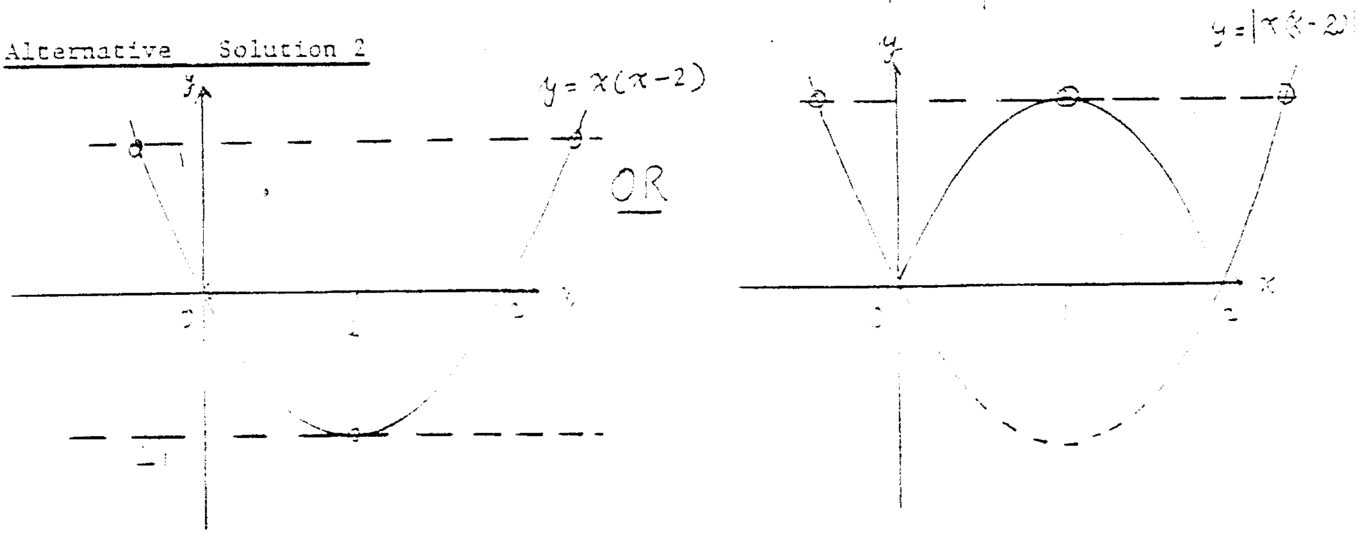
1983 PAPER I

Solution	Marks	Remarks
1. $x^2 - 4x + 2 + \lambda(2x - 1) = 0$		
$\Rightarrow x^2 - (4 + 2\lambda)x + (2 - \lambda) = 0$	1A	
For the equation to have no real roots,		
$(4 + 2\lambda)^2 - 4(2 + \lambda) < 0$	1M	
$4\lambda^2 + 12\lambda - 8 < 0$		
$4(\lambda - 1)(\lambda + 2) < 0$	1A	
$-2 < \lambda < -1$	1M+1A	
	5	
2. a, b, c in A.P. $\Rightarrow$ $b = \frac{1}{2}(a+c)$	1A	
x, y, z in G.P. $\Rightarrow$ $y = \sqrt{xz}$	1A	
$(b - c)\log x + (c - a)\log y + (a - b)\log z$		
$= [\frac{1}{2}(a+c) - c]\log x + (c-a)\log\sqrt{xz} + [a - \frac{1}{2}(a+c)]\log z$	1M	elimination of y and b, etc
$= \left(\frac{a-c}{2}\right)\log x + (c-a)\frac{1}{2}(\log x + \log z) + \left(\frac{a-c}{2}\right)\log z$	1M+1M+1A	1M for $\log MN = \log M + \log N$; 1M for $\log M^n = n\log M$
$= 0$		
	6	
Alternatively		
Let $b = a + d,\ c = a + 2d$	1A	
$y = xr,\ z = xr^2$	1A	
$(b - c)\log x + (c - a)\log y + (a - b)\log z$		
$= -d\log x + 2d\log xr - d\log xr^2$	1M	
$= -d\log x + 2d(\log x + \log r) - d(\log x + 2\log r)$	1M+1M+1A	
$= 0$		
	6	

