Marking Scheme

Module 1 (Calculus and Statistics)

This document was prepared for markers' reference. It should not be regarded as a set of model answers. Candidates and teachers who were not involved in the marking process are advised to interpret its contents with care.

General Marking Instructions

1. It is very important that all markers should adhere as closely as possible to the marking scheme. In many cases, however, candidates will have obtained a correct answer by an alternative method not specified in the marking scheme. In general, a correct answer merits ***all the marks*** allocated to that part, unless a particular method has been specified in the question. Markers should be patient in marking alternative solutions not specified in the marking scheme.

2. In the marking scheme, marks are classified into the following three categories:

'M' marks	awarded for correct methods being used;
'A' marks	awarded for the accuracy of the answers;
Marks without 'M' or 'A'	awarded for correctly completing a proof or arriving at an answer given in a question.

In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. However, 'A' marks for the corresponding answers should NOT be awarded (unless otherwise specified).

3. For the convenience of markers, the marking scheme was written as detailed as possible. However, it is still likely that candidates would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, markers should exercise their discretion in marking candidates' work. In general, marks for a certain step should be awarded if candidates' solution indicated that the relevant concept/technique had been used.

4. In marking candidates' work, the benefit of doubt should be given in the candidates' favour.

5. In the marking scheme, 'r.t.' stands for 'accepting answers which can be rounded off to' and 'f.t.' stands for 'follow through'. Steps which can be skipped are shaded whereas alternative answers are enclosed with rectangles . All fractional answers must be simplified.

6. Unless otherwise specified in the question, numerical answers should either be exact or given to 4 decimal places. Answers not accurate up to the required degree of accuracy should not be accepted.

Solution	Marks	Remarks
1. (a) $a+0.15+0.15+b+0.05+0.25=1$	1M	
$a+b=0.4$		
$E(5X+1)=10$		
$5E(X)+1=10$	1M	
$E(X)=1.8$		
$(-1)(a)+(0)(0.15)+(1)(0.15)+(2)(b)+(3)(0.05)+(4)(0.25)=1.8$	1M	
$-a+2b=0.5$		
Solving, we have $a=0.1$ and $b=0.3$.	1A	for both correct
(b) $P(C \mid D)$		
$=\dfrac{P(C\cap D)}{P(D)}$		
$=\dfrac{0.15+0.3}{0.1+0.15+0.15+0.3}$	1M	
$=\dfrac{9}{14}$	1A	r.t. 0.6429
	----------(6)	
2. (a) The required probability		
$=(0.12)(0.97)+(1-0.12)(0.89)$	1M	
$=0.8996$	1A	
(b) The required probability		
$=(0.12)(0.97)+(1-0.12)(1-0.89)$	1M	
$=0.2132$	1A	
(c) The required probability		
$=\dfrac{(0.12)(0.97)}{0.2132}$	1M	
≈ 0.545966228		
< 0.6		
Thus, the required probability is less than 0.6 .	1A	f.t.
	----------(6)	

Solution	Marks	Remarks
3. (a) P(Peter answers 1 question correctly) $= (0.8)(1-0.1) = 0.72$	1M	can be absorbed
The required probability $= (0.72)^{10} + C_1^{10}(0.72)^9(1-0.72) + C_2^{10}(0.72)^8(1-0.72)^2$	1M	
≈ 0.437829035		
≈ 0.4378	1A	r.t. 0.4378
(b) The required probability $= (0.8)^{10}((1-0.1)^{10} + C_1^{10}(1-0.1)^9(0.1) + C_2^{10}(1-0.1)^8(0.1)^2)$	1M	
≈ 0.099837499		
≈ 0.0998	1A	r.t. 0.0998
(c) The required probability $\approx \dfrac{0.099837499}{0.437829035}$	1M	
≈ 0.228028503		
≈ 0.2280	1A ----------(7)	r.t. 0.2280
4. (a) (i) The sample proportion $= \dfrac{28}{40} = 0.7$	1A	
(ii) An approximate 90% confidence interval for p is $= \left(0.7 - 1.645\left(\sqrt{\dfrac{(0.7)(1-0.7)}{40}}\right), 0.7 + 1.645\left(\sqrt{\dfrac{(0.7)(1-0.7)}{40}}\right)\right)$	1M+1A	
$\approx (0.580808426, 0.819191573)$		
$\approx (0.5808, 0.8192)$	1A	r.t. (0.5808, 0.8192)
(b) Let n be the number of children that Mary surveys. $(2)(2.575)\sqrt{\dfrac{(0.7)(1-0.7)}{n}} \leq 0.1$	1M	
$\dfrac{n}{0.21} \geq \left(\dfrac{(2)(2.575)}{0.1}\right)^2$		
$n \geq 556.9725$		
Thus, the least number of children that Mary should survey is 557.	1A ----------(6)	

