香港考試局

HONG KONG EXAMINATIONS AUTHORITY

一九八一年香港中學會考

HONG KONG CERTIFICATE OF EDUCATION EXAMINATION, 1981

Additional Mathematics I

MARKING SCHEME

評卷參考

Solution	Marks	Remarks
1. $(1+2x)^4(1-x)^7$		
$= [1+4(2x)+6(2x)^2+\cdots][1+7(-x)+21(-x)^2+\cdots]$	2+1A	2A for first given factor
$= 1+x-11x^2+\cdots$	1A	
Coefficient of x^2 is -11.	1A	
	5	
Alternatively,		
$(1+2x)^4(1-x)^7$		
$= [1+4(2x)+6(2x)^2+\cdots][1+7(-x)+21(-x)^2+\cdots]$	2+1A	
Coeff. of $x^2 = 1\times 21 + 8\times(-7) + 24\times 1$	1A	
$= -11$	1A	
	5	
Alternatively,		
Coeff of x^2		
$= {}_4C_2(2)^2\times 1 + {}_4C_1\times 2\times {}_7C_1\times(-1) + 1\times {}_7C_2$	1+1+1A	
$= 24\times 1 + 8\times(-7) + 1\times 21$	1A	
$= -11$	1A	
	5	
2. $\log_{78} 52 = \dfrac{\log_3 52}{\log_3 78}$	1M	or any ~~suitable~~ base
$= \dfrac{\log_3 4 + \log_3 13}{\log_3 2 + \log_3 3 + \log_3 13}$	1M	for $\log(ab) = \log a + \log b$
$= \dfrac{2\log_3 2 + \log_3 13}{\log_3 2 + \log_3 3 + \log_3 13}$	1M	for $\log(a^p) = p\log a$
$= \dfrac{2a+b}{1+a+b}$	2A	有[illegible] award
	5	
Alternatively,		
$\left.\begin{array}{l}\log_3 2 = a \Rightarrow 2 = 3^a \\ \log_3 13 = b \Rightarrow 13 = 3^b\end{array}\right\}$	1A	
Let $\log_{78} 52 = x$	1M	
$78^x = 52$		
$(2\times 3\times 13)^x = 2\times 2\times 13$		
$2^{x-2}\times 3^x\times 13^{x-1} = 1$		
$(3^a)^{x-2}\times 3^x\times(3^b)^{x-1} = 3^0$	1M	Put $2=3^a$, $13=3^b$
$ax-2a+x+bx-b = 0$	1M	Equate indices
$x = \dfrac{2a+b}{1+a+b}$	1A	
	5	

Solution	Marks	Remarks
3. Put $u = \frac{x-1}{x+1}$		
$\frac{dy}{du} = \sec^2 u$	1A	
$\frac{du}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2}$		
$= \frac{2}{(x+1)^2}$	1A	
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ Chain Rule	2M	
$= \frac{2}{(x+1)^2} \cdot \sec^2\left(\frac{x-1}{x+1}\right)$	1A	
	5	
4. $x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$		
$= \cos\theta \pm i\sin\theta$	1A	De Moivre's Theorem
$x^n = \cos n\theta \pm i\sin n\theta$	1M+1A	
$\therefore$ Reqd eqt. is		Alternatively, Sum of roots $= 2\cos n\theta$
$(x - \cos n\theta - i\sin n\theta)(x - \cos n\theta + i\sin n\theta) = 0$	1M	Prod. of roots $= 1$ $\longrightarrow$ 1M + 1A
$x^2 - 2\cos n\theta \cdot x + 1$ $= 0$ Deduct 1 mark	2A	
	6	$x^2 - 2\cos n\theta \cdot x + 1 = 0$ 1A
5. $f(x) = (x + \frac{a}{2})^2 + (b - \frac{a^2}{4})$	1+1A	Alternatively,
一定高 $\geq b - \frac{a^2}{4}$ (realize $(x+\frac{a}{2})^2 \geq 0$	1M	$\frac{df}{dx} = 2x + a$ 1A
$= f(-\frac{a}{2})$	1A	$\frac{df}{dx} = 0$
		$\Rightarrow x = -\frac{a}{2}$ 1A
Alternatively,		
$f(x) - f(-\frac{a}{2})$ 相減	1M	$\frac{d^2f}{dx^2} = 2 > 0$
$= (x^2 + ax + b) - (\frac{a^2}{4} - \frac{a^2}{2} + b)$		$\therefore f$ is min. at $-\frac{a}{2}$ 1A
$= x^2 + ax + \frac{a^2}{4}$		
$= (x + \frac{a}{2})^2$	1A	Since f is quadratic, $\therefore f(x) \geq f(-\frac{a}{2})$. 1M
≥ 0 ~~for all x~~	1A	
$\therefore f(x) \geq f(-\frac{a}{2})$ for all x	1A	explain this is absolute minimum
The min. value of $x^2 - \sqrt{13}x + 5$ is		
$(\frac{\sqrt{13}}{2})^2 - \sqrt{13} \cdot \frac{\sqrt{13}}{2} + 5$ or $5 - \frac{1}{4}(\sqrt{13})^2$	1M	
$= \frac{7}{4}$	1A	
	6	

