUTIONS	MARKS	REMARKS
10. (a) Alt. Solution :		
Let $z = x + iy$		
$w = \dfrac{(x-1)+iy}{(x+1)+iy}$	1M	
$= \dfrac{x^2+y^2-1}{(x+1)^2+y^2} + \dfrac{2yi}{(x+1)^2+y^2}$	1A	
$\overline{w} = \dfrac{x^2+y^2-1}{(x+1)^2+y^2} - \dfrac{2yi}{(x+1)^2+y^2}$		
$\therefore \dfrac{1}{2}(w+\overline{w}) = \dfrac{x^2+y^2-1}{(x+1)^2+y^2}$		
$= p$	1A	
$\dfrac{1}{2i}(w-\overline{w}) = \dfrac{1}{2i}\left(\dfrac{4yi}{(x+1)^2+y^2}\right)$		
$= q$	1A	
$\dfrac{z\overline{z}-1}{z\overline{z}+z+\overline{z}+1} = \dfrac{x^2+y^2-1}{x^2+y^2+(x+iy)+(x-iy)+1}$	1A	
$= \dfrac{x^2+y^2-1}{x^2+y^2+2x+1}$		
$= p$	1A	
$\dfrac{i(\overline{z}-z)}{z\overline{z}+z+\overline{z}+1} = \dfrac{i(-2iy)}{x^2+y^2+2x+1}$		
$= q$	1A	
	7	
(b) (i) w is real $\Leftrightarrow q = 0$	1	Optional
$\therefore y = 0$	1A	
The locus of z is the real axis, excluding $z = -1$	1A+1A	
(ii) w is purely imaginary $\Leftrightarrow p = 0,\ q \neq 0$	1	Optional
$\therefore x^2 + y^2 = 1$	1A	
The locus of z is the circle, centre 0, radius 1, excluding $z = \pm 1$.	1A+1A	
(iii) $\|w\|^2 = w\overline{w}$		
$= \dfrac{1}{[(x+1)^2+y^2]^2}[(x^2+y^2-1)^2+4y^2]$	1A	
$\|w\| = 1 \Leftrightarrow [(x+1)^2+y^2]^2 = (x^2+y^2-1)^2+4y^2$	1M	
$[(x+1)^2+y^2+x^2+y^2-1][2x+2] = 4y^2$		
$x[(x+1)^2+y^2] = 0$		
$\therefore x = 0$ (as $z = x+iy \neq -1$)	1A	
The locus is the imaginary axis.	2A	
	13	

SOLUTIONS	MARKS	REMARKS
11. (a) $\tan\theta = \tan(BAC - DAC)$	1	
$= \dfrac{\tan BAC - \tan DAC}{1 + \tan BAC \tan DAC}$	1M	
$= \dfrac{\frac{30}{x} - \frac{10}{x}}{1 + \frac{30}{x}\cdot\frac{10}{x}}$		Show working
$= \dfrac{20x}{x^2 + 300}$	1A 3	
(b) Differentiating both sides w.r.t. x,		
$\dfrac{d}{dx}(\tan\theta) = \dfrac{d}{dx}\left(\dfrac{20x}{x^2 + 300}\right)$,		
$\sec^2\theta \dfrac{d\theta}{dx} = \dfrac{20(x^2 + 300) - 20x(2x)}{(x^2 + 300)^2}$	1A+1A	
But $\sec^2\theta = 1 + \tan^2\theta$		
$= \dfrac{(x^2 + 300)^2 + (20x)^2}{(x^2 + 300)^2}$	1A	
$\therefore \dfrac{d\theta}{dx} = \dfrac{(x^2 + 300)^2}{(x^2 + 300)^2 + (20x)^2} \cdot \dfrac{20(x^2 + 300) - 20x(2x)}{(x^2 + 300)^2}$		
$= \dfrac{20(300 - x^2)}{x^4 + 1000x^2 + 90\ 000}$	1A	
$\dfrac{d\theta}{dx} = 0 \Leftrightarrow x = \sqrt{300}$ ($\doteqdot$ 17.3) (-ve root rejected)	1A	Accept $x = \pm\sqrt{300}$
When $x < \sqrt{300}$ slightly, $\dfrac{d\theta}{dx} > 0$;		
When $x > \sqrt{300}$ slightly, $\dfrac{d\theta}{dx} < 0$.		
$\therefore$ θ is maximum when $x = \sqrt{300}$	1A 6	

SOLUTIONS	MARKS	REMARKS
11. (c) If $x = 50$, $\frac{d\theta}{dx} = \frac{20(300 - 50^2)}{50^4 + 1000(50)^2 + 90\ 000}$	1M	
$= \frac{-44\ 000}{3\ 840\ 000}$		
$= -0.0050$ (correct to 4 d.p.)	1A	Follow through for −0.005
$1° = 0.0175$ radians		
Since $\Delta x \doteqdot \Delta\theta \frac{1}{\frac{d\theta}{dx}}$ (or $\Delta\theta \doteqdot \frac{d\theta}{dx}\Delta x$), ...	1M	
at $x = 50$,		
$\Delta x \doteqdot \frac{-0.0175}{-0.005}$	1M+1A	
$= 3.5$ (correct to the nearest $\frac{1}{10}$ m)	1A	
	6	



SOLUTIONS	MARKS	REMARKS
(d) At $x = 0$, $\theta = 0$.	1A	May be indicated in diag
At $x = \sqrt{300}$,		
$\tan\theta = 0.577$		
$\theta = 0.524$ (or 30°)	1A	
As $x \to \infty$, $\theta \to 0$	1A	
	2	1 shape, 1 tail
	5	