11. (a) Let $|z|$ and $\bar{z}$ denote, respectively, the modulus and conjugate of the complex number z. Show that

(i) $|z|^2 = z\bar{z}$,

(ii) the imaginary part of $z = -\frac{1}{2}i(z - \bar{z})$.

(5 marks)

(b) Let p and q be non-zero, distinct complex numbers such that $|p - q| = |p + q|$.

(i) Using the results in (a), or otherwise, show that

$$p\bar{q} + \bar{p}q = 0$$

and the imaginary part of $\frac{ip}{q} = 0$.

(ii) Let O, P and Q be three points on the Argand plane representing the complex numbers 0, p and q, respectively. By considering the argument of $\frac{p}{q}$, or otherwise, show that $OP \perp OQ$.

(15 marks)

12. Let λ_1 and λ_2 be the roots of the quadratic equation

$$t^2 - (b + 1)t + (b - 1) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (*),$$

where b is a real number.

(a) (i) Show that λ_1 and λ_2 are real and distinct.

(ii) By proving $(1 - \lambda_1)(1 - \lambda_2) < 0$, deduce that either $\lambda_1 < 1 < \lambda_2$ or $\lambda_2 < 1 < \lambda_1$.

(7 marks)

(b) Let λ be one of the roots of $(*)$. Find b in terms of λ and hence express $(1 - \lambda)[(x^2 + 2x + b) - \lambda(x^2 + 1)]$ as a perfect square.

$b = \frac{\lambda^2 - \lambda - 1}{\lambda - 1}$

$= \{x(1-\lambda)+1]^2$

(5 marks)

(c) Using the results of (a) and (b), show that if $\lambda_1 < \lambda_2$, then

$$\lambda_1 < \frac{x^2 + 2x + b}{x^2 + 1} < \lambda_2 \quad \text{for all real values of } x.$$

(8 marks)

END OF PAPER

香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八二年香港中學會考

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1982

附加數學 試卷二

二小時完卷

上午十一時十五分至下午一時十五分

本試卷必須用英文作答

ADDITIONAL MATHEMATICS PAPER II

Two hours

11.15 a.m.—1.15 p.m.

This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

SECTION A (40 marks)
Answer ALL questions in this section.

1. By using the substitution $u = \sqrt{x+9}$, or otherwise, find the indefinite integral

$$\int \frac{x}{\sqrt{x+9}}\,dx\ .$$

(5 marks)

2. Find the ratio in which the line segment joining $A(3,\,-1)$ and $B(-1,\,1)$ is divided by the straight line $x - y - 1 = 0$.

(5 marks)

3. Find the general solution of the equation

$$\cos 2\theta - \sqrt{3}\cos\theta + 1 = 0\ .$$

(6 marks)

4. Figure 1 shows the curve

$$C:\ x = 2 + \sin y\,,$$

where $0 \leqslant y \leqslant 2\pi$. A vessel is formed by rotating OA and C about the y-axis. Find the capacity of the vessel in terms of π.

(6 marks)

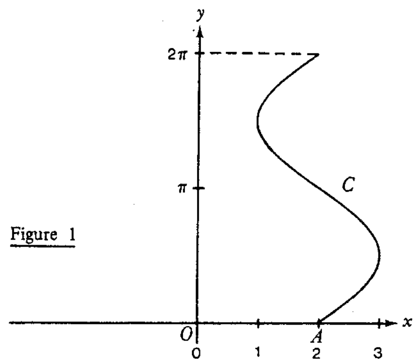

Figure 1

5. Given that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x + \sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\cos x + \sin x}\,dx$. By considering the sum of these integrals, determine their common value in terms of π.

(6 marks)

6. A is the point $(3,\,0)$. $P(x_1,\,y_1)$ is a variable point on the circle $x^2 + y^2 = 4$. If AP is divided internally in the ratio $2:3$ at Q, find the equation of the locus of Q.

(6 marks)

7. In Figure 2, P and S are variable points on the line OA while Q and R are variable points on the line OB such that $PQ \perp OB$, $RS \perp OA$ and $OQ = QR$. θ is constant. Let $OP = x$.

   (a) Find the areas of ΔOPQ and ΔORS in terms of x and θ.

   (b) If the rates of change of area (with respect to x) of the two triangles are equal, find θ.