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L.F. 405 (2/77)

Solution $\log_a x^y = y \log_a x$	Marks	Remarks

6. $a^x = (ab)^z$, $b^y = (ab)^z$

$x \log a = z \log(ab)$, $y \log b = z \log(ab)$ — 1+1 M

$x = \dfrac{\log(ab)}{\log a} \cdot z \qquad y = \dfrac{\log(ab)}{\log b} \cdot z$ — 1+1 A

$$\frac{xy}{x+y} = \frac{\frac{\log(ab)}{\log a}\cdot z \cdot \frac{\log ab}{\log b}\cdot z}{\frac{\log ab}{\log a}\cdot z + \frac{\log ab}{\log b}\cdot z}$$ — 1M

$$= \frac{\log(ab)\cdot z}{\log a + \log b}$$

$$= z$$ — 1A

6

Remarks: Alternatively,

$x \log a = y \log b$ — 1M

$x \log a = z(\log a + \log b)$ — 1M

$(x-z)\log a = z \log b$ — 1M + 1A

$\therefore \dfrac{x}{x-z} = \dfrac{y}{z}$ — 1M

$z = \dfrac{xy}{x+y}$ — 1A

eliminate $\log a$, $\log b$

Alternatively,

$a^x = b^y \Rightarrow b = a^{\frac{x}{y}}$ — 1A

$(ab)^z = (a \cdot a^{\frac{x}{y}})^z$ — 代 $b = a^{\frac{x}{y}}$ — 1M

$= a^{\frac{yz+xz}{y}}$ — 1A

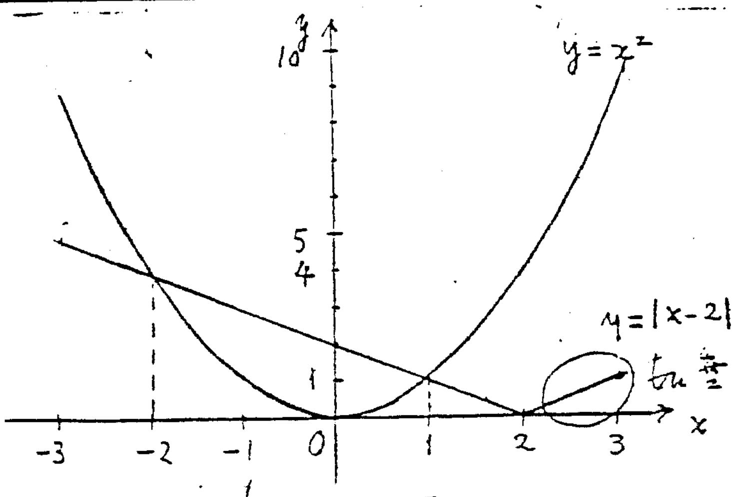
$\therefore a^x = a^{\frac{yz+xz}{y}}$ — equate — 1M