Solution	Marks	Remarks
5. (a) $\frac{\mathrm{d}g(u)}{\mathrm{d}u}$		
$= e^{-u}(2u+2)+(-e^{-u})(u^2+2u+2)$	1M	
$= -u^2e^{-u}$		
$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{3}x^{\frac{-2}{3}}$	1M	
$\frac{\mathrm{d}g(u)}{\mathrm{d}x}$		
$=\frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$		
$=\left(-u^2e^{-u}\right)\left(\frac{1}{3}x^{\frac{-2}{3}}\right)$		
$=\left(-x^{\frac{2}{3}}e^{-x^{\frac{1}{3}}}\right)\left(\frac{1}{3}x^{\frac{-2}{3}}\right)$		
$=\frac{-1}{3}e^{-x^{\frac{1}{3}}}$		
$\beta=\frac{-1}{3}$	1A	
$\mathrm{g}(u)=e^{-x^{\frac{1}{3}}}(x^{\frac{2}{3}}+2x^{\frac{1}{3}}+2)$		
$\frac{\mathrm{d}g(u)}{\mathrm{d}x}$		
$=\frac{-1}{3}x^{\frac{-2}{3}}e^{-x^{\frac{1}{3}}}(x^{\frac{2}{3}}+2x^{\frac{1}{3}}+2)+e^{-x^{\frac{1}{3}}}\left(\frac{2}{3}x^{\frac{-1}{3}}+\frac{2}{3}x^{\frac{-2}{3}}\right)$	1M+1M	
$=\frac{-1}{3}e^{-x^{\frac{1}{3}}}$		
$\beta=\frac{-1}{3}$	1A	
(b) The required area		
$=\int_0^8 \mathrm{f}(x)\mathrm{d}x$	1A	
$=-3\int_0^8\frac{\mathrm{d}g(u)}{\mathrm{d}x}\mathrm{d}x$		
$=-3[\mathrm{g}(u)]_0^2$	1M	
$=-3\left[e^{-u}\left(u^2+2u+2\right)\right]_0^2$		
$=-3\left(e^{-2}(4+4+2)-e^0(2)\right)$		
$=6-\frac{30}{e^2}$	1A	
	----------(6)	

Solution	Marks	Remarks
6. (a) e^{-6x}		
$=1+(-6x)+\frac{(-6x)^2}{2!}+\frac{(-6x)^3}{3!}+\frac{(-6x)^4}{4!}+\cdots$	1M	
$=1-6x+18x^2-36x^3+54x^4-\cdots$	1A	
(b) $(1-kx^2)^5$		
$=1-5kx^2+10k^2x^4-\cdots-k^5x^{10}$	1M	
$(1)(10k^2)+(18)(-5k)+(54)(1)=-26$	1M	
$10k^2-90k+54=-26$		
$k^2-9k+8=0$		
$(k-1)(k-8)=0$		
$k=1$ or $k=8$	1A	
	----------(5)	
7. Note that $x^3-x+2>0$ for all $0\le x\le 5$.		
(a) $\frac{dy}{dx}$		
$=\frac{(x^3-x+2)e^x-e^x(3x^2-1)}{(x^3-x+2)^2}$	1M	
$=\frac{(x^3-3x^2-x+3)e^x}{(x^3-x+2)^2}$	1A	

Solution	Marks	Remarks
(b) $\frac{dy}{dx}=0$		
$x^3-3x^2-x+3=0$ $(x-3)(x-1)(x+1)=0$ $x=3$, $x=1$ or $x=-1$ (rejected)	1M	

x	$0\le x<1$	$x=1$	$1<x<3$	$x=3$	$3<x\le 5$
$\frac{dy}{dx}$	+	0	–	0	+
y	↗	$\frac{e}{2}$	↘	$\frac{e^3}{26}$	↗

1M — for testing

For $x=0$, we have $y=\frac{1}{2}$. For $x=5$, we have $y=\frac{e^5}{122}$. — 1M

Note that $\frac{1}{2}<\frac{e^3}{26}<\frac{e^5}{122}<\frac{e}{2}$.