Solution	Marks	Remarks
3. $AB = AC = 1 - x$		
$\therefore \quad AD = \sqrt{(1-x)^2 - x^2}$		
$= \sqrt{1 - 2x}$	1A	
Volume formed $= 2 \times \frac{1}{3}\pi AD^2 \times BD$	1M	
$= \frac{2}{3}\pi(1-2x)x$	1A	
$V = \frac{2}{3}\pi(x - 2x^2)$		
$\frac{dV}{dx} = \frac{2}{3}\pi(1 - 4x)$	1A	
$\frac{dV}{dx} = 0$	1M	
$\Rightarrow x = \frac{1}{4}$		
$\frac{d^2V}{dx^2} = \frac{2}{3}\pi(-4) < 0$	1A	
$\therefore$ V is maximum at $x = \frac{1}{4}$		
	6	
4. $(1+ax)^4(1-4x)^3 = (1 + 4ax + 6a^2x^2 + \ldots)(1 - 12x + 48x^2 + \ldots)$	1+1+1A	1 for "+ ... "
$= 1 + (4a-12)x + (6a^2-48a+48)x^2 + \ldots$	2A	-1 for 1 wrong term
As the coefficient of x is zero, $4a - 12 = 0$, $a = 3$	1A	
$\therefore$ coefficient of x^2 is $54 - 144 + 48 = -42$	1A	
	7	
5. $\lvert x(x-2)\rvert < 1$		
$\Leftrightarrow -1 < x(x-2) < 1$	1A	
$\Leftrightarrow x^2 - 2x + 1 > 0$ and $x^2 - 2x - 1 < 0$	1+1+1A	
$\Leftrightarrow (x-1)^2 > 0$ and $[x - (1+\sqrt{2})][x - (1-\sqrt{2})] < 0$	1A	
$\Leftrightarrow x \neq 1$ and $1 - \sqrt{2} < x < 1 + \sqrt{2}$	1+1+1A	OR $1-\sqrt{2} < x < 1$ or $1 < x < 1+\sqrt{2}$
	8	

Solution	Marks	Remarks
5. <u>Alternative Solution 1</u>		
Case (i) $x(x-2) \geq 0$ and $x(x-2) < 1$	1A	
$\Rightarrow x(x-2) \geq 0$ and $x^2 - 2x - 1 < 0$	1A	
$\Rightarrow [x \geq 2$ or $x \leq 0]$ and $(1-\sqrt{2}) < x < (1+\sqrt{2})$		
$\Rightarrow [1-\sqrt{2} < x \leq 0]$ or $(2 \leq x < 1+\sqrt{2})$	1+1A	
Case (ii) $x(x-2) < 0$ and $-x(x-2) < 1$	1A	
$\Rightarrow 2 > x > 0$ and $x^2 - 2x + 1 > 0$	1A	
$\Rightarrow 2 > x > 0$ and $x \neq 1$	1A	
Solution of $\lvert x(x-2) \rvert < 0$ is $x \neq 1$ and $(1-\sqrt{2}) < x < 1+\sqrt{2}$	1A	
	8	

<u>Alternative Solution 2</u>



2 Marks for curve.
2 Marks for necessary line(s) or points.
4 Marks for answers.

$x \neq 1$ and $-0.4 < x < 2.4$
1   1   2 (deduct one mark if there is equality sign).
必須全對

Solution	Marks	Remarks
5. $\quad \lvert z - (3 + i) \rvert = \lvert z - (5 + 5i) \rvert$		
$\Rightarrow \lvert (x - 3) + (y - 1)i \rvert = \lvert (x - 5) + (y - 5)i \rvert$	1A	
$\Rightarrow (x - 3)^2 + (y - 1)^2 = (x - 5)^2 + (y - 5)^2$	1M	
$4x + 8y - 40 = 0$	1A	
i.e. $\quad x + 2y - 10 = 0$		
As the locus of z is a line of slope $-\frac{1}{2}$, the required z with the smallest modulus corresponds to the foot of the perpendicular from the origin to this line.	2M	-1 if omitted but continued as below
Equation of perpendicular is $y = 2x$	1A	
Solving this with the locus of z,		
$x + 4x - 10 = 0$		
$x = 2$	1A	
$y = 4$	1A	
$\therefore z = 2 + 4i$		
	8	
<u>Alternatively</u>		
$\lvert z \rvert = \sqrt{x^2 + y^2}$	1A	
$= \sqrt{(10 - 2y)^2 + y^2}$	1M	
$= \sqrt{5y^2 - 40y + 100}$		
$\frac{d\lvert z \rvert}{dy} = \frac{10y - 40}{2\sqrt{5y^2 - 40y + 100}}$	1M	
$\frac{d\lvert z \rvert}{dy} = 0$ when $y = 4$ and $\frac{d\lvert z \rvert}{dy}$ changes sign at $y = 4$.	1M	
$\lvert z \rvert$ is minimum at $x = 2$, $y = 4$.	1A	