$x = \dfrac{yz+xz}{y}$ — equate index — 1M

$z = \dfrac{xy}{x+y}$ — 1A

6

7.



drawn result

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$\lvert x-2 \rvert$	5	4	3	2	1	0	1

1A — For graph of $y = x^2$

2A — For graph of $y = |x-2|$

如畫張則給 2A 及 1+1+1 A

From the graphs, $|x-2| < x^2$ — 1M

iff $\underline{x < -2}$ $\underline{\text{or}}$ $\underline{x > 1}$ — 1+1+1A — If equality sign included in the answer, deduct 1 mark.

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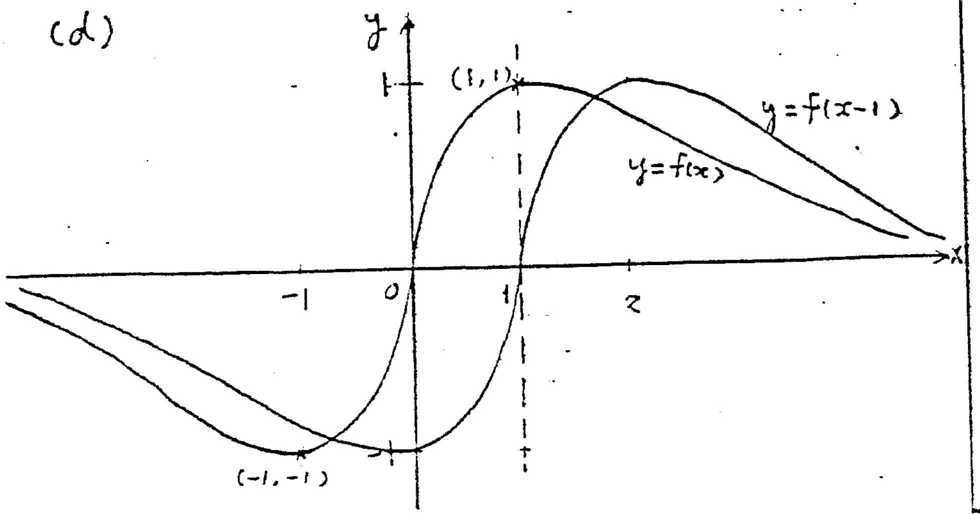
Solution	Marks	Remarks
8. (a) $f(-x) = \frac{2(-x)}{(-x)^2+1}$ 代入 $(-x)$ →	1M	$\mathcal{L}$ must
$= \frac{-2x}{x^2+1}$		
$= -f(x)$ for all x	1A	
	2	
(b) $\frac{dy}{dx} = \frac{2(x^2+1) - 2x \cdot 2x}{(x^2+1)^2}$	1M	← For Quotient Rule
$= \frac{2-2x^2}{(x^2+1)^2}$	1A	
$\frac{d^2y}{dx^2} = \frac{-4x(x^2+1)^2 - (2-2x^2) \cdot 2(x^2+1) \cdot 2x}{(x^2+1)^4}$	1M	← For Quotient Rule
$= \frac{4x(x^2+1)[-(x^2+1)-(2-2x^2)]}{(x^2+1)^4}$		
$= \frac{4x(x^2-3)}{(x^2+1)^3}$	2A	Accept $\frac{d^2y}{dx^2} = \frac{4(x^5-2x^3-3x)}{(x^2+1)^4}$
	5	
(c) Put $\frac{dy}{dx} = 0$ Put $\frac{dy}{dx} = 0$	1M	
$x = \pm 1$	1+1A	
At $x=1$, $\frac{d^2y}{dx^2} < 0$, ∴ f is max.	1M	← Test max or min. from $\frac{d^2y}{dx^2}$
Awarded independent of the accuracy of $\frac{d^2y}{dx^2}$ → ∴ Max. point $=(1,1)$ 撞都有分	1A	
At $x=-1$, $\frac{d^2y}{dx^2} > 0$, ∴ f is min.	1M	← and realise that max. or min. may be determined from y''
→ ∴ Min. point $= (-1,-1)$	1A	
	7	
(d) 	1A	For (0,0)
	1A	For the 2 turning pts
	1A	For 2 tails not cutting x-axis
	1M	shifting horizontally
	1M	shifting 1 unit towards the right
	1A	for graph of $y=f(x-1)$
	6	

