10. A straight line through the point $R(-1, -1)$ has a variable slope m. It intersects the circle

$$x^2 + y^2 = 1$$

at A and B. Let P be the mid-point of AB.

(a) Find the coordinates of P in terms of m. (9 marks)

(b) The locus of P is a part of a curve C. Find the equation of C and name it. (6 marks)

(c) Sketch the locus of P. (5 marks)

11. (a) Show that $\frac{\sin 3\theta}{\sin\theta} = 2\cos 2\theta + 1$.

By putting $\theta = \frac{\pi}{4} + \phi$ in the above identity, show that

$$\frac{\cos 3\phi - \sin 3\phi}{\cos\phi + \sin\phi} = 1 - 2\sin 2\phi .$$

(7 marks)

(b) Using the substitution $\phi = \frac{\pi}{2} - u$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi = \int_{\frac{\pi}{2}}^{0} \frac{\sin 3u}{\cos u + \sin u}\, du .$$

Hence, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi - \sin 3\phi}{\cos\phi + \sin\phi}\, d\phi .$$

(8 marks)

(c) Using the results in (a) and (b), evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos 3\phi}{\cos\phi + \sin\phi}\, d\phi .$$

(5 marks)

12. Let $f(x)$ be a function of x and let k and s be constants.

(a) By using the substitution $y = x + ks$, show that

$$\int_0^{s} f(x + ks)\, dx = \int_{ks}^{(k+1)s} f(x)\, dx .$$

Hence show that, for any positive integer n,

$$\int_0^{s} [f(x) + f(x + s) + \ldots + f(x + (n-1)s)]\, dx = \int_0^{ns} f(x)\, dx .$$

(10 marks)

(b) Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}}$ by using the substitution $x = \sin\theta$.

Using this result together with (a), evaluate

$$\int_0^{\frac{1}{2n}} \left(\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - (x + \frac{1}{2n})^2}} + \frac{1}{\sqrt{1 - (x + \frac{2}{2n})^2}} + \ldots + \frac{1}{\sqrt{1 - (x + \frac{n-1}{2n})^2}} \right) dx .$$

(10 marks)

END OF PAPER

HONG KONG EXAMINATIONS AUTHORITY

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1984

附加數學　試卷一

ADDITIONAL MATHEMATICS PAPER I

8.30 am–10.30 am (2 hours)

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

SECTION A (39 marks)
Answer ALL questions in this section.

1. Given $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j}$,

$\overrightarrow{OB} = -\mathbf{i} + \mathbf{j}$.

(a) Find the unit vector in the direction of $\overrightarrow{AB}$.

(b) If P is a point such that $\overrightarrow{AP} = m\overrightarrow{AB}$, express $\overrightarrow{OP}$ in terms of m .

(7 marks)

2. The surface area of a sphere is increasing at a rate of $8 \text{ cm}^2/\text{s}$. How fast is the volume of the sphere increasing when the surface area is $36\pi \text{ cm}^2$?

(8 marks)

3. Let $z = (1 - 2i)^5$.

(a) Using the binomial theorem, express z in the form $a + bi$, where a , b are real.

(b) Find the real part of $\frac{1}{z}$.

Hence write down the real part of $z + \frac{1}{z}$, correct to the nearest integer.

(6 marks)

4. Solve for x :

$$\left| 2|x| - 1 \right| \leqslant 2 .$$

(5 marks)

5. Let α and β be the roots of the equation

$$x^2 - 2x - (m^2 - m + 1) = 0 ,$$

where m is a real number.

(a) Show that $(\alpha - \beta)^2 > 0$ for any value of m .

(b) Find the minimum value of $|\alpha - \beta|$.

(7 marks)

6. ABC is a triangle in which $AB = AC$ and $\angle BAC = 2\theta$. The median $AD = h$. Find a point P on AD so that the product of the distances from P to the three sides of $\triangle ABC$ is a maximum.

(6 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

7. In Figure 1, $ABCD$ is a square with $\overrightarrow{AB} = \mathbf{i}$ and $\overrightarrow{AD} = \mathbf{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.

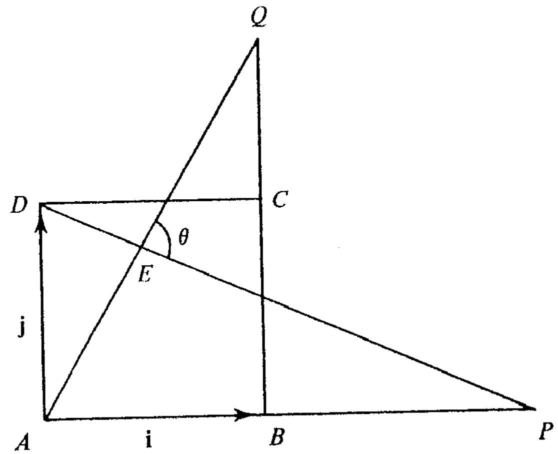

Figure 1

(a) By calculating $\overrightarrow{AQ} \cdot \overrightarrow{DP}$, find $\cos\theta$ in terms of m and k.

(8 marks)

(b) Given that $\dfrac{DE}{EP} = \dfrac{1}{4}$.

(i) Express $\overrightarrow{AE}$ in terms of k.

(ii) Let $\dfrac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m.