Solution	Marks	Remarks

7. (a) $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta +$ — 1A

$6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta +$

$i^4\sin^4\theta$

$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) +$

$(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\,i$ — 1+1A

But $(\cos\theta + i\sin\theta)^4 = \cos4\theta + i\sin4\theta$ — 1A

Comparing real and imaginary parts, we have — 1M

(i) $\cos4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii) $\sin4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

5

(b) $\tan4\theta = \dfrac{\sin4\theta}{\cos4\theta}$ — 1M

$= \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ — 1M

$= \dfrac{4\dfrac{\sin\theta}{\cos\theta} - 4\dfrac{\sin^3\theta}{\cos^3\theta}}{1 - 6\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$ — Remarks: some working must be shown.

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ — 1A

3

Alternatively

$\tan4\theta = \dfrac{2\tan2\theta}{1 - \tan^2 2\theta}$ — 1A

$= \dfrac{\dfrac{4\tan\theta}{1 - \tan^2\theta}}{1 - \dfrac{4\tan^2\theta}{1 - 2\tan^2\theta + \tan^4\theta}}$ — 1A

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ — 1A

3

Solution	Marks	Remarks
7. (c) Putting $x = \tan\theta$ in		
$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$		
$\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$	1A	
$4\tan\theta - 4\tan^3\theta = 1 - 6\tan^2\theta + \tan^4\theta$	1A	
$\therefore \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 1$	1A	
By (b) $\tan 4\theta = 1$	1A	
$4\theta = \dfrac{4n + 1}{4}\pi$	1A	
$\theta = \dfrac{(4n + 1)\pi}{16}$, $n = 0, \pm 1, \pm 2, \ldots$	1+1A	
$\therefore x = \tan\theta$	1M	
$= \tan\dfrac{(4n + 1)\pi}{16}$,	1A	
$x_1 = \tan\dfrac{\pi}{16}$ (0.063π)		
$x_2 = \tan\dfrac{5\pi}{16}$ (0.476π)		
$x_3 = \tan\dfrac{9\pi}{16}$ (-1.60π)		
$x_4 = \tan\dfrac{13\pi}{16}$ (-0.213π)	2A	-1 for each wrong answer
As these are all distinct, they are the four roots of (*)	12	

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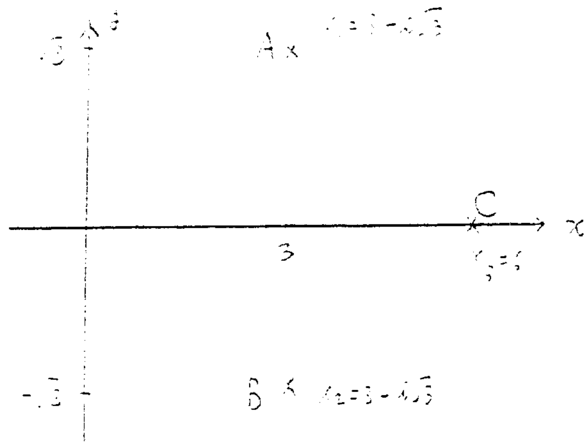
Solution	Marks	Remarks
8. (a) $f(x) = x^3 + ax^2 + bx - 72$		
$f'(x) = 3x^2 + 2ax + b$	1A	
If $x = 4$ is a double root of $f'(x) = 0$		
$\frac{2a}{3} = -8$	1M	Or $4a^2 - 12b = 0$.
$\frac{b}{3} = 16$	1M	$8a + b = -48$.
$\therefore$ $a = -12$	1A	
$b = 48$	1A	
	5	
(b) $x^3 - 12x^2 + 48x - 72 = (x + p)^3 + q$		OR
$= x^3 + 3px^2 + 3p^2x + p^3 + q$	1A	$x^3 - 12x^2 + 48x - 72$
$\Longleftrightarrow \begin{cases} 3p = -12 \\ 3p^2 = 48 \\ p^3 + q = -72 \end{cases}$	1M	$= x^3 - 3\times4\times x^2 + 3\times4^2x - 4^3$
The system is consistent (or rejecting $p = 4$)	1M	$+(4^3 - 72)$ 1M+1A
$p = -4$, $q = -8$	1A+1A	$= (x-4)^3 - 8$ 1A
$\therefore f(x) = (x - 4)^3 - 8$		$p = -4$ 1A
		$q = -8$ 1A
$x^3 - 12x^2 + 48x - 72 = 0$		
$\Rightarrow (x - 4)^3 - 8 = 0$	1M	
$\Rightarrow (x - 4 - 2)[(x - 4)^2 + 2(x - 4) + 4] = 0$		
$\Rightarrow (x - 6)(x^2 - 6x + 12) = 0$	1A	
$\therefore x = 6$ or $3 \pm i\sqrt{3}$	1+1A	−1 for each wrong ans.
	9	
(c) Let $x_1 = 3 + i\sqrt{3}$, $x_2 = 3 - i\sqrt{3}$	1A	
	2A	All 3 points correct.