Solutions	Marks	Remarks
9 (a) $V = \underline{\frac{4}{3}\pi r^3} + \underline{\pi r^2 h}$	1+1A	
$\therefore h = \frac{V}{\pi r^2} - \frac{4}{3}r$	1A	
	3	
(b) (i) Surface area of cylinder $= 2\pi r h$	1A	
$= 2\pi r\left(\frac{V}{\pi r^2} - \frac{4}{3}r\right)$	1M	Sub. h
Surface area of ends $= 4\pi r^2$	1A	
$\therefore C = 2\pi r\left(\frac{V}{\pi r^2} - \frac{4}{3}r\right)k + 4\pi r^2 \times 2k$	1+1M	$\leftarrow C = \underset{\text{cylinder}}{S_C}\, k + \underset{\text{ends}}{S_E}\, 2k$
$= \frac{2kV}{r} + \frac{16}{3}k\pi r^2$	1A	
(ii) $\frac{dC}{dr} = -\frac{2kV}{r^2} + \frac{32}{3}k\pi r$	1A	
用一次 $\frac{dc}{dr}$		
$\frac{dC}{dr} = 0 \Rightarrow (32k\pi r^3 - 6kV) = 0$	1M	Put $\frac{dc}{dr} = 0$
$r = \sqrt[3]{\frac{3V}{16\pi}}$	1A	
$\frac{d^2C}{dr^2} = \frac{4kV}{r^3} + \frac{32}{3}k\pi$ $\frac{d^2c}{dr^2}$ 用	1M+1A	Alternatively, Checking sign of $\frac{dc}{dr}$ 2M
If $r = \sqrt[3]{\frac{3V}{16\pi}}$, $\frac{d^2C}{dr^2} > 0$ Test sign of $\frac{d^2c}{dr^2}$	1M+1A	Correct working 2A
$\therefore C$ is min. awarded only if the above is correct		
(iii) $\frac{r}{h} = \frac{r}{\frac{V}{\pi r^2} - \frac{4}{3}r}$ 代 h from (a)	1M	
$= \frac{3\pi r^3}{3V - 4\pi r^3}$		
$= \frac{3\pi \cdot \frac{3V}{16\pi}}{3V - 4\pi \cdot \frac{3V}{16\pi}}$	1M	
$= \frac{1}{4}$	2A	
$\therefore r : h = 1 : 4$		
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Solutions	Marks	Remarks
(10) (a) $\frac{df}{dx} = 3x^2 + 2ax + b$ 試 $\frac{df}{dx}$	1M+1A	
f has stationary values at α, β		
$\Rightarrow \alpha, \beta$ are roots of $3x^2+2ax+b=0$	2M	← may be omitted if the following are correct
$\therefore \alpha+\beta = -\frac{2a}{3}$	1A	
$\alpha\beta = \frac{b}{3}$	1A	
	6	
(b) Discriminant of $f'(x)=0$ is	1M	Alternatively,
$(2a)^2 - 4(3b)$ 搵 D		$(\alpha-\beta)^2 > 0$ 1M
$= 4a^2 - 12b$		$\alpha^2+\beta^2 > 2\alpha\beta$
Since $\alpha \neq \beta$ (α, β real)		$(\alpha+\beta)^2 > 4\alpha\beta$ 1A
$\therefore 4a^2-12b > 0$ 知 $D > 0$	1M	$\therefore \frac{4a^2}{9} > 4 \cdot \frac{b}{3}$
i.e. $a^2 > 3b$	1A	i.e. $a^2 > 3b$ 1A
	3	
(c) $\frac{f(\alpha)-f(\beta)}{\alpha-\beta} = \frac{(\alpha^3+a\alpha^2+b\alpha+c)-(\beta^3+a\beta^2+b\beta+c)}{\alpha-\beta}$	1M 代 α, β	$\left.\begin{array}{l} a = -\frac{3}{2}(\alpha+\beta) \\ b = 3\alpha\beta \end{array}\right\}$ 1M
$= \frac{(\alpha^3-\beta^3)+a(\alpha^2-\beta^2)+b(\alpha-\beta)}{\alpha-\beta}$		$a^2-3b = \frac{9}{4}(\alpha+\beta)^2 - 9\alpha\beta$
		$= \frac{9}{4}(\alpha-\beta)^2$ 1A
		> 0 1A
$= \alpha^2+\alpha\beta+\beta^2+a(\alpha+\beta)+b$	2A	
$= (\alpha+\beta)^2 - \alpha\beta + a(\alpha+\beta) + b$	1M	For $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
代 $\alpha+\beta$, $\alpha\beta$ $\longrightarrow$ $= \frac{4a^2}{9} - \frac{b}{3} - \frac{2a^2}{3} + b$	1M+1A	
$= \frac{2}{9}(3b-a^2)$	1A	
	7	
(d) Since $a^2 > 3b$, $\frac{2}{9}(3b-a^2) < 0$	1M	
$\therefore \frac{f(\alpha)-f(\beta)}{\alpha-\beta} < 0$	1M	
$\therefore f(\alpha)-f(\beta) < 0$		
if $\alpha-\beta > 0$	2A	
i.e. $f(\alpha) < f(\beta)$		
if $\alpha > \beta$		
	4	