Thus, the greatest value and the least value of y are $\frac{e}{2}$ and $\frac{1}{2}$ respectively. — 1A+1A

$\frac{dy}{dx}=0$

$x^3-3x^2-x+3=0$ — 1M

$(x-3)(x-1)(x+1)=0$

$x=3$, $x=1$ or $x=-1$ (rejected)

$$\frac{d^2y}{dx^2}=\frac{(x^6-6x^5+10x^4+12x^3-17x^2-18x+10)e^x}{(x^3-x+2)^3}$$

$\left.\frac{d^2y}{dx^2}\right|_{x=1}=-e<0$ — 1M — for testing

$\left.\frac{d^2y}{dx^2}\right|_{x=3}=\frac{2e^3}{169}>0$

For $x=0$, we have $y=\frac{1}{2}$. For $x=5$, we have $y=\frac{e^5}{122}$. — 1M

Note that $\frac{1}{2}<\frac{e^3}{26}<\frac{e^5}{122}<\frac{e}{2}$.

Thus, the greatest value and the least value of y are $\frac{e}{2}$ and $\frac{1}{2}$ respectively. — 1A+1A

----------(7)

Solution	Marks	Remarks
8. (a) $f'(1)=\frac{8}{9}$		
$\frac{2^k}{1+2^k}=\frac{8}{9}$	1M	
$2^k=8$		
$k=3$	1A	
(b) Let $u=1+2^{3x}$.		
So, we have $\frac{du}{dx}=3(2^{3x})\ln 2$.	1M	
$f(x)$		
$=\int\frac{2^{3x}}{1+2^{3x}}dx$		
$=\frac{1}{3\ln 2}\int\frac{1}{u}du$	1M	
$=\frac{1}{3\ln 2}\ln\lvert u\rvert+C$	1M	
$=\frac{1}{3\ln 2}\ln(1+2^{3x})+C$		
When $x=1$, we have $y=2$.		
$f(1)=2$		
$\frac{1}{3\ln 2}\ln(1+2^3)+C=2$	1M	
$C=2-\frac{2\ln 3}{3\ln 2}$		
Thus, we have $f(x)=\frac{1}{3\ln 2}\ln(1+2^{3x})+2-\frac{2\ln 3}{3\ln 2}$.	1A	
	----------(7)	

Solution	Marks	Remarks
9. (a) $\mathrm{P}\left(Z<\frac{180-200}{\sigma}\right)=0.2119$		
$\frac{180-200}{\sigma}=-0.8$	1M	
$\sigma=25$		
$\mathrm{P}\left(\frac{180-200}{25}\leq Z<\frac{230-200}{25}\right)$	1M	
$=\mathrm{P}(-0.8\leq Z<1.2)$		
$=0.2881+0.3849$		
$=0.673$		
Thus, the required percentage is 67.3%.	1A ----------(3)	
(b) P(a potato is *big*) $=1-0.2119-0.673=0.1151$	1M	
The required probability		
$=C_1^3(0.1151)(1-0.1151)^2(0.1151)$	1M	
≈ 0.031121483		
≈ 0.0311	1A ----------(3)	r.t. 0.0311
(c) (i) The required probability		
$=C_1^5(0.1151)C_2^4(0.2119)^2(0.673)^2$	1M	
≈ 0.070224494		
≈ 0.0702	1A	r.t. 0.0702
(ii) The required probability		
$\approx\frac{0.070224494+C_1^5(0.1151)C_3^4(0.2119)^3(0.673)+C_1^5(0.1151)(0.2119)^4}{C_1^5(0.1151)(1-0.1151)^4}$	1M+1M	
≈ 0.244066836		
≈ 0.2441	1A ----------(5)	r.t. 0.2441