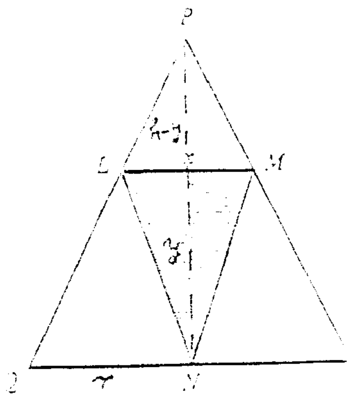
The three roots of $z^3 = 8$ form an equilateral triangle with 0 as the centre and $2\text{cis}120°$, $2\text{cis}240°$ (or $2\text{cis}-120°$), $2\text{cis}0°$ as vertics.

Putting $z = x - 4$, the three roots x_1, x_2, x_3 of $f(x) = 0$ form an equilateral (1A) triangle with $4 + 0i$ as the centre.

Solution	Marks	Remarks
	1A	OR
$\therefore \arg\left(\frac{x_2 - 4}{x_1 - 4}\right) = 120°$ (or $-240°$)	1A	(1A) $\arg\left(\frac{x_2-4}{x_1-4}\right) = \arg\left(\frac{-1-\sqrt{3}i}{-1-\sqrt{3}i}\right)$
	6	(1A) $= \arg\frac{1}{2}(-1+\sqrt{3}i)$
		(1A) $= 120°$ (or $-240°$)

Solution	Marks	Remarks
9. (a) For $n = 1$, L.S. $= 1\cdot 2$		
R.S. $= \frac{1}{3}\,1(2)(3)$		
$=$ L.S.	1A	
Assume that for some $k \geqslant 1$, $1\cdot 2 + 2\cdot 3 + \ldots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$	1M	
For $n = k+1$, L.S. $= 1\cdot 2 + 2\cdot 3 + \ldots + k(k+1) + (k+1)(k+2)$	1A	
$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$	1M	
$= \frac{1}{3}(k+1)(k+2) \times [k+3]$	1A	
$= \frac{1}{3}(k+1)[(k+1)+1][(k+1)+2]$		
$=$ R.S.		
$\therefore$ by induction, $1\cdot 2 + 2\cdot 3 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ for all $n \geqslant 1$.	1M	
	6	
(b) (i) The number of balls in the r-th layer		
$= 1 + 2 + \ldots + r$	1A	
$= \frac{r}{2}(1+r)$	1A	
(ii) The total number of balls in a heap of n layers		
$= \sum_{r=1}^{n} \frac{r}{2}(1+r)$	1M+1M	1M for Σ; 1M for r = 1 to n
$= \frac{1}{2}\sum_{r=1}^{n} r(1+r)$		
$= \frac{1}{2}\left[\frac{1}{3}n(n+1)(n+2)\right]$		
$= \frac{1}{6}n(n+1)(n+2)$	1A	
(iii) The time required to deliver and fire all balls in the r-th layer		
$= \frac{1}{2}r(r+1) \times \frac{2}{r}$ minutes	1M	
$= (r+1)$ minutes	1A	
The total time required $= \sum_{r=1}^{10}(r+1)$	1M	
$= \sum_{r=1}^{10} r + 10$	1A	
$= \frac{1}{2}\,10\cdot 11 + 10$	1A	
$= 65$ minutes	1A	
	14	