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Solutions	Marks	Remarks
(11) (a) Let $P(n): 1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$		
$P(1)$ is true since $R.S. = \frac{1}{6}\times 1\times 2\times 3 = 1 = L.S.$	1A	
Assume $P(n)$ is true for $n=k$,	1M	Assume $n=k$ is true
i.e. $1^2+2^2+\dots+k^2 = \frac{1}{6}k(k+1)(2k+1)$		
$1^2+2^2+\dots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$	1A	Add $(k+1)^2$ to both sides
$= \frac{(k+1)}{6}[(2k^2+k)+6(k+1)]$		
$= \frac{(k+1)}{6}(k+2)(2k+3)$	1A	
$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$	1A	
$\therefore P(n)$ is also true for $n=k+1$. By M.I., $P(n)$ is true $\forall n\in\mathbb{N}$	1M	Only awarded if above correct
	6	
(b) (i) Length of each brick $= \frac{x}{n}$	1A	
~~$\therefore$ vol. of each brick $= \left(\frac{x}{n}\right)^3$~~ ~~No. of bricks in the r-th layer $= r^2$~~	~~1A~~	
vol. of the r-th layer $= r^2\times\left(\frac{x}{n}\right)^3$	1A	
There are altogether n layers	1A	$V = V_1+V_2+\dots V_n$
$\therefore$ vol. of the solid $= \left(\frac{x}{n}\right)^3(1^2+2^2+\dots+n^2)$	1M+1A	
(ii) Height of pyramid $= n\times\frac{x}{n} = x$	1A	
$\therefore$ vol. of pyramid $= \frac{1}{3}x^2\cdot x = \frac{x^3}{3}$	1A	
Vol. of solid $-$ vol. of pyramid		
$= \left(\frac{x}{n}\right)^3(1^2+2^2+\dots+n^2) - \frac{x^3}{3}$	1M	Volume 相減
$= \left(\frac{x}{n}\right)^3\cdot\frac{1}{6}n(n+1)(2n+1) - \frac{x^3}{3}$	1M	代(a) result 入
$= \frac{x^3}{3}\left[\frac{1}{2n^2}(n+1)(2n+1)-1\right]$		
$= \frac{x^3}{3}\left(\frac{3n+1}{2n^2}\right) \quad \left(\text{or } \frac{x^3}{6}\left(\frac{3}{n}+\frac{1}{n^2}\right)\right)$	2A	
>0	1A	
$\therefore$ vol. of solid is always greater than vol. of pyramid. If n is very large, the difference is close to zero.	1A	quite independent of above