Solution	Marks	Remarks
10. (a) The required probability		
$= \frac{0.9^0 e^{-0.9}}{0!} + \frac{0.9^1 e^{-0.9}}{1!} + \frac{0.9^2 e^{-0.9}}{2!}$	1M+1M	
$= 2.305e^{-0.9}$	1A	r.t. 0.9371
	----------(3)	
(b) The required probability		
$= \frac{7.8^5 e^{-7.8}}{5!}$	1M	
$= 240.597864e^{-7.8}$	1A	r.t. 0.0986
	----------(2)	
(c) The required probability		
$= \left(\frac{1.3^2 e^{-1.3}}{2!}\right)\left(\frac{0.9^0 e^{-0.9}}{0!}\right) + \left(\frac{1.3^1 e^{-1.3}}{1!}\right)\left(\frac{0.9^1 e^{-0.9}}{1!}\right) + \left(\frac{1.3^0 e^{-1.3}}{0!}\right)\left(\frac{0.9^2 e^{-0.9}}{2!}\right)$	1M+1M	
$= 2.42e^{-2.2}$	1A	r.t. 0.2681
	----------(3)	
(d) The required probability		
$= \frac{\left(\frac{1.3^0 e^{-1.3}}{0!}\right)\left(\frac{0.9^2 e^{-0.9}}{2!}\right)}{2.42e^{-2.2}}$	1M+1M	
$= \frac{81}{484}$	1A	r.t. 0.1674
	----------(3)	
(e) P(the number of emails John receives in a certain hour is fewer than 3)		
$= \left(\frac{1.3^0 e^{-1.3}}{0!}\right)\left(\frac{0.9^0 e^{-0.9}}{0!}\right) + \left(\frac{1.3^1 e^{-1.3}}{1!}\right)\left(\frac{0.9^0 e^{-0.9}}{0!}\right)$		
$+ \left(\frac{1.3^0 e^{-1.3}}{0!}\right)\left(\frac{0.9^1 e^{-0.9}}{1!}\right) + 2.42e^{-2.2}$		
$= 5.62e^{-2.2}$		
The required probability		
$= \frac{\left(\frac{1.3^0 e^{-1.3}}{0!}\right)\left(2.305e^{-0.9}\right)}{5.62e^{-2.2}}$	1M+1M	
$= \frac{461}{1124}$	1A	r.t. 0.4101
	----------(3)	

Solution	Marks	Remarks
11. (a) $f'(x)$		
$=\frac{1}{2}\left(\frac{x}{2-x}\right)^{\frac{-1}{2}}\frac{(2-x)+x}{(2-x)^2}$		
$=x^{\frac{-1}{2}}(2-x)^{\frac{-3}{2}}$	1A	
$f''(x)$		
$=x^{\frac{-1}{2}}\left(\frac{-3}{2}\right)(2-x)^{\frac{-5}{2}}(-1)+\left(\frac{-1}{2}\right)x^{\frac{-3}{2}}(2-x)^{\frac{-3}{2}}$	1M	
$=\frac{1}{2}x^{\frac{-3}{2}}(2-x)^{\frac{-3}{2}}\left(\frac{3x}{2-x}-1\right)$		
$=x^{\frac{-3}{2}}(2-x)^{\frac{-5}{2}}(2x-1)$	1A ----------(3)	
(b) (i) J		
$\approx\frac{1}{2}\left(\frac{0.5}{5}\right)(f(0)+f(0.5)+2(f(0.1)+f(0.2)+f(0.3)+f(0.4)))$	1M	
≈ 0.177150823		
≈ 0.1772	1A	r.t. 0.1772
(ii) $\int_0^{0.5}f(x)dx+\int_{0.5}^1 f(x)dx=\int_0^1 f(x)dx$		
$J+K=\frac{\pi-2}{2}$	1M	
K		
$=\frac{\pi-2}{2}-J$		
$\approx\frac{\pi-2}{2}-0.177150823$		
≈ 0.393645504		
≈ 0.3936	1A	r.t. 0.3936
(iii) By (a), we have $f''(x)=x^{\frac{-3}{2}}(2-x)^{\frac{-5}{2}}(2x-1)<0$ for $0<x<0.5$.		
By (b)(i), we have $J>0.177150823$.	1	
Since $J=\frac{\pi-2}{2}-K$, we have $K<0.393645504$.	1M	
Note that $K>0$.		
$\frac{J}{K}$		
$>\frac{0.177150823}{0.393645504}$	1M	
≈ 0.450026282		
>0.44		
Thus, the claim is disagreed.	1A ---------(8)	f.t.