Solution	Marks	Remarks
10. (a) Since LM // QR , $\triangle PQR \sim \triangle PLM$	1M	
Let the heights of $\triangle PQR$ and $\triangle LMN$ be h and y, respectively,		
$\frac{h-y}{x} = \frac{h}{2r}$	1A	
$h - y = \frac{h}{2r}x$		
$y = h - \frac{h}{2r}x$	1A	
Area of $\triangle LMN$ = A = $\frac{1}{2}x\,y$		
$= \frac{1}{2}(h - \frac{h}{2r}x)x$	1M	
$\frac{dA}{dx} = \frac{1}{2}(h - \frac{h}{r}x)$	1A	
$\frac{dA}{dx} = 0$ if $x = r$	1M+1A	
$\frac{d^2A}{dx^2} = -\frac{h}{2r} < 0$		
$\therefore$ A is maximum at $x = r$.	1A	
	8	
(b) Volume of cone = V = $\frac{1}{3}\pi\left(\frac{x}{2}\right)^2 \cdot y$	1A	
$= \frac{1}{3}\pi\frac{x^2}{4}\cdot(h - \frac{h}{2r}x)$	1M	
$= \frac{\pi}{12}(hx^2 - \frac{h}{2r}x^3)$		
$\frac{dV}{dx} = \frac{\pi h}{12}(2x - \frac{3}{2r}x^2)$	1A	
$\frac{dV}{dx} = 0$ if $x = 0$ or $\frac{4}{3}r$	1A	
$\frac{d^2V}{dx^2} = \frac{\pi h}{12}(2 - \frac{3}{r}x)$	1M	
< 0 at $x = \frac{4}{3}r$		
$\therefore$ V is maximum at $x = \frac{4}{3}r$	1A	
	6	

Solution	Marks	Remarks
10. (c) Volume of cone generated by revolving △LMN in (a) about PN $= \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 y$	1A	
where $y = h - \frac{h}{2r}r$		
$= \frac{h}{2}$	1A	
Volume of cone in (b) $= \frac{1}{3}\pi\left(\frac{2}{3}r\right)^2 y$,	1A	
where $y = h - \frac{h}{2r}\left(\frac{4}{3}r\right)$		
$= \frac{h}{3}$	1A	
∴ ratio of 2 volumes $= \dfrac{\frac{1}{3}\pi\left(\frac{r}{2}\right)^2\frac{h}{2}}{\frac{1}{3}\pi\left(\frac{2}{3}r\right)^2\frac{h}{3}}$	1M	
$= \frac{27}{32}$	1A	
	6	

Solution	Marks	Remarks
11. (a) (i) $x^2 + y^2 - 2xy\cos 120^\circ = 7$	2A	
$x^2 + y^2 + xy = 7$ .... (*)		
$x = 2$, $y^2 + 2y - 3 = 0$	1M	
$(y-1)(y+3) = 0$	1A+1A	1A for $y = 1$ or -3; 1A for $y = 1$ (-3 rejected)
$y = 1$ (−ve value rejected)		
(ii) Diff. (*) w.r.t. x,		
$2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$	1M+1A	
$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$	1A	
$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$	1M	
$= -\left(\frac{2x+y}{x+2y}\right)\left(\frac{dx}{dt}\right)$		
	10	
(b) At $x = 2$, $\frac{dx}{dt} = -\frac{1}{2}$	1M	accept $\frac{1}{2}$
$\frac{dy}{dt} = -\left(\frac{2x+y}{x+2y}\right)\left(\frac{dx}{dt}\right)$		
$= -\left(\frac{4+1}{2+2}\right)\left(-\frac{1}{2}\right)$	1M	
$= \frac{5}{8}$	1A	
The speed of B is $\frac{5}{8}$ m/s.	1A	
	4	
(c) Area of $\triangle ABO = \frac{1}{2}xy\sin 120^\circ$		
$= \frac{\sqrt{3}}{4}xy$	1A	
Area of $\triangle ABO$ is also equal to $\frac{1}{2}p\sqrt{7}$	1A	
$\therefore \frac{\sqrt{3}}{4}xy = \frac{1}{2}p\sqrt{7}$	1M	
$p = \frac{xy}{2}\sqrt{\frac{3}{7}}$		
$\frac{dp}{dt} = \frac{1}{2}\sqrt{\frac{3}{7}}\left[y\frac{dx}{dt} + x\frac{dy}{dt}\right]$	1A	
When $x = 2$, $y = 1$, $\frac{dx}{dt} = -\frac{1}{2}$, $\frac{dy}{dt} = \frac{5}{8}$		
$\therefore \frac{dp}{dt} = \frac{1}{2}\sqrt{\frac{3}{7}}\left(-\frac{1}{2} + 2\left(\frac{5}{8}\right)\right)$	1M	
$= \frac{3}{8}\sqrt{\frac{3}{7}}$	1A	
	6	