11. (a) Prove, by mathematical induction, that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

for all positive integers n.

(6 marks)

(b) Identical cubical bricks are piled up in layers to form a pyramid-like solid with a square base of side x metres as shown in Figure 2. The side of the bottom layer consists of n bricks whereas each side of the square layer immediately above has $n-1$ bricks, and so on. There is only one brick in the top layer.

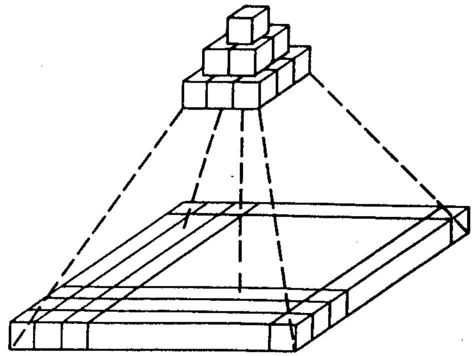

Figure 2

(i) Find the volume of the rth layer counting from the top. Hence find the volume of the solid.

(ii) Using the results of (a) and (b)(i), show that the volume of the solid is always greater than that of a pyramid of the same height, standing on the same base.

When n is very large, what value will the difference in volumes be close to?

(14 marks)

12. Let $\omega = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$, where $i^2 = -1$ and k is a given integer such that $\omega \neq 1$.

(a) Show that $\omega^n + \omega^{-n} = 2\cos\frac{2nk\pi}{5}$ for any integer n. (3 marks)

(b) Prove that $\omega^5 = 1$.
Hence, or otherwise, show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$. (6 marks)

(c) Making use of the results in (b), show that

$$(\omega + \omega^{-1})^2 + (\omega^2 + \omega^{-2})^2 = 3.$$ (6 marks)

(d) Deduce from (a) and (c) that $\left(\cos\frac{2k\pi}{5}\right)^2 + \left(\cos\frac{4k\pi}{5}\right)^2 = \frac{3}{4}$. (5 marks)

END OF PAPER

香港考試局
HONG KONG EXAMINATIONS AUTHORITY

一九八一年香港中學會考
HONG KONG CERTIFICATE OF EDUCATION EXAMINATION 1981

附加數學 試卷二

二小時完卷
上午十一時十五分至下午一時十五分
本試卷必須用英文作答

ADDITIONAL MATHEMATICS PAPER II

Two hours
11.15 a.m.—1.15 p.m.
This paper must be answered in English

Answer ALL questions in Section A and any THREE questions from Section B.

All working must be clearly shown.

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SECTION A (40 marks)

Answer ALL questions in this section.

1. Find the indefinite integral $\int (1 + \cos\theta)^2 \, d\theta$. (5 marks)

2. Figure 1 shows the curves

$C : y = \cos 2x$ and

$S : y = \sin \frac{x}{2}$,

where $0 < x < \pi$. Given that the curves meet at the points P and Q whose x-coordinates are $\frac{\pi}{5}$ and π , respectively, find the area of the region bounded by S and C . (5 marks)

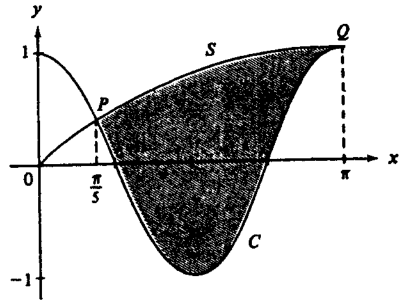

Figure 1

3. Using the substitution $u^2 = 9 - x$, evaluate $\int_0^9 \frac{x}{\sqrt{9 - x}} \, dx$. (6 marks)

4. If $12\cos 3x - 5\sin 3x = r\cos(3x + \theta)$, where $r > 0$ and $0° < \theta < 90°$, find r and θ .

Hence find the general solution of

$12\cos 3x - 5\sin 3x = 13$,

giving your final answer to the nearest degree. (6 marks)

5. If $\sin\theta$ and $\cos\theta$ $(0° < \theta < 90°)$ are the roots of the equation

$2x^2 - hx + 1 = 0$,

find the value of h , leaving your answer in surd form. (6 marks)

6. The circles

$C_1 : x^2 + y^2 + 7y + 11 = 0$ and

$C_2 : x^2 + y^2 + 6x + 4y + 8 = 0$

touch each other externally at P .