Solutions	Marks	Remarks
2) (a) $\omega^n + \omega^{-n} = \left[\cos\frac{2nk\pi}{5} + i\sin\frac{2nk\pi}{5}\right] + \left[\cos\frac{-2nk\pi}{5} + i\sin\frac{-2nk\pi}{5}\right]$	1+1M	De Moivre's Thm
$= 2\cos\frac{2nk\pi}{5}$	1A	
	3	
(b) $\omega^5 = \cos 5\times\frac{2k\pi}{5} + i\sin 5\times\frac{2k\pi}{5}$	1A	
$= \cos 2k\pi + i\sin 2k\pi$		
$= 1$	1A	Alt
		$\omega^5 - 1 = 0$ 1A
$\therefore 1+\omega+\omega^2+\omega^3+\omega^4 = \frac{1-\omega^5}{1-\omega}$	3A	$(\omega-1)(1+\omega+\omega^2+\omega^3+\omega^4) = 0$ 1A
$= 0$ $\boxed{\cancel{(\because \omega \neq 1)}}$ X	1A	$\because \omega \neq 1$, $1+\omega+\omega^2+\omega^3+\omega^4 = 0$ 1A
	6	
Alt		
$1+\omega+\omega^2+\omega^3+\omega^4 = \omega^5+\omega+\omega^2+\omega^3+\omega^4$	1A	代 無 分
$= \omega(\omega^4 + 1 + \omega + \omega^2 + \omega^3)$	1A	
$\therefore (1-\omega)(1+\omega+\omega^2+\omega^3+\omega^4) = 0$	1A	
$1+\omega+\omega^2+\omega^3+\omega^4 = 0$ $\cancel{(\because \omega \neq 1)}$	1A	
(c) $(\omega+\omega^{-1})^2 + (\omega^2+\omega^{-2})^2 = \omega^2+2+\omega^{-2}+\omega^4+2+\omega^{-4}$	1A	Alt
$\omega^{-2}+1 = \omega^{-2}\times\omega^5$; $= \omega^2+2+\omega^3+\omega^4+2+\omega$	2M* + 1A	$(\omega+\omega^{-1})^2 + (\omega^2+\omega^{-2})^2$
		$= \omega^2+2+\frac{1}{\omega^2}+\omega^4+2+\frac{1}{\omega^4}$ 1A
$= 3 + (1+\omega+\omega^2+\omega^3+\omega^4)$	1A	$= \frac{\omega^6+2\omega^4+\omega^2+\omega^8+2\omega^4+1}{\omega^4}$
$= 3$	1A	$= \frac{\omega+2\omega^4+\omega^2+\omega^3+2\omega^4+1}{\omega^4}$ 2M+1A
	6	$= \frac{(1+\omega+\omega^2+\omega^3+\omega^4)+3\omega^4}{\omega^4}$ 1A
* 2M for $\omega^{-n} = \omega^{-n+5}$		$= 3$ 1A
		2M for $\omega^n = \omega^{n-5}$
(d) From (a) $\cos\frac{2k\pi}{5} = \frac{\omega+\omega^{-1}}{2}$, $\cos\frac{4k\pi}{5} = \frac{\omega^2+\omega^{-2}}{2}$	1+1A	
$\therefore \left(\cos\frac{2k\pi}{5}\right)^2 + \left(\cos\frac{4k\pi}{5}\right)^2 = \left(\frac{\omega+\omega^{-1}}{2}\right)^2 + \left(\frac{\omega^2+\omega^{-2}}{2}\right)^2$	1M	Sub.
可以 omit $= \frac{1}{4}\left[(\omega+\omega^{-1})^2 + (\omega^2+\omega^{-2})^2\right]$	1A	
$= \frac{3}{4}$ from (c)	1A	
	5	

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