Solution	Marks	Remarks
12. (a) $\frac{d}{dt}\left(\frac{dV}{dt}\right)$		
$=\frac{1}{2\sqrt{t+1}}\sqrt{3-\sqrt{t+1}}+\sqrt{t+1}\left(\frac{1}{2\sqrt{3-\sqrt{t+1}}}\right)\left(\frac{-1}{2\sqrt{t+1}}\right)$	1M	
$=\frac{6-3\sqrt{t+1}}{4\sqrt{t+1}\sqrt{3-\sqrt{t+1}}}$		
$\frac{d}{dt}\left(\frac{dV}{dt}\right)=0$ $6-3\sqrt{t+1}=0$ $t=3$	1M	
(see table below)	1M	for testing
So, $\frac{dV}{dt}$ attains its maximum value when $t=3$. Thus, we have $T=3$.	1A	

	$0\le t<3$	$t=3$	$3<t\le 7$
$\frac{d}{dt}\left(\frac{dV}{dt}\right)$	+	0	–

Solution	Marks	Remarks
$\frac{d}{dt}\left(\frac{dV}{dt}\right)$ $=\frac{1}{2\sqrt{t+1}}\sqrt{3-\sqrt{t+1}}+\sqrt{t+1}\left(\frac{1}{2\sqrt{3-\sqrt{t+1}}}\right)\left(\frac{-1}{2\sqrt{t+1}}\right)$	1M	
$=\frac{6-3\sqrt{t+1}}{4\sqrt{t+1}\sqrt{3-\sqrt{t+1}}}$ $\frac{d}{dt}\left(\frac{dV}{dt}\right)=0$ $6-3\sqrt{t+1}=0$ $t=3$	1M	
$\frac{d^2}{dt^2}\left(\frac{dV}{dt}\right)$ $=\frac{-3t+18\sqrt{t+1}-39}{16(t+1)^{\frac{3}{2}}(3-\sqrt{t+1})^{\frac{3}{2}}}$ $\left.\frac{d^2}{dt^2}\left(\frac{dV}{dt}\right)\right\vert_{t=3}=\frac{-3}{32}<0$	1M	for testing
So, $\frac{dV}{dt}$ attains its maximum value when $t=3$. Thus, we have $T=3$.	1A	
	----------(4)	

Solution	Marks	Remarks
(b) Let $u=3-\sqrt{t+1}$.	1M	
So, we have $\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{-1}{2\sqrt{t+1}}$.		
Exact value of V when $t=T$		
$=\int_0^3\sqrt{t+1}\sqrt{3-\sqrt{t+1}}\,\mathrm{d}t$	1M	
$=-2\int_0^3(t+1)\sqrt{3-\sqrt{t+1}}\left(\frac{-1}{2\sqrt{t+1}}\right)\mathrm{d}t$		
$=-2\int_2^1(3-u)^2\sqrt{u}\,\mathrm{d}u$	1M	
$=2\int_1^2(9u^{\frac{1}{2}}-6u^{\frac{3}{2}}+u^{\frac{5}{2}})\mathrm{d}u$		
$=2\left[6u^{\frac{3}{2}}-\frac{12}{5}u^{\frac{5}{2}}+\frac{2}{7}u^{\frac{7}{2}}\right]_1^2$	1M	
$=\frac{328\sqrt{2}-272}{35}$	1A	
	----------(5)	
(c) (i) Let r m be the radius of the surface of rain water in the tank.		
$\frac{h}{r}=\frac{1}{6}$		
$r=6h$		
V		
$=\frac{1}{3}\pi r^2h$		
$=\frac{1}{3}\pi(6h)^2h$	1M	
$=12\pi h^3$		
$\frac{\mathrm{d}V}{\mathrm{d}t}=36\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}$	1M	
$Q=36\pi$	1A	

Solution	Marks	Remarks
(ii) $V\big\|_{t=T}=\dfrac{328\sqrt{2}-272}{35}$		
$12\pi(h\big\|_{t=T})^3=\dfrac{328\sqrt{2}-272}{35}$		
$h\big\|_{t=T}=\left(\dfrac{82\sqrt{2}-68}{105\pi}\right)^{\frac{1}{3}}$		
$\dfrac{\mathrm{d}V}{\mathrm{d}t}\bigg\|_{t=T}$		
$=\sqrt{3+1}\sqrt{3-\sqrt{3+1}}$		
$=2$		
At $t=T$, we have		
$2=36\pi\left(\dfrac{82\sqrt{2}-68}{105\pi}\right)^{\frac{2}{3}}\dfrac{\mathrm{d}h}{\mathrm{d}t}\bigg\|_{t=T}$	1M	
$\dfrac{\mathrm{d}h}{\mathrm{d}t}\bigg\|_{t=T}=\dfrac{1}{18\pi^{\frac{1}{3}}}\left(\dfrac{105}{82\sqrt{2}-68}\right)^{\frac{2}{3}}$	1A	r.t. 0.0640
	----------(5)	