(a) Find the coordinates of P .

(b) Find the equation of the common tangent at P . (6 marks)

7. $S(s, 3s)$ and $T(t, -3t)$ are variable points on the lines

$y = 3x$ and

$y = -3x$,

respectively, such that the length of ST is always equal to 2 units. If P is the mid-point of ST , find the equation of the locus of P . (6 marks)

SECTION B (60 marks)

Answer any THREE questions from this section.
Each question carries 20 marks.

8. (a) Using the substitution $y = \sin x$, evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \, dx$. (6 marks)

(b) (i) Show that $\frac{1}{x^2 + 3} - \frac{1}{(x + 1)^2} \equiv \frac{2(x - 1)}{(x^2 + 3)(x + 1)^2}$ for $x \neq -1$.

(ii) Using the substitution $x = \sqrt{3}\tan\theta$, show that $\int_0^3 \frac{dx}{x^2 + 3} = \frac{\pi\sqrt{3}}{9}$.

(iii) Using the results of (i) and (ii), evaluate $\int_0^3 \frac{2(x - 1)}{(x^2 + 3)(x + 1)^2} \, dx$.

(14 marks)

9. The gradient of a curve at any point (x, y) is given by

$$\frac{dy}{dx} = k(x - \frac{1}{4}) ,$$

where k is a constant.

(a) Find the value of k if the curve passes through the points $(-1, 4)$ and $(0, 1)$. Find also the equation of the curve. (6 marks)

(b) Find the area of the region in the first quadrant bounded by the curve, the y-axis and the line $y = 2x + 3$. (7 marks)

(c) If the region in (b) is rotated about the x-axis, find the volume generated. (7 marks)

10. In Figure 2, $ABCDE$ is a right pyramid with a square base $ABCD$. Each of the eight edges of the pyramid is of length k. F, G and H are points on AB, AC and AD, respectively, such that FGH is a straight line and $BF = DH = rk$, where $0 < r < 1$. $EG \perp HF$, $\angle EGC = \theta$ and N is the foot of the perpendicular from E to the base.

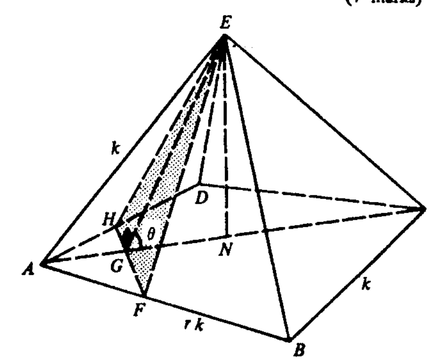

Figure 2

(a) Express FE^2 and FG^2 in terms of k and r. (8 marks)

(b) Express EG and EN in terms of k and r .

Hence, or otherwise, show that $\sin\theta = \frac{1}{\sqrt{1 + r^2}}$. (8 marks)

(c) Using the results of (b), find the range of the inclination of the plane EFH to the base as r varies from 0 to 1 . (4 marks)

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11. The lines

$L_1 : x - y - 2 = 0$ and

$L_2 : x + 2y - 8 = 0$

intersect at A.

(a) B and C are points on L_1 and L_2, respectively. If the centroid of ΔABC is $G(t, t-6)$, find, in terms of t, the coordinates of the mid-point D of BC. (5 marks)

(b) If AD passes through $H(0, -10)$, find the length of AD and $\tan \angle BAD$. (7 marks)

(c) Given that $AB = 14\sqrt{2}$ units, use the result of (b) to find the area of ΔABC. (4 marks)

(d) A point P moves such that the area of ΔAPD is equal to that of ΔACD. It is known that the locus of P consists of a pair of lines; find the equations of these lines. (4 marks)

12. The line $L : y = mx + 2$ meets the circle $C : x^2 + y^2 = 1$ at the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(a) Show that the length of the chord AB is $2\sqrt{\dfrac{m^2 - 3}{m^2 + 1}}$. (6 marks)

(b) Find the values of m such that

(i) L meets C at two distinct points,

(ii) L is a tangent to C,

(iii) L does not meet C. (5 marks)

(c) For the two tangents in (b)(ii), let the corresponding points of contact be P and Q. Find the equation of PQ. (5 marks)

(d) Find the equation of the family of circles of which PQ is a chord. (4 marks)

END OF PAPER

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