(iii) If $\theta = 90°$, use the above results to find the values of k, m and r.

(12 marks)

8. Let $f(x) = 5x^2 + bx + c$, where b and c are real, $c > 0$ and $f(\frac{1}{2}) < 0$.

(a) Show that the equation

$$f(x) = 0$$

has two distinct real roots.

(6 marks)

(b) Let α and β $(\alpha < \beta)$ be the roots of $f(x) = 0$.

(i) By expressing $f(x)$ in factor form, show that

$$0 < \alpha < \frac{1}{2} < \beta .$$

(ii) If $\left|\alpha - \frac{1}{2}\right| = \left|\beta - \frac{1}{2}\right|$, find the value of b and hence the range of values of c.

(14 marks)

9. Let $\omega\ (\neq 1)$ be a cube root of 1.

(a) (i) Prove that $1 + \omega + \omega^2 = 0$.

(ii) Prove that for any integer k,

$$1 + \omega^{3k+1} + (\omega^2)^{3k+1} = 0 ,$$

$$1 + \omega^{3k+2} + (\omega^2)^{3k+2} = 0 .$$

(6 marks)

(b) Making use of the property of complex numbers: $|\alpha|^2 = \alpha\bar{\alpha}$, or otherwise, show that for any complex number z,

$$|1 - \omega\bar{z}| = |z - \omega| .$$

(5 marks)

(c) If z represents a variable point on the Argand diagram and c is a positive constant, what kind of curves does the equation

$$|1 - \omega\bar{z}| = c$$

represent? Sketch the locus of z on the same diagram for each of the possible values of ω when $c = \frac{1}{2}$.

(9 marks)

10. In Figure 2, $ABCD$ is a square tin plate of side $2\sqrt{2}$ m. $PQRS$ is a square whose centre coincides with that of $ABCD$. The shaded parts are cut off and the remaining part is folded to form a right pyramid with base $PQRS$. Let $PQ = 2x$ metres and let the volume of the pyramid $= V$ cubic metres.

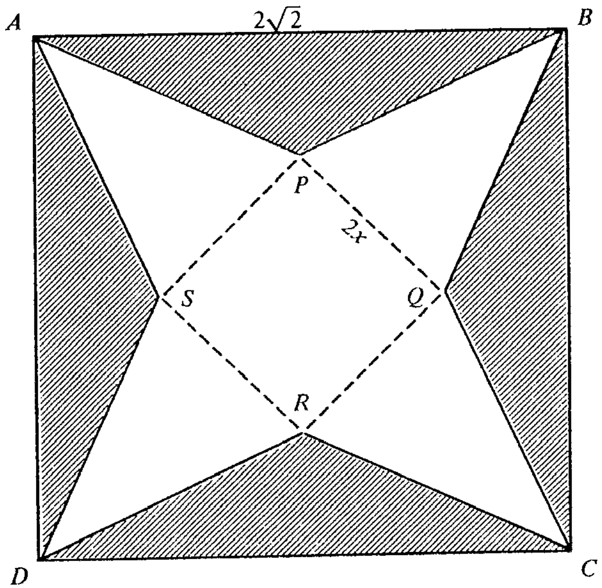

Figure 2

(a) Show that the height of the pyramid is given by $2\sqrt{1-x}$ metres.

Hence express V as a function of x.

(8 marks)

(b) Find the stationary points of the graph of V.

Find the equations of the tangents to the graph at the stationary points and at $x = 1$.

Hence sketch the graph for $0 \leqslant x \leqslant 1$.

(12 marks)

11. In Figure 3, AB is a railway 50 km long. C is a factory h kilometres from B such that $\angle ABC = 90°$. Goods are to be transported from C to A. The transportation cost per tonne of goods across the country by truck is \$2 per km, whereas by railway it is \$1 per km.

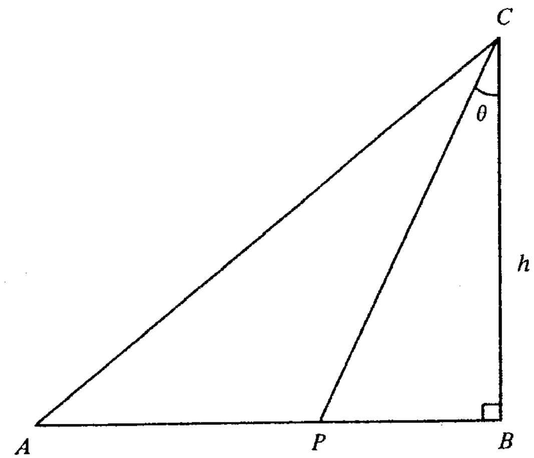

Figure 3

(a) Let P be a point on the railway, $\angle PCB = \theta$, and let \$$N$ be the total transportation cost for 1 tonne of goods from C to P and then to A. Find N in terms of θ and h.

(4 marks)

(b) If $h = 50$, show that the least transportation cost for 1 tonne of goods from C to A is \$$50(\sqrt{3}+1)$.

(7 marks)

(c) (i) Suppose $h > 50\sqrt{3}$. Show that $\tan\theta < \frac{1}{\sqrt{3}}$, and deduce that $\frac{dN}{d\theta} < 0$ for all possible values of θ.

(ii) If $h = 200$, what route should be taken so that the transportation cost is the least?

(9 marks)

END OF PAPER