(6 marks)

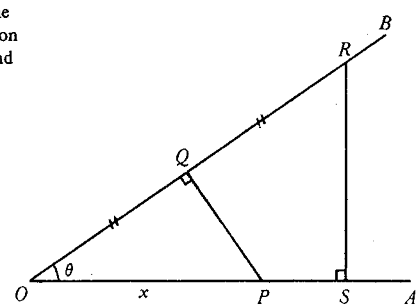

Figure 2

SECTION B (60 marks)
Answer any THREE questions from this section.
Each question carries 20 marks.

8. M is the point $(5,\,6)$, L is the line $5x + 12y = 32$ and C is the circle with M as centre and touching L.

   (a) Find the equation of C. (4 marks)

   (b) Show that C also touches the y-axis. (2 marks)

   (c) Find the equation of the tangent (other than the y-axis) to C from the origin. (6 marks)

   (d) $P(2,\,2)$ is a point on C. Q is another point on C such that PQ is a diameter. Find the equation of the line PQ and write down the equation of the family of circles passing through P and Q.

   Hence, or otherwise, find the equation of the circle which passes through P, Q and the origin. (8 marks)

9. $P(s^2,\,2s)$ and $Q(t^2,\,2t)$ are distinct points on the parabola $y^2 = 4x$, where s and t are non-zero. The tangents at P and Q meet at R.

   (a) Find the equations of PR and QR and hence find the coordinates of R in terms of s and t. (6 marks)

   (b) If s and t vary such that the sum of the slopes of PR and QR is always equal to 2, show that R must lie on a straight line and find the equation of this line L. (4 marks)

   (c) Find the area of the region bounded by L and the parabola. (6 marks)

   (d) If the region in (c) is rotated about the x-axis, find the volume generated. (4 marks)

10. (a) The lines $3x - 2y - 8 = 0$ and $x - y - 2 = 0$ meet at a point P. L_1 and L_2 are lines passing through P and having slopes $\frac{1}{2}$ and 2, respectively. Find their equations. (6 marks)

   (b) A line L through the point $Q(2,\,0)$ intersects L_1 and L_2 at two distinct points A and B, respectively. If the slope of L is m, show that the area of ΔPAB is

$$\frac{6(m-1)^2}{(m-2)(2m-1)}\ .$$

   As m varies, find the equation of L such that the area of ΔPAB is a minimum. (14 marks)

11. (a) Using the substitution $u = \cos\theta$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \sin^3\theta \cos^2\theta \, d\theta$. (6 marks)

(b) Given the curve $C : y = x^3\sqrt{1 - x^2}$.

(i) Write down the range of values of x for which y is real.

(ii) Find the points where C meets the x-axis.

(iii) Find the coordinates of those points on C at which the tangents are parallel to the x-axis. (6 marks)

(c) Using the results in (b), sketch the curve C. (3 marks)

(d) Using the substitution $x = \sin\theta$ and the result in (a), find the total area bounded by the curve C and the x-axis. (5 marks)

12. (a) Express $\tan 2\theta$ in terms of $\tan\theta$.
Hence find an expression for $\tan 15°$ in surd form. $2-\sqrt{3}$ (4 marks)

(b) In Figure 3, XT is the horizontal ground and TY is a tower, perpendicular to XT. A and B are flower pots on the ground between X and T. A man ascends the tower. When he reaches a point C, at height 10 metres from the ground, he observes the angles of depression of A and B to be $30°$ and $15°$ respectively.

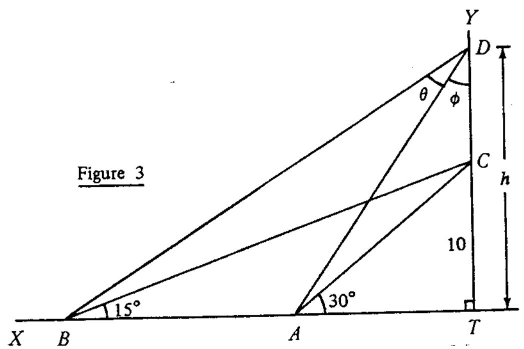

Figure 3

(i) Find the distance between A and B. $\frac{10}{\tan 15°} - \frac{10}{\tan 30°}$

(ii) If the man continues to climb up the tower until he reaches a point D, at height h metres, such that $\angle ADB = \theta$ and $\angle ADT = \phi$, express $\tan\phi$ in terms of h and hence show that

$\tan\phi = \frac{10}{h\tan 30°} = \frac{10\sqrt{3}}{h}$

$$\tan\theta = \frac{20h}{h^2 + 100(3 + 2\sqrt{3})}.$$

(iii) Find the value of h so that AB subtends equal angles at D and C.
What is the value of h when the angle subtended by AB at D is a maximum? (16 marks)

END OF